SOME APPLICATIONS OF ALGEBRAIC CYCLES TO AFFINE ALGEBRAIC GEOMETRY

V. SRINIVAS

In this series of talks, I will discuss some applications of the theory of algebraic cycles to affine algebraic geometry (i.e., to commutative algebra).

1. The Chow ring and Chern classes

First, we recall the definition of the graded *Chow ring* $CH^*(X) = \bigoplus_{p\geq 0} CH^p(X)$ of a non-singular variety X over a field k (see [9] for more details; see also [3]). We will usually (but not always) take k to be algebraically closed; X need not be irreducible. The graded components $CH^p(X)$ generalize the more familiar notion of the *divisor class group*, which is just the group $CH^1(X)$.

If $Z \subset X$ is irreducible, let $\mathcal{O}_{Z,X}$ be the local ring of Z on X (*i.e.*, the local ring of the generic point of Z, in the terminology of Hartshorne's book [12]). The codimension of Z in X, denoted $\mathrm{codim}_X Z$, is the dimension of the local ring $\mathcal{O}_{Z,X}$. Now let

$$\begin{aligned} Z^p(X) &= \text{Free abelian group on irreducible subvarieties of } X \text{ of codimension } p \\ &= \text{Group of algebraic cycles on } X \text{ of codimension } p. \end{aligned}$$

For an irreducible subvariety $Z \subset X$, let $[Z]$ denote its class in $Z^p(X)$ (where $p = \mathrm{codim}_X Z$).

Let $Y \subset X$ be irreducible of codimension $p-1$, and let $k(Y)^*$ denote the multiplicative group of non-zero rational functions on Y ($k(Y)$, which is the field of rational functions on Y, is the residue field of $\mathcal{O}_{Y,X}$). For each irreducible divisor $Z \subset Y$, we have a homomorphism $\mathrm{ord}_Z : k(Y)^* \to \mathbb{Z}$, given by

$$\mathrm{ord}_Z(f) = \ell(\mathcal{O}_{Z,Y}/a\mathcal{O}_{Z,Y}) - \ell(\mathcal{O}_{Z,Y}/b\mathcal{O}_{Z,Y}),$$

for any expression of f as a ratio $f = a/b$ with $a, b \in \mathcal{O}_{Z,Y} \setminus \{0\}$. Here $\ell(M)$ denotes the length of an Artinian module M.

For $f \in k(Y)^*$, let $(f)_Y$ denote the divisor of f on Y, defined by

$$(f)_Y = \sum_{Z\subset Y} \mathrm{ord}\,_Z(f)\cdot [Z],$$

where Z runs over all irreducible divisors in Y; the sum has only finitely many non-zero terms, and is hence well-defined. Clearly we may also view $(f)_Y$ as an element of $Z^p(X)$.

Let $R^p(X) \subset Z^p(X)$ be the subgroup generated by cycles $(f)_Y$ as (Y, f) ranges over all irreducible subvarieties Y of X of codimension $p-1$, and all $f \in k(Y)^*$.

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We refer to elements of $R^p(X)$ as cycles *rationally equivalent to 0* on X. The p-th Chow group of X is defined to be

$$CH^p(X) = \frac{Z^p(X)}{R^p(X)}$$
$$= \text{group of rational equivalence classes of codimension } p\text{-cycles on } X.$$

We will abuse notation and also use $[Z]$ to denote the class of an irreducible subvariety Z in $CH^p(X)$.

The graded abelian group

$$CH^*(X) = \bigoplus_{0\leq p\leq \dim X} CH^p(X)$$

can be given the structure of a commutative (graded) ring via the *intersection product*. This product is characterized by the following property — if $Y \subset X$, $Z \subset X$ are irreducible of codimensions p, q respectively, and $Y \cap Z = \cup_i W_i$, where each $W_i \subset X$ is irreducible of codimension $p + q$ (we then say Y and Z *intersect properly* in X), then the intersection product of the classes $[Y]$ and $[Z]$ is

$$[Y] \cdot [Z] = \sum_i I(Y, Z; W_i)[W_i]$$

where $I(Y, Z; W_i)$ is the *intersection multiplicity* of Y and Z along W_i, defined by Serre's formula

$$I(Y, Z; W_i) = \sum_{j\geq 0} (-1)^j \ell \left(\mathrm{Tor}_j^{\mathcal{O}_{W_j,X}} (\mathcal{O}_{W_j,Y}, \mathcal{O}_{W_j,Z}) \right).$$

One of the important results proved in the book [9] is that the above procedure does give rise to a well-defined ring structure on $CH^*(X)$.

The Chow ring is an algebraic analogue for the even cohomology ring

$$\bigoplus_{i=0}^{n} H^{2i}(X, \mathbb{Z})$$

defined in algebraic topology. To illustrate this, we note the following 'cohomology-like' properties. Here, we follow the convention of [12], and use the term "vector bundle on X" to mean "(coherent) locally free sheaf of $\mathcal{O}_X$-modules", and use the term "geometric vector bundle on X", as in [12] II Ex. 5.18, to mean a Zariski locally trivial algebraic fiber bundle $V \to X$ whose fibres are affine spaces, with linear transition functions. With this convention, we can also identify vector bundles on an affine variety $X = \operatorname{Spec} A$ with finitely generated projective A-modules; as in [12], we use the notation $\widetilde{M}$ to denote the coherent sheaf corresponding to a finitely generated A-module M.

Theorem 1.1. *(***Properties of the Chow ring and Chern classes***)*

(1) $X \mapsto \bigoplus_p CH^p(X)$ *is a contravariant functor from the category of smooth varieties over k to graded rings. If $X = \coprod_i X_i$, where X_i are the irreducible (= connected) components, then $CH^*(X) = \prod_i CH^*(X_i)$.*

(2) *If X is irreducible and projective (or more generally, proper) over an algebraically closed field k and $d = \dim X$, there is a well defined* degree homomorphism

$$\deg : CH^d(X) \to \mathbb{Z}$$

given by $\deg(\sum_i n_i[x_i]) = \sum_i n_i$. *This allows one to define* **intersection numbers** *of cycles of complementary dimension, in a purely algebraic way, which agree with those defined via topology when $k = \mathbb{C}$ (see (7) below).*

(3) *If $f : X \to Y$ is a proper morphism of smooth varieties, there are "Gysin" (or "push-forward") maps $f_* : CH^p(X) \to CH^{p+r}(Y)$ for all p, where $r = \dim Y - \dim X$; here if $p + r < 0$, we define f_* to be 0. The induced map $CH^*(X) \to CH^*(Y)$ is $CH^*(Y)$-linear (***projection formula***), where $CH^*(X)$ is regarded as a $CH^*(Y)$-module via the (contravariant) ring homomorphism $f^* : CH^*(Y) \to CH^*(X)$. If $f : X \hookrightarrow Y$ is the inclusion of a closed subvariety, then f_* is induced by the natural inclusions $Z^p(X) \hookrightarrow Z^{p+r}(Y)$.*

(4) $f^* : CH^*(X) \xrightarrow{\cong} CH^*(V)$ *for any geometric vector bundle $f : V \to X$* **(homotopy invariance)**. *In particular, $CH^*(X \times \mathbb{A}^n) = CH^*(X)$, and $CH^*(\mathbb{A}^n) = \mathbb{Z}$.*

(5) *If V is a vector bundle (i.e., locally free sheaf) of rank r on X, then there are* **Chern classes** $c_p(V) \in CH^p(X)$, *such that*
   (a) $c_0(V) = 1$,
   (b) $c_p(V) = 0$ *for* $p > r$, *and*
   (c) *for any exact sequence of vector bundles*

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

   we have $c(V_2) = c(V_1)c(V_3)$, where $c(V_i) = \sum_p c_p(V_i)$ are the corresponding **total Chern classes**
   (d) $c_p(V^\vee) = (-1)^p c_p(V)$, *where $V^\vee$ is the dual vector bundle.*

   Moreover, we also have the following properties.

(6) *If $f : \mathbb{P}(V) = \mathbf{Proj}\, S(V) \to X$ is the projective bundle associated to a vector bundle of rank r (where $S(V)$ is the symmetric algebra of the sheaf V over $\mathcal{O}_X$), then $CH^*(\mathbb{P}(V))$ is a $CH^*(X)$-algebra generated by $\xi = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$, the first Chern class of the tautological line bundle, which is subject to the relation*

$$\xi^r - c_1(V)\xi^{r-1} + \cdots + (-1)^r c_n(V) = 0;$$

in particular, $CH^*(\mathbb{P}(V))$ is a free $CH^*(X)$-module with basis $1, \xi, \xi^2, \ldots, \xi^{r-1}$.

(7) *If $k = \mathbb{C}$, there are cycle class homomorphisms $CH^p(X) \to H^{2p}(X, \mathbb{Z})$ such that the intersection product corresponds to the cup product in cohomology, and for a vector bundle E, the cycle class of $c_p(E)$ is the topological p-th Chern class of E.*

(8) *The first Chern class determines an isomorphism $\mathrm{Pic} X \to CH^1(X)$ from the Picard group of line bundles on X to the first Chow group (i.e., the divisor class group) of X. For an arbitrary vector bundle V, of rank n, we have $c_1(V) = c_1(\det V)$, where $\det V = \bigwedge^n V$.*

(9) *If $f : X \to Y$ is a morphism between non-singular varieties, V a vector bundle on Y, then the Chern classes of the pull-back vector bundle f^*V on X are given by $c(f^*V) = f^*c(V)$, where on the right, f^* is the ring homomorphism $CH^*(Y) \to CH^*(X)$* **(functoriality of Chern classes)**. *In particular, taking Y = point, we see that $c(\mathcal{O}_X) = 1 \in CH^*(X)$.*

(10) *If $i : Y \hookrightarrow X$ is the inclusion of an irreducible smooth subvariety of codimension r in a smooth variety, with normal bundle $\mathcal{N} = (\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee$ (where $\mathcal{I}_Y \subset \mathcal{O}_X$ is the ideal sheaf of Y in X), then $\mathcal{N}$ is a vector bundle on Y of rank r with top Chern class*

$$c_r(\mathcal{N}) = i^* \circ i_*[Y],$$

where $[Y] \in CH^0(Y) = \mathbb{Z}$ is the generator **(self-intersection formula)**.

Remark 1.2. If $X = \mathrm{Spec}\, A$ is affine, we will also sometimes write $CH^*(A)$ in place of $CH^*(X)$; similarly, by the Chern classes $c_i(P)$ of a finitely generated projective A-module P, we mean $c_i(\widetilde{P})$ where $\widetilde{P}$ is the associated vector bundle on X.

We remark that the total Chern class of a vector bundle on a smooth variety X is a *unit* in the Chow ring $CH^*(X)$, since it is of the form $1+$(nilpotent element). Thus the assignment $V \mapsto c(V)$ gives a homomorphism of groups from the Grothendieck group $K_0(X)$ of vector bundles (locally free sheaves) on X to the multiplicative group of those units in the graded ring $CH^*(X)$, which are expressible as $1 +$ (higher degree terms).

On a non-singular variety X, every coherent sheaf has a resolution by locally free sheaves (vector bundles) of finite rank, and the Grothendieck group $K_0(X)$ of vector bundles coincides with the Grothendieck group of coherent sheaves. There is a finite decreasing filtration $\{F^pK_0(X)\}_{p\geq 0}$ on $K_0(X)$, where $F^pK_0(X)$ is the subgroup generated by classes of sheaves supported in codimension $\geq p$. Further, $F^pK_0(X)/F^{p+1}K_0(X)$ is generated, as an abelian group, by the classes $\mathcal{O}_Z$ for irreducible subvarieties $Z \subset X$ of codimension p – for example, if $X = \mathrm{Spec}\, A$ is affine, we can see this using the fact that any finitely generated A-module M has a finite filtration whose quotients are of the form $A/\wp$ for prime ideals $\wp$, such that the minimal primes in $\mathrm{supp}\,(M)$ all occur, and their multiplicities in the filtration are independent of the choice of filtration. Thus, we have a natural surjection $Z^p(X) \to F^pK_0(X)/F^{p+1}K_0(X)$.

If $\mathcal{F}$ is any coherent sheaf on X whose support is of codimension p, recall that we can associate to it a codimension p cycle

$$|\mathcal{F}| \in Z^p(X),$$

by

$$|\mathcal{F}| = \sum_{W\subset X} \ell(\mathcal{F}_{\eta_W}),$$

where W ranges over the irreducible, codimension p subvarieties of X in the support of $\mathcal{F}$, and η_W is the generic point of W, so that the stalk $\mathcal{F}_{\eta_W}$ is a module of finite length over the local ring $\mathcal{O}_{W,X} = \mathcal{O}_{\eta_W,X}$. If V is a vector bundle of rank p, and s a section, then s induces a map of sheaves $V^\vee \to \mathcal{O}_X$, whose image is an ideal sheaf $\mathcal{I}_Y$, where Y is the zero scheme of s. Since the ideal sheaf of Y is locally generated by p elements, each irreducible component of Y_{red} has codimension $\leq p$. If Y is purely of codimension p, there is thus an associated cycle $|Y| = |\mathcal{O}_Y| \in Z^p(X)$.

Now we can state the following result, part (d) of which is sometimes called the *Riemann-Roch theorem without denominators* (see the book [9] for a proof). Note that (c) is consistent with the self-intersection formula, in the case when Y is nonsingular, since in this case, $V \otimes \mathcal{O}_Y$ is identified with the normal bundle.

Theorem 1.3. *Let X be a non-singular variety.*

(a) *If $x \in F^pK_0(X)$, then $c_i(x) = 0$ for $i < p$, and $c_p : F^pK_0(X) \to CH^p(X)$ is a group homomorphism. Let $\overline{c_p} : F^pK_0(X)/F^{p+1}K_0(X) \to CH^p(X)$ be the induced homomorphism.*
(b) *The natural surjection $Z^p(X) \twoheadrightarrow F^pK_0(X)/F^{p+1}K_0(X)$ factors through rational equivalence, yielding a map $\psi_p : CH^p(X) \twoheadrightarrow F^pK_0(X)/F^{p+1}K_0(X)$.*
(c) *Let V be a vector bundle of rank p on X, and s a section with zero scheme Y, which has codimension p. Then $|Y| \in Z^p(X)$ is a cycle representing the p-th Chern class $c_p(V) \in CH^p(X)$.*
(d) *The compositions $\overline{c_p}\circ\psi_p$ and $\psi_p\circ\overline{c_p}$ both equal multiplication by the integer $(-1)^{p-1}(p-1)!$. In particular, both $\overline{c_p}$ and ψ_p are isomorphisms $\otimes\mathbb{Q}$.*

In particular, if $Z \subset X$ is an irreducible subvariety of codimension p, then $c_i([\mathcal{O}_Z]) = 0$ for $i < p$, and $c_p([\mathcal{O}_Z]) = (-1)^{p-1}(p-1)![Z] \in CH^p(X)$.

Remark 1.4. If $X = \operatorname{Spec} A$ is affine, any element $\alpha \in K_0(X)$ can be expressed as a difference $\alpha = [P] - [A^{\oplus m}]$ for some finitely generated projective A-module P and some positive integer m. Hence the total Chern class $c(\alpha)$ coincides with $c(P)$. The above theorem now implies that for any element $a \in CH^p(X)$, there is a finitely generated projective A module P with $c_p(P) = (p-1)!a$. By the Bass stability theorem, which implies that any projective A-module of rank $> d = \dim A$ has a free direct summand of positive rank, we can find a projective A-module P with $\operatorname{rank} P \leq d$ and $c_p(P) = (p-1)!a$.

Incidentally, this statement cannot be improved, in general: for any $p > 2$, there are examples of affine non-singular varieties X and elements $a \in CH^p(X)$ such that $ma \in \operatorname{image} c_p$ for some integer $m \iff (p-1)!|m$.

2. An example of a graded ring

We now discuss our first application of these constructions, due to N. Mohan Kumar (unpublished). It is a counterexample to the "principle": if a commutative algebra problem with graded data has a solution, then it also has a graded solution.

Let $k = \overline{k}$. We give an example of a 3-dimensional, regular, graded integral domain $A = \bigoplus_{n\geq 0} A_n$, with the following properties:

(1) A is generated by A_1 as an A_0-algebra, where A_0 is a regular affine k-algebra of dimension 1
(2) the "irrelevant graded prime ideal" $P = \bigoplus_{n>0} A_n$ is the radical of an ideal generated by 2 elements (i.e., the subvariety of $Z = \operatorname{Spec} A$ defined by P is a set-theoretic complete intersection in Z)
(3) P cannot be expressed as the radical of an ideal generated by 2 *homogeneous* elements (i.e., the subvariety is not a "homogeneous" set-theoretic complete intersection).

For the example, take A_0 to be affine coordinate ring of a non-singular curve $C \subset \mathbb{A}^3_k$ such that the canonical module $\omega_{A_0} = \Omega_{A_0/k}$ is a non-torsion element of the divisor class group of A_0 (this implies k is not the algebraic closure of a finite field). In fact, if we choose A_0 to be a non-singular affine k-algebra of dimension 1 such that ω_{A_0} is non-torsion in the class group, then $C = \operatorname{Spec} A_0$ can be realized as a curve embedded in $\mathbb{A}^3_k$, by more or less standard arguments (see [12], IV, or [26], for example).

Let $R = k[x, y, z]$ denote the polynomial algebra, and let $\varphi : R \to A_0$ be the surjection corresponding to $C \hookrightarrow \mathbb{A}^3_k$. Let $I = \ker \varphi$ be the ideal of C. Then I/I^2 is a projective A_0-module of rank 2; we let

$$A = S(I/I^2) = \bigoplus_{n\geq 0} S^n(I/I^2)$$

be its symmetric algebra over A_0. We claim this graded ring A has the properties stated above.

Consider the exact sequence of projective A_0-modules

$$0 \to I/I^2 \xrightarrow{\psi} \Omega_{R/k} \otimes A_0 \xrightarrow{\overline{\varphi}} \omega_{A_0} \to 0 \tag{2.1}$$

with $\overline{\varphi}$ induced by φ, and ψ by the derivation $d : R \to \Omega_{R/k}$. Let

$$h : \Omega_{R/k} \otimes A_0 \to I/I^2$$

be a splitting of ψ. Use h to define a homomorphism of k-algebras

$$\Phi : R \to A,$$

by setting

$$\Phi(t) = \phi(t) + h(dt) \in A_0 \oplus A_1 = A_0 \oplus I/I^2$$

for $t = x, y, z$; this uniquely specifies a k-algebra homomorphism Φ defined on the polynomial algebra R.

Clearly $\Phi(I) \subset P = \bigoplus_{n>0} A_n$, the irrelevant graded ideal, and one verifies that Φ induces isomorphisms $R/I \to A/P$ and $I/I^2 \to P/P^2$, and in fact an isomorphism between the I-adic completion of R and the P-adic completion of A.

Since $C \subset \mathbb{A}^3_k$ is a non-singular curve, it is a set-theoretic complete intersection, from a theorem of Ferrand and Szpiro (see [32], for example). If $a, b \in I$ with $\sqrt{(a,b)} = I$, then clearly we have $\sqrt{(\Phi(a), \Phi(b))} = P \cap Q$, for some (radical) ideal Q with $P + Q = A$. We can correspondingly write $(\Phi(a), \Phi(b)) = J \cap J'$ with $\sqrt{J} = P$, $\sqrt{J'} = Q$. Then $J/J^2 \cong (A/J)^{\oplus 2}$. This implies (by an old argument of Serre) that $\mathrm{Ext}^1_A(J, A) \cong \mathrm{Ext}^2_A(A/J, A) \cong A/J$ is free of rank 1, and any generator determines an extension

$$0 \to A \to V \to J \to 0$$

where V is a projective A-module of rank 2, and such that the induced surjection $V \otimes A/J \to J/J^2 \cong (A/J)^{\oplus 2}$ is an isomorphism.

We claim the projective module V is necessarily of the form $V = V_0 \otimes_{A_0} A$; this implies $V_0 = V \otimes_A A/P \cong J/PJ \cong (A/P)^{\oplus 2}$ is free, so that V is a free A-module, and J is generated by 2 elements. To prove the claim, note that I/I^2 is a direct summand of a free $A/I = A_0$-module of finite rank; hence there is an affine A-algebra $A' \cong A_0[x_1, \ldots, x_n]$, which is a polynomial algebra over A_0, such that A is an algebra retract of A'. Now it suffices to observe that any finitely generated projective A'-module is of the form $M \otimes_{A_0} A'$, for some projective A_0-module M; this is the main result of [13], the solution of the so-called Bass-Quillen Conjecture (see also [14]).

On the other hand, we claim that it is impossible to find two *homogeneous* elements $x, y \in P$ with $\sqrt{(x,y)} = P$. Indeed, let $X = \mathrm{Proj}\, A$, and $\pi : X \to C = \mathrm{Spec}\, A_0$ be the natural morphism. Then $X = \mathbb{P}(V)$ is the $\mathbb{P}^1$-bundle over C associated to the locally free sheaf $V = \widetilde{I/I^2}$ (the sheaf determined by the projective A_0-module I/I^2). Let $\xi = c_1(\mathcal{O}_X(1)) \in CH^1(X)$ be the 1st Chern class of the tautological line bundle $\mathcal{O}_X(1)$. Then by Theorem 1.1(6) above, $CH^*(X)$ is a free $CH^*(C)$-module with basis $1, \xi$, and ξ satisfies the monic relation

$$\xi^2 - c_1(V)\xi + c_2(V) = 0.$$

Since $\dim C = 1$, $CH^i(C) = 0$ for $i > 1$, and so this relation reduces to

$$\xi^2 = c_1(V)\xi.$$

From the exact sequence (2.1), we have a relation in $CH^*(C)$

$$1 = c(\mathcal{O}_C)^3 = c(\mathcal{O}_C^{\oplus 3}) = c(\Omega_{\mathbb{A}^3/k} \otimes \mathcal{O}_C) = c(V) \cdot c(\omega_C).$$

Hence $c_1(V) = -c_1(\omega_C)$, which by the choice of C is a non-torsion element of $CH^1(C)$ (which is the divisor class group of A_0). Thus $\xi^2 \in CH^2(X)$ is a non-torsion element of $CH^2(X)$.

If homogeneous elements $x, y \in P$ exist, say of degrees r and s respectively, such that $\sqrt{(x,y)} = P$, then we may regard x, y as determining global sections of the sheaves $\mathcal{O}_X(r)$ and $\mathcal{O}_X(s)$ respectively, *which have no common zeroes on* X. Let $D_x \subset X$, $D_y \subset X$ be the divisors of zeroes of $x \in \Gamma(X, \mathcal{O}_X(r))$ and $y \in \Gamma(X, \mathcal{O}_X(s))$ respectively. Then we have equations in $CH^1(X)$

$$[D_x] = c_1(\mathcal{O}_X(r)) = rc_1(\mathcal{O}_X(1)) = r\xi, \;\; [D_y] = c_1(\mathcal{O}_X(s)) = sc_1(\mathcal{O}_X(1)) = s\xi.$$

But $D_x \cap D_y = \emptyset$. Hence in $CH^2(X)$, we have a relation

$$0 = [D_x] \cdot [D_y] = rs\xi^2,$$

contradicting that $\xi^2 \in CH^2(X)$ is a non-torsion element.

Remark 2.1. The construction of the homomorphism from the polynomial ring R to the graded ring A is an algebraic analogue of the *exponential map* in Riemannian geometry, which identifies a tubular neighbourhood of a smooth submanifold of a Riemannian manifold with the normal bundle of the submanifold (see [15, Theorem 11.1], for example). The exponential map is usually constructed using geodesics on the ambient manifold; here we use the global structure of affine space, where "geodesics" are lines, to make a similar construction algebraically. This idea appears in a paper[6] of Boratynski, who uses it to argue that a smooth subvariety of $\mathbb{A}^n$ is a set-theoretic complete intersection if and only if the zero section of its normal bundle is a set-theoretic complete intersection in the total space of the normal bundle.

3. Zero cycles on non-singular proper and affine varieties

In this section, we discuss results of Mumford and Roitman, which give criteria for the non-triviality of $CH^d(X)$ where X is a non-singular variety over $\mathbb{C}$ of dimension $d \geq 2$, which is either proper, or affine.

If X is non-singular and irreducible, and $\dim X = d$, then $Z^d(X)$ is just the free abelian group on the (closed) points of X. Elements of $Z^d(X)$ are called *zero cycles* on X (since they are linear combinations of irreducible subvarieties of dimension 0). In the presentation $CH^d(X) = Z^d(X)/R^d(X)$, the group $R^d(X)$ of relations is generated by divisors of rational functions on irreducible curves in X.

The main non-triviality result for zero cycles is the following result, called the *infinite dimensionality theorem for 0-cycles.* It was originally proved (without $\otimes\mathbb{Q}$) by Mumford [17], for surfaces, and extended to higher dimensions by Roitman [22]; the statement with $\otimes\mathbb{Q}$ follows from [23].

Theorem 3.1. (Mumford, Roitman) *Let X be an irreducible, proper, non-singular variety of dimension d over $\mathbb{C}$. Suppose X supports a non-zero regular q-form (i.e., $\Gamma(X, \Omega^q_{X/\mathbb{C}}) \neq 0$), for some $q > 0$. Then for any closed algebraic subvariety $Y \subset X$ with $\dim Y < q$, we have $CH^d(X - Y) \otimes \mathbb{Q} \neq 0$.*

Corollary 3.2. *Let X be an irreducible, proper, non-singular variety of dimension d over $\mathbb{C}$, such that $\Gamma(X, \omega_X) \neq 0$. Then for any affine open subset $V \subset X$, we have $CH^d(V) \otimes \mathbb{Q} \neq 0$.*

The corollary results from the identification of ω_X with the sheaf $\Omega^d_{X/\mathbb{C}}$ of d-forms.

Bloch [3] gave another proof of the above result, using the action of algebraic correspondences on the étale cohomology, and generalized the result to arbitrary characteristics. In [27] and [28], Bloch's argument (for the case of characteristic 0) is recast in the language of differentials, extending it as well to certain singular varieties. One way of stating the infinite dimensionality results of [27] and [28], in the smooth case, is the following. The statement is technical, but it will be needed below when discussing M. Nori's construction of indecomposable projective modules.

We recall the notion of a k-generic point of an irreducible variety; we do this in a generality sufficient for our purposes. If X_0 is an irreducible k-variety, where $k \subset \mathbb{C}$ is a countable algebraically closed subfield, a point $x \in X = (X_0)_{\mathbb{C}}$ determines an irreducible subvariety $Z \subset X$, called the k-closure of X, which is the smallest subvariety of X which is defined over k (*i.e.*, of the form $(Z_0)_{\mathbb{C}}$ for some subvariety $Z_0 \subset X_0$) and contains the chosen point x. We call x a *k-generic point* if its k-closure is X itself.

In the case X_0 (and thus also X) is affine, say $X_0 = \operatorname{Spec} A$, and $X = \operatorname{Spec} A_{\mathbb{C}}$ with $A_{\mathbb{C}} = A \otimes_k \mathbb{C}$, then a point $x \in X$ corresponds to a maximal ideal $\mathfrak{m}_x \subset A_{\mathbb{C}}$. Let $\wp_x = A \cap \mathfrak{m}_x$, which is a prime ideal of A, not necessarily maximal. Then, in the earlier notation, $\wp_x$ determines an irreducible subvariety $Z_0 \subset X_0$. The k-closure $Z \subset X$ of x is the subvariety determined by the prime ideal $\wp_x A_{\mathbb{C}}$ (since k is algebraically closed, $\wp_x A_{\mathbb{C}}$ is a prime ideal). In particular, x is a k-generic point $\iff \wp_x = 0$. In this case, x determines an inclusion $A \hookrightarrow A_{\mathbb{C}}/\mathfrak{m}_x = \mathbb{C}(x) \cong \mathbb{C}$. This in turn gives an inclusion $i_x : K \hookrightarrow \mathbb{C}$ of the quotient field K of A (*i.e.*, of the function field $k(X_0)$) into the complex numbers.

In general, even if X is not affine, if we are given a k-generic point $x \in X$, we can replace X by any affine open subset defined over k, which will (because x is k-generic) automatically contain x; one verifies easily that the corresponding inclusion $K \hookrightarrow \mathbb{C}$ does not depend on the choice of this open subset. Thus we obtain an inclusion $i_x : K \hookrightarrow \mathbb{C}$ of the function field $K = k(X_0)$ into $\mathbb{C}$, associated to any k-generic point of X.

It is easy to see that the procedure is reversible: any inclusion of k-algebras $i : K \hookrightarrow \mathbb{C}$ determines a unique k-generic point of X. Indeed, choose an affine open subset $\operatorname{Spec} A = U_0 \subset X_0$, so that K is the quotient field of A. The induced inclusion $A \hookrightarrow \mathbb{C}$ induces a *surjection* of $\mathbb{C}$-algebras $A_{\mathbb{C}} \to \mathbb{C}$, whose kernel is a maximal ideal, giving the desired k-generic point.

Suppose now that X_0 is proper over k, and so X is proper over $\mathbb{C}$ (e.g., X is projective). Let $\dim X_0 = \dim X = d$. Then by the Serre duality theorem, the sheaf cohomology group $H^d(X, \mathcal{O}_X)$ is the dual $\mathbb{C}$-vector space to

$$\Gamma(X, \Omega^d_{X/\mathbb{C}}) = \Gamma(X, \omega_X) = \Gamma(X_0, \omega_{X_0}) \otimes_k \mathbb{C}.$$

Hence we may identify $H^d(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^d_{\mathbb{C}/k}$ with

$$\operatorname{Hom}_{\mathbb{C}}(\Gamma(X, \omega_X), \Omega^d_{\mathbb{C}/k}) = \operatorname{Hom}_k(\Gamma(X_0, \omega_{X_0}), \Omega^d_{\mathbb{C}/k}).$$

Note that a k-generic point x determines, via the inclusion $i_x : K \hookrightarrow \mathbb{C}$, a k-linear inclusion $\Omega^n_{K/k} \hookrightarrow \Omega^n_{\mathbb{C}/k}$, and hence, via the obvious inclusion

$$\Gamma(X_0, \omega_{X_0}) = \Gamma(X_0, \Omega^n_{X_0/k}) \hookrightarrow \Omega^n_{K/k},$$

a canonical element

$$di_x \in \operatorname{Hom}_k(\Gamma(X_0, \omega_{X_0}), \Omega^d_{\mathbb{C}/k}) = H^d(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^d_{\mathbb{C}/k}.$$

Theorem 3.3. *Let $k \subset \mathbb{C}$ be a countable algebraically closed subfield, and X_0 an irreducible non-singular proper k-variety of dimension d, with $\Gamma(X_0, \omega_{X_0}) \neq 0$. Let $U_0 \subset X_0$ be any Zariski open subset. Let $X = (X_0)_{\mathbb{C}}$, $U = (U_0)_{\mathbb{C}}$ be the corresponding complex varieties. Then there is a homomorphism of graded rings*

$$CH^*(U) \to \bigoplus_{p\geq 0} H^p(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^p_{\mathbb{C}/k},$$

with the following properties.

(i) *If $x \in U$ is a point, which is not k-generic, then the image in $H^d(X, \mathcal{O}_X) \otimes \Omega^d_{\mathbb{C}/k}$ of $[x] \in CH^d(U)$ is zero.*

(ii) *If $x \in U$ is a k-generic point, then the image in $H^d(X, \mathcal{O}_X) \otimes \Omega^d_{\mathbb{C}/k}$ of $[x] \in CH^d(U)$ is (up to sign) the canonical element di_x described above.*

As stated earlier, the above more explicit form of the infinite dimensionality theorem follows from results proved in [27] and [28].

4. Zero cycle obstructions to embedding and immersing affine varieties

We now consider the following two problems, which turn out to have some similarities. We will show how, in each case, the problem reduces to finding an example for which the Chern classes of the cotangent bundle (*i.e.*, the sheaf of Kähler differentials) have appropriate properties. We will then see, in Example 4.4, how to construct examples with these properties. The discussion is based on the article [4] of Bloch, Murthy and Szpiro.

Problem 4.1. Find examples of n-dimensional, non-singular affine algebras A over (say) the complex number field $\mathbb{C}$, for each $n \geq 1$, such that A cannot be generated by $2n$ elements as a $\mathbb{C}$-algebra, or such that the module of Kähler differentials cannot be generated by $2n-1$ elements $da_1, \ldots, da_{2n-1}$ (in contrast, it is a "classical" result that such an algebra A can always be generated by $2n+1$ elements, and its Káhler differentials can always be generated by $2n$ exact 1-forms; see, for example, [26]).

Problem 4.2. Find examples of prime ideals I of height $< N$ in a polynomial ring $\mathbb{C}[x_1, \ldots, x_N]$ such that $\mathbb{C}[x_1, \ldots, x_N]/I$ is regular, but I cannot be generated by $N-1$ elements (a theorem of Mohan Kumar [16] implies that such an ideal I can always be generated by N elements).

First we discuss Problem 4.1. Suppose A is an affine smooth $\mathbb{C}$-algebra which is an integral domain of dimension n. Assume $X = \operatorname{Spec} A$ can be generated by $2n$ elements, *i.e.*, that there is a surjection $f : \mathbb{C}[x_1, \ldots, x_{2n}] \to A$ from a polynomial ring. Let $I = \ker f$. If $i : X \hookrightarrow \mathbb{A}^{2n}_{\mathbb{C}}$ is the embedding corresponding to the surjection f, then the normal bundle to i is the sheaf $V^\vee$, where $V = \widetilde{I/I^2}$.

From the self-intersection formula, and the formula for the Chern class of the dual of a vector bundle, we see that

$$(4.1) \qquad (-1)^n c_n(V) = c_n(V^\vee) = i^* i_*[X] = 0,$$

since $CH^n(\mathbb{A}^{2n}_{\mathbb{C}}) = 0$.

On the other hand, suppose $j : X \hookrightarrow Y$ is any embedding as a closed subvariety of a non-singular affine variety Y whose cotangent bundle (*i.e.*, sheaf of Kähler differentials) $\Omega_{Y/\mathbb{C}}$ is a trivial bundle. For example, we could take $Y = \mathbb{A}^{2n}_{\mathbb{C}}$, and $j = i$, but below we will consider a different example as well.

Let W be the conormal bundle of X in Y (if $Y = \operatorname{Spec} B$, and $J = \ker j^* : B \twoheadrightarrow A$, then $W = \widetilde{J/J^2}$). We then have an exact sequence of vector bundles on X

$$0 \to W \to j^*\Omega_{Y/\mathbb{C}} \to \Omega^1_{X/\mathbb{C}} \to 0.$$

Since $\Omega_{Y/\mathbb{C}}$ is a trivial vector bundle, we get that

$$c(W) = c(\Omega_{X/\mathbb{C}})^{-1} \in CH^*(X). \tag{4.2}$$

Note that this expression for $c(W)$, and hence the resulting formula for $c_n(W)$ as a polynomial in the Chern classes of $\Omega_{X/\mathbb{C}}$, is in fact *independent* of the embedding j. In particular, from (4.1), we see that $c_n(W) = 0$ for *any* such embedding $j : X \hookrightarrow Y$.

Remark 4.3. In fact, the stability and cancellation theorems of Bass and Suslin imply that in the above situation, the vector bundle W itself is, up to isomorphism, independent of j, and is thus an invariant of the variety X. We call it the *stable normal bundle* of X; this is similar to the case of embeddings of smooth manifolds into Euclidean spaces. We will not need this fact in our computations below.

Returning to our discussion, we see that to find a $\mathbb{C}$-algebra A with $\dim A = n$, and which cannot be generated by $2n$ elements as a $\mathbb{C}$-algebra, it suffices to produce an embedding $j : X \hookrightarrow Y$ of $X = \operatorname{Spec} A$ into a smooth variety Y of dimension $2n$, such that

(i) $\Omega_{Y/\mathbb{C}}$ is a trivial bundle, and
(ii) if W is the conormal bundle of j, then $c_n(W) \neq 0$; in fact it suffices to produce such an embedding such that $j_*c_n(W) \in CH^{2n}(Y)$ is non-zero.

We see easily that the same example $X = \operatorname{Spec} A$ will have the property that $\Omega_{A/\mathbb{C}}$ is not generated by $2n-1$ elements; in fact if $P = \ker(f : A^{\oplus 2n-1} \twoheadrightarrow \Omega_{A/\mathbb{C}})$ for some surjection f, then $\widetilde{P}$ is a vector bundle of rank $n-1$, so that $c_n(\widetilde{P}) = 0$, while on the other hand, the exact sequence

$$0 \to P \to A^{\oplus 2n-1} \xrightarrow{f} \Omega_{A/\mathbb{C}} \to 0$$

implies that

$$c(\widetilde{P}) = c(\Omega_{X/\mathbb{C}})^{-1},$$

so that we would have

$$0 = c_n(\widetilde{P}) = c_n(W) \neq 0,$$

a contradiction.

Next we discuss the Problem 4.2 of finding an example of a "non-trivial" prime ideal $I \subset \mathbb{C}[x_1, \ldots, x_N]$ in a polynomial ring such that the quotient ring $A = \mathbb{C}[x_1, \ldots, x_N]/I$ is smooth of dimension > 0, while I cannot be generated by $N-1$

elements (by the Eisenbud-Evans conjectures, proved by Sathaye and Mohan Kumar, I can always be generated by N elements).

Suppose I can be generated by $N-1$ elements, and $\dim A/I = n > 0$. Then $I/I^2 \oplus Q = A^{\oplus N-1}$ for some projective A-module Q of rank $n-1$; hence

$$(I/I^2 \oplus Q \oplus A) \cong A^{\oplus N} \cong (I/I^2 \oplus \Omega_{A/\mathbb{C}}).$$

Hence we have an equality between total Chern classes

$$c(\Omega_{X/\mathbb{C}}) = c(\widetilde{Q}),$$

and in particular, $c_n(\Omega_{X/\mathbb{C}}) = 0$.

So if $X = \operatorname{Spec} A$ is such that $c_n(\Omega_{X/\mathbb{C}}) \in CH^n(X)$ is non-zero, then for any embedding $X \hookrightarrow \mathbb{A}^N_{\mathbb{C}}$, the corresponding prime ideal I cannot be generated by $N-1$ elements.

Example 4.4. We now show how to construct an example of an n-dimensional affine variety $X = \operatorname{Spec} A$ over $\mathbb{C}$, for any $n \geq 1$, such that, for some embedding $X \hookrightarrow Y = \operatorname{Spec} B$ with $\dim Y = 2n$, and ideal $I \subset B$, the projective module $P = I/I^2$ has the following properties:

(i) $c_n(P) \neq 0$ in $CH^n(X) \otimes \mathbb{Q}$
(ii) if $c(P) \in CH^*(X)$ is the total Chern class, then $c(P)^{-1}$ has a non-torsion component in $CH^n(X) \otimes \mathbb{Q}$.

Then, by the discussion earlier, the affine ring A will have the properties that

(a) A cannot be generated by $2n$ elements as a $\mathbb{C}$-algebra
(b) $\Omega_{A/\mathbb{C}}$ is not generated by $2n-1$ elements
(c) for any way of writing $A = \mathbb{C}[x_1, \ldots, x_N]/J$ as a quotient of a polynomial ring (with n necessarily at least $2n+1$), the ideal J requires N generators (use the formula (4.2)).

The technique is that given in [4]. Let E be an elliptic curve (*i.e.*, a non-singular, projective plane cubic curve over $\mathbb{C}$), for example,

$$E = \operatorname{Proj} \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3).$$

Let $E^{2n} = E \times \cdots \times E$, the product of $2n$ copies of E. Let $Y = \operatorname{Spec} B \subset E^{2n}$ be any affine open subset. By the Mumford-Roitman infinite dimensionality theorem (Theorem 3.1 above), $CH^{2n}(Y) \otimes \mathbb{Q} \neq 0$. Also, since $Y \subset E^{2n}$, clearly the $2n$-fold intersection product

$$CH^1(Y)^{\otimes 2n} \to CH^{2n}(Y)$$

is surjective. Hence we can find an element $\alpha \in CH^1(Y)$ with $\alpha^{2n} \neq 0$ in $CH^{2n}(Y) \otimes \mathbb{Q}$. Let P be the projective B-module of rank 1 corresponding to α. Since Y is affine, by Bertini's theorem, we can find elements $a_1, \ldots, a_n \in P$ such that the corresponding divisors $H_i = \{a_i = 0\} \subset Y$ are non-singular, and intersect transversally; take $X = H_1 \cap \cdots \cap H_n$. Then $X = \operatorname{Spec} A$ is non-singular of dimension n, and the ideal $I \subset B$ of $X \subset Y$ is such that $I/I^2 \cong (P \otimes_B A)^{\oplus n}$. Thus, if $j : X \hookrightarrow Y$ is the inclusion, then we have a formula between total Chern classes

$$c(I/I^2) = j^* c(P)^n = (1 + j^* c_1(P))^n = (1 + j^* \alpha)^n.$$

Hence

$$c_n(I/I^2) = j^*(\alpha)^n,$$

and so by the projection formula,

$$j_*c_n(I/I^2) = j_*(1)\alpha^n = \alpha^{2n},$$

since

$$j_*(1) = [X] = [H_1]\cdot[H_2]\cdot\cdots\cdots[H_n] = \alpha^n \in CH^n(Y),$$

as X is the complete intersection of divisors H_i, each corresponding to the class $\alpha \in CH^1(Y)$. By construction, $j_*c_n(I/I^2) \neq 0$ in $CH^{2n}(Y)\otimes\mathbb{Q}$, and so we have that $c_n(I/I^2) \neq 0$ in $CH^n(X)\otimes\mathbb{Q}$, as desired.

Similarly

$$c(I/I^2)^{-1} = (1 + j^*\alpha)^{-n}$$

has a non-zero component of degree n, which is a non-zero integral multiple of $j^*\alpha^n$.

Remark 4.5. The existence of n-dimensional non-singular affine varieties X which do not admit closed embeddings into affine $2n$-space is in contrast to the situation of differentiable manifolds — the "hard embedding theorem" of Whitney states that any smooth n-manifold has a smooth embedding in the Euclidean space $\mathbb{R}^{2n}$.

5. Indecomposable projective modules, using 0-cycles

Now we discuss M. Nori's (unpublished) construction of indecomposable projective modules of rank d over any affine $\mathbb{C}$-algebra $A_{\mathbb{C}}$ of dimension d, such that $U = \operatorname{Spec} A_{\mathbb{C}}$ is an open subset of a non-singular projective (or proper) $\mathbb{C}$-variety X with $H^0(X,\omega_X) = H^0(X,\Omega^d_{X/\mathbb{C}}) \neq 0$.

The idea is as follows. Fix a countable, algebraically closed subfield $k \subset \mathbb{C}$ such that X and U are defined over k; in particular, we are given an affine k-subalgebra $A \subset A_{\mathbb{C}}$ such that $A_{\mathbb{C}} = A \otimes_k \mathbb{C}$. We also have a k-variety X_0 containing $U_0 = \operatorname{Spec} A$ as an affine open subset, such that $X = (X_0)_{\mathbb{C}}$.

Let K_n be the function field of $X_0^n = X_0 \times_k \cdots \times_k X_0$ (equivalently, K_n is the quotient field of $A^{\otimes n} = A \otimes_k \cdots \otimes_k A$). We have n induced embeddings $\varphi_i : K \hookrightarrow K_n$, where $K = K_1$ is the quotient field of A, given by $\varphi_i(a) = 1\otimes\cdots\otimes 1\otimes a\otimes 1\otimes\cdots\otimes 1$ with a in the i-th position.

Choose an embedding $K_n \hookrightarrow \mathbb{C}$ as a k-subalgebra. The inclusions φ_i then determine n inclusions $K \hookrightarrow \mathbb{C}$, or equivalently, k-generic points $x_1,\ldots,x_n \in X$ (in algebraic geometry, these are called "n independent generic points of X"). Let $\mathfrak{m}_i$ be the maximal ideal of $A_{\mathbb{C}}$ determined by x_i, and let $I = \cap_{i=1}^n \mathfrak{m}_i$. Clearly I is a local complete intersection ideal of height d in the d-dimensional regular ring $A_{\mathbb{C}}$. Thus we can find a projective resolution of I

$$0 \to P \to F_{d-1} \to \cdots \to F_1 \to I \to 0,$$

where F_i are free. By construction, $c(P) = c(A/I)^{(-1)^d}$. By theorem 1.3, we have

$$c(A/I) = 1 + (-1)^{d-1}(d-1)!(\sum_{i=1}^n [x_i]) \in CH^*(U).$$

Hence $c_i(P) = 0$ for $i < d$, while $c_d(P)$ is a non-zero integral multiple of the class $\sum_i [x_i] \in CH^d(U)$. This class is non-zero, from theorem 3.3 (we will get a stronger conclusion below). Hence $\operatorname{rank} P \geq d$.

By Bass' stability theorem, if $\operatorname{rank} P = d+r$, we may write $P = Q \oplus A^{\oplus r}$, where Q is projective of rank d. Then P and Q have the same Chern classes. So we can find a projective module Q of rank d with $c(Q) = 1 + m(\sum_i [x_i]) \in CH^*(U)$, for some non-zero integer m.

Suppose $Q = Q_1 \oplus Q_2$ with $\operatorname{rank} Q_1 = p$, $\operatorname{rank} Q_2 = d - p$, and $1 \leq p \leq d$. Then in $CH^*(U) \otimes \mathbb{Q}$, the class $\sum_i [x_i]$ is expressible as

$$\sum_i [x_i] = \alpha \cdot \beta, \;\; \alpha \in CH^p(U) \otimes \mathbb{Q}, \; \beta \in CH^{d-p}(U) \otimes \mathbb{Q}.$$

Using the homomorphism of graded rings of Theorem 3.3,

$$CH^*(U) \otimes \mathbb{Q} \to \bigoplus_{j \geq 0} H^j(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^j_{\mathbb{C}/k},$$

we see that the element

$$\xi = \sum_{i=1}^n di_{x_i} \in H^d(X, \mathcal{O}_X) \otimes \Omega^d_{\mathbb{C}/k}$$

is expressible as a product

$$\xi = \sum_{i=1}^n di_{x_i} = \alpha \cdot \beta, \;\; \alpha \in H^p(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^p_{\mathbb{C}/k}, \;\; \beta \in H^{d-p}(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^{d-p}_{\mathbb{C}/k}.$$

Let L be the algebraic closure of K_n in $\mathbb{C}$. The graded ring

$$\bigoplus_{j=0}^d H^j(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^j_{\mathbb{C}/k} = \bigoplus_{j=0}^d H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{\mathbb{C}/k}$$

has a graded subring

$$\bigoplus_{j=0}^d H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{L/k}$$

which contains the above element ξ. We claim that ξ is then expressible as a product $\alpha \cdot \beta$ of homogeneous elements of degrees $p, d-p$ with α, β lying in this subring. Indeed, since $\mathbb{C}$ is the direct limit of its subrings B which are finitely generated L-subalgebras, we can find such a subring B, and homogeneous elements $\widetilde{\alpha}$, $\widetilde{\beta}$ of degrees p, $d-p$ in $\bigoplus_{j=0}^d H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{B/k}$ such that $\xi = \widetilde{\alpha} \cdot \widetilde{\beta}$. Choosing a maximal ideal in B, we can find an L-algebra homomorphism $B \twoheadrightarrow L$, giving rise to a graded ring homomorphism

$$f : \bigoplus_{j=0}^d H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{B/k} \to \bigoplus_{j=0}^d H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{L/k}.$$

Then $\xi = f(\widetilde{\alpha}) \cdot f(\widetilde{\beta})$ holds in $\bigoplus_{j=0}^d H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{L/k}$ itself.

Now
$$\Omega^1_{L/k} = \Omega^1_{K_n/k} \otimes_{K_n} L = \bigoplus_{j=1}^{n} \Omega^1_{K/k} \otimes_K L,$$
where the j-th summand corresponds to the j-th inclusion $K \hookrightarrow K_n$. We may write this as
$$\Omega^1_{L/k} = \Omega^1_{K/k} \otimes_K W,$$
where $W \cong L^{\oplus n}$ is an n-dimensional L-vector space with a distinguished basis. Then there are natural surjections
$$\Omega^r_{L/k} = \bigwedge_L^r (\Omega^1_{K/k} \otimes_K W) \twoheadrightarrow \Omega^r_{K/k} \otimes_K S^r(W),$$
where $S^r(W)$ is the r-th symmetric power of W as an L-vector space. In particular, since $\Omega^d_{K/k}$ is 1-dimensional over K, we get a surjection $\Omega^d_{L/k} \twoheadrightarrow S^d(W)$. This determines the component of degree d of a graded ring homomorphism
$$\Phi : \bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{L/k} \to \bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{K/k} \otimes_K S^j(W).$$

As in the discussion preceeding Theorem 3.3, by Serre duality on X_0, the natural inclusion $H^0(X_0, \Omega^d_{X_0/k}) \hookrightarrow \Omega^d_{K/k}$ determines a canonical element $\theta \in H^d(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^d_{K/k}$. Identifying the symmetric algebra $S^\bullet(W) = S^\bullet(L^{\oplus n})$ with the polynomial algebra $L[t_1, \dots, t_n]$, we have that $\Phi(\xi) = \theta \cdot (t_1^d + \cdots + t_n^d)$. Hence, in the graded ring
$$\bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{K/k} \otimes_K S^j(W),$$
the element $\theta \cdot (t_1^d + \cdots + t_n^d)$ is expressible as a product of homogeneous elements α, β of degrees p and $d-p$. Hence, by expressing
$$\alpha \in H^p(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^p_{K/k} \otimes_K S^p(W), \;\; \beta \in H^{d-p}(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^{d-p}_{K/k} \otimes_K S^{d-p}(W)$$
in terms of K-bases of $H^p(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^p_{K/k}$ and $H^{d-p}(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^{d-p}_{K/k}$, we deduce that in the polynomial ring $S^\bullet(W) = L[t_1, \dots, t_n]$, the "Fermat polynomial" $t_1^d + \cdots + t_n^d$ is expressible as a sum of pairwise products of homogeneous polynomials
$$t_1^d + \cdots + t_n^d = \sum_{m=1}^{N} a_m(t_1, \dots, t_n) b_m(t_1, \dots, t_n)$$
with
$$N = \binom{d}{p}\binom{d}{d-p} (\dim_k H^p(X_0, \mathcal{O}_{X_0}))(\dim_k H^{d-p}(X_0, \mathcal{O}_{X_0})).$$
If $n > 2N$, the system of homogeneous polynomial equations $a_1 = b_1 = \cdots = a_N = b_N = 0$ defines a non-empty subset of the projective variety $t_1^d + \cdots + t_n^d = 0$ in $\mathbb{P}^{n-1}_L$, along which this Fermat hypersurface is clearly singular — and this is a contradiction!

6. Stably trivial vector bundles on affine varieties, using cycles

In this section, we give a construction of stably trivial non-trivial vector bundles on affine varieties, following an argument of Mohan Kumar and Nori. Here, instead of topology, the properties of Chow rings and Chern classes provide the invariants to prove non-triviality of the vector bundles; thus the arguments are valid over any base field.

Theorem 6.1. *Let k be a field, and let*

$$A = \frac{k[x_1, \dots, x_n, y_1, \dots, y_n]}{(x_1y_1 + \cdots + x_ny_n - 1)}.$$

Let $\mathbf{m} = (m_1, \dots, m_n)$ be an n-tuple of positive integers, and set

$$P(\mathbf{m}) = \ker\left(\psi(\mathbf{m}) : A^{\oplus n} \to A\right),$$

$$(a_1, \dots, a_n) \mapsto a_1x_1^{m_1} + \cdots + a_nx_n^{m_n}.$$

Then $P(\mathbf{m})$ is a stably free projective A-module of rank $n-1$. If $\prod_{i=1}^n m_i$ is not divisible by $(n-1)!$, then $P(\mathbf{m})$ is not free.

Note that in particular, if all the m_i are 1, and $n \geq 3$, the corresponding projective module is not free.

Conversely, a theorem of Suslin [31] implies that, if $\prod_{i=1}^n m_i$ is divible by $(n-1)!$, then P is free, since we can find an invertible $n \times n$ matrix over A whose first row has entries $x_1^{m_1}, \dots, x_n^{m_n}$.

The theorem above is proved by relating the freeness of $P(\mathbf{m})$ to another property. Let

$$R = \frac{k[z, x_1, \dots, x_n, y_1, \dots, y_n]}{(z(1-z) - x_1y_1 - \cdots - x_ny_n)},$$

and consider the ideals

$$I = (x_1, \dots, x_n, z) \subset R, \quad I' = (x_1, \dots, x_n, 1-z).$$

Note that I, I' are prime ideals in R, and $I + I' = R$ (i.e. the corresponding subvarieties of $\operatorname{Spec} R$ are disjoint); further, $I \cap I' = II' = (x_1, \dots, x_n)$ is a complete intersection.

Consider the ideal $(x_1^{m_1}, \dots, x_n^{m_n}) \subset R$. Clearly its radical is the complete intersection $(x_1, \dots, x_n) = II'$. Hence we may uniquely write

$$(x_1^{m_1}, \dots, x_n^{m_n}) = J(\mathbf{m}) \cap J'(\mathbf{m})$$

where $J(\mathbf{m})$, $J'(\mathbf{m})$ have radicals I, I' respectively. In fact $J(\mathbf{m})$, $J'(\mathbf{m})$ are the contractions to R of the complete intersection $(x_1^{m_1}, \dots, x_n^{m_n})$ from the overrings $R[\frac{1}{1-z}]$, $R[\frac{1}{z}]$ respectively.

Lemma 6.2. *Let $X = \operatorname{Spec} R$, and $W(\mathbf{m}) \subset X$ the subscheme with ideal $J(\mathbf{m})$. If $P(\mathbf{m})$ is a free A-module, then there exists a vector bundle V on X of rank n and a section, whose zero scheme on X is $W(\mathbf{m})$.*

Proof. We write J, J' instead of $J(\mathbf{m})$, $J'(\mathbf{m})$ to simplify notation. Consider the surjection

$$\alpha : R^{\oplus n} \to J \cap J' = (x_1^{m_1}, \ldots, x_n^{m_n}).$$

If we localize to $R[\frac{1}{z(1-z)}]$, we obtain a surjection

$$\widetilde{\alpha} : R_{z(1-z)}^{\oplus n} \to R_{z(1-z)},$$

whose kernel Q is a stably free projective $R_{z(1-z)}$-module, fitting into a split exact sequence of projective modules

$$0 \to Q \to R_{z(1-z)}^{\oplus n} \xrightarrow{\widetilde{\alpha}} R_{z(1-z)} \to 0.$$

It is easy to see that the projective module Q is free if and only if there is some $\varphi \in \mathrm{GL}_n(R_{z(1-z)})$ so that $\widetilde{\alpha} \circ \varphi$ is projection onto the first coordinate.

If this is the case, the surjection

$$R[\frac{1}{1-z}]^{\oplus n} \to JJ'[\frac{1}{1-z}] = J[\frac{1}{1-z}]$$

clearly "patches" over $\operatorname{Spec} R_{z(1-z)}$ with the "trivial" surjection

$$R[\frac{1}{z}]^{\oplus n} \to R[\frac{1}{z}] = J[\frac{1}{z}]$$

given by projection on the first coordinate, to yield a projective R-module $\widetilde{P}$ of rank n, with a surjection

$$\widetilde{P} \to J.$$

We then take V to be the vector bundle on X associated to the dual projective module $\widetilde{P}^\vee$, and the section of V to be dual to the map $\widetilde{P} \to J$.

Finally, we observe that there is a homomorphism $A \to R[\frac{1}{z(1-z)}]$ given by

$$x_i \mapsto x_i,$$
$$y_i \mapsto \frac{y_i}{z(1-z)},$$

which gives an identification

$$Q \cong P(\mathbf{m}) \otimes_A R_{z(1-z)}.$$

Hence if $P(\mathbf{m})$ is free, so is the projective $R_{z(1-z)}$-module Q, and we can thus construct V as above. □

Lemma 6.3. *$CH^i(X) = 0$ for $1 \leq i \leq n-1$, and $CH^n(X) \cong \mathbb{Z}$ is generated by the class of W, the irreducible subvariety defined by the ideal I.*

Proof. Consider the smooth $2n$-dimensional projective quadric hypersurface $\overline{X}$ in $\mathbb{P}_k^{2n+1}$ given by the equation

$$\sum_{i=1}^{n} X_i Y_i = Z_0(Z_1 - Z_0).$$

We may identify $X = \operatorname{Spec} R$ with the affine open subset of $\overline{X}$ obtained by setting $Z_1 = 1$ in the above homogeneoues polynomial equation. Hence the complement of X is the smooth hyperplane section $Z_1 = 0$. From the known structure of the

Chow ring of a smooth "split" quadric hypersurface in any dimension, we know that

(i) $CH^i(\overline{X})$ is spanned by the complete intersection with any projective linear subspace of codimension i, for any $i < n$, while
(ii) $CH^n(\overline{X})$ is free abelian of rank 2, spanned by the classes of 2 linear (projective) subvarieties, which are the irreducible components of a suitable complete intersection with a projective linear subspace.

For $i < n$, we may choose a complete intersection generator of $CH^i(\overline{X})$ as in (i) to be contained in the linear subspace intersection $\overline{X} \cap \{Z_1 = 0\}$. From the exact localization sequence

$$CH^{i-1}(\overline{X} \cap \{Z_1 = 0\}) \to CH^i(\overline{X}) \to CH^i(X) \to 0$$

it follows that $CH^i(X) = 0$ for $1 \leq i \leq n-1$. Also, one sees easily that if W' is the subscheme defined by I', then the closures of W, W' in $\overline{X}$ are generators for $CH^n(\overline{X}) = \mathbb{Z}^{\oplus 2}$ as in (ii), and $W + W'$ is rationally equivalent to 0 on X, so that $CH^n(X)$ is generated by the class of W. Finally, $CH^{n-1}(\overline{X} \cap \{Z_1 = 0\}) = \mathbb{Z}$, so $CH^n(X)$ must have rank 1, again from the above exact localization sequence of Chow groups. □

Thus, from Theorem 1.3 we must also have $F^1K_0(X) = F^nK_0(X)$ where $F^iK_0(X)$ is the subgroup generated by the classes of modules supported in codimension $\geq i$; also, $F^nK_0(X)/F^{n+1}K_0(X)$ is generated by the class of $\mathcal{O}_W$.

The vector bundle V constructed in lemma 6.2 above thus has the following properties:

(i) $c_n(V) = [W(\mathbf{m})] = (\prod_{i=1}^n m_i)[W]$ in $CH^n(X)$.
(ii) $[V] - [\mathcal{O}_X^{\oplus n}] \in F^1K_0(X) = F^nK_0(X)$, so that for some $r \in \mathbb{Z}$, we must have

$$[V] - [\mathcal{O}_X^{\oplus n}] = r[\mathcal{O}_W] \pmod{F^{n+1}K_0(X)},$$

which yields

$$c_n(V) = rc_n([\mathcal{O}_W]) \in CH^n(X)$$

(iii) from the Riemann-Roch theorem without denominators,

$$c_n(\mathcal{O}_W) = (-1)^{n-1}(n-1)![W] \in CH^n(X) = \mathbb{Z} \cdot [W].$$

Hence we must have

$$\prod_{i=1}^n m_i = r(-1)^{n-1}(n-1)!,$$

which proves Theorem 6.1.

7. 0-cycles and the complete intersection property for affine varieties

Let A be a finitely generated reduced algebra over a field k, which we assume to be algebraically closed, for simplicity. Let $d = \dim A$. A point $x \in X = \operatorname{Spec} A$ is called a *complete intersection point* if the corresponding maximal ideal $\mathfrak{M} \subset A$ has height d, and is generated by d elements of A. Any such point is necessarily a smooth point (of codimension d) in X (we will take points of codimension $< d$

to be singular, by definition, in this context). We will refer to the corresponding maximal ideals as smooth maximal ideals.

In this section, we want to discuss the following problem:

Problem 7.1. Characterize reduced affine k-varieties X such that all smooth points are complete intersections.

It is easy to show, using the theory of the Jacobian (suitably generalized in the singular case), that for $d = \dim X = 1$, we have a complete answer, as follows. Any such curve X can be written as $X = \overline{X} \setminus S$ where $\overline{X}$ is a reduced projective curve over k, and S a finite set of non-singular points of X, in a unique way. Then: all smooth points of X are complete intersections $\Leftrightarrow H^1(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$.

So the interesting case of the problem is in dimensions $d > 1$. There are several conjectures and results related to this problem. We first state a general "positive" result.

Theorem 7.2. *Let $k = \overline{\mathbb{F}}_p$ be the algebraic closure of the finite field $\mathbb{F}_p$. Then for any reduced finitely generated k-algebra A of dimension $d > 1$, every smooth maximal ideal is a complete intersection.*

In the case when $\dim A \geq 3$, or A is smooth of dimension 2, this is a result essentially due to M. P. Murthy. The higher dimensional case is reduced to the 2-dimensional case by showing that any smooth point of $V = \operatorname{Spec} A$ lies on a smooth affine surface $W \subset V$ such that the ideal of W in A is generated by $d-2$ elements (i.e., W is a complete intersection surface in V). This argument (see [20] for details) depends on the fact that we are dealing here with *affine* algebraic varieties.

The case of an arbitrary 2-dimensional algebra is a corollary of results of Amalendu Krishna and mine [1]; the details are worked out in [2]. I will make a few remarks about this later in this paper.

Next, we state two conjectures, which are "affine versions" of famous conjectures on 0-cycles.

Conjecture 7.3. *(Affine Bloch Conjecture).* Let $k = \mathbb{C}$, the complex numbers. Let $V = \operatorname{Spec} A$ be a non-singular affine $\mathbb{C}$-variety of dimension $d > 1$, and let $X \supset V$ be a smooth proper (or projective) $\mathbb{C}$-variety containing V as a dense open subset. Then:
all maximal ideals of A are complete intersections
$\Leftrightarrow$ X does not support any global regular (or holomorphic) differential d-forms
$\Leftrightarrow H^d(X, \mathcal{O}_X) = 0$.

Here, $\mathcal{O}_X$ is the sheaf of algebraic regular functions on X. The non-existence of d-forms is equivalent to the cohomology vanishing condition, by Serre duality; the open question is the equivalence of either of these properties with the complete intersection property for maximal ideals.

This conjecture has been verified in several "non-trivial" examples (for example, if $V = \operatorname{Spec} A$ is a "small enough" Zariski open subset of the Kummer variety of an odd (> 1) dimensional abelian variety over $\mathbb{C}$, all smooth maximal ideals of A are complete intersections).

One consequence of the conjecture is that, for smooth affine $\mathbb{C}$-varieties, the property that all maximal ideals are complete intersections is a *birational* invariant (that is, it depends only on the quotient field of A, as a $\mathbb{C}$-algebra). This birational invariance can be proved to hold in dimension 2, using a result of Roitman; in dimensions ≥ 3, it is unknown in general.

Conjecture 7.4. *(Affine Bloch-Beilinson Conjecture)* Let $k = \overline{\mathbb{Q}}$ be the field of algebraic numbers (algebraic closure of the field of rational numbers). Then for any finitely generated *smooth* k-algebra of dimension $d > 1$, every maximal ideal is a complete intersection.

This very deep conjecture has not yet been verified in any "nontrivial" example (i.e., one where there do exist smooth maximal ideals of $A \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ which are not complete intersections).

However, it is part of a more extensive set of interrelated conjectures relating *K-groups of motives over algebraic number fields* and *special values of L-functions*, and there are nontrivial examples where some other parts of this system of conjectures can be verified. This is viewed as indirect evidence for the above conjecture.

I will now relate these "affine" conjectures to the more standard forms of these, in terms of algebraic cycles and K-theory. The first step is a fundamental result of Murthy, giving a K-theoretic interpretation of the complete intersection property.

Recall that $K_0(A)$ denotes the Grothendieck group of finitely generated projective A-modules. It coincides with the Grothendieck group of finitely generated A-modules of finite projective dimension: recall that M has *finite projective dimension* if there exists a finite projective resolution of M, i.e., an exact sequence

$$0 \to P_r \to P_{r-1} \to \cdots \to P_0 \to M \to 0$$

where the P_i are finitely generated projective A-modules. Then such a module M has a well-defined class $[M] \in K_0(A)$, obtained by choosing any such resolution, and defining

$$[M] = \sum_{i=0}^{r} (-1)^i [P_i] \in K_0(A).$$

Recall also that a maximal ideal $\mathfrak{M}$ has finite projective dimension precisely when the local ring $A_{\mathfrak{M}}$ is a regular local ring.

Theorem 7.5. (M. P. Murthy) *Let A be a reduced finitely generated algebra over an algebraically closed field, of dimension d. Assume $F^d K_0(A)$ has no torsion of exponent $(d-1)!$. Then a smooth maximal ideal $\mathfrak{M}$ of A is a complete intersection $\Leftrightarrow$ $[\mathfrak{M}] = [A]$ in $K_0(A)$.*

The main point in the proof of Murthy's theorem is to show that any smooth maximal ideal $\mathfrak{M}$ is a quotient of a projective A-module P of rank d, satisfying the addiitonal condition that $[P] - [A^{\oplus d}] \in F^d K_0(A)$. Then $[\mathfrak{M}] = [A]$ in $K_0(A)$ implies that the above projective module P has $c_d(P) = 0$, so that P is in fact stably trivial, from Riemann-Roch (Theorem 1.3), since we assumed $F^d K_0(A)$ has no $(d-1)!$-torsion. Now Suslin's cancellation theorme for projective modules implies that P is a free module, so that $\mathfrak{M}$ is a complete intersection.

Let A be a reduced, finitely generated algebra, of Krull dimension d, over an algebraically closed field k. We can associate to it the group

$$F^dK_0(A) = \text{ subgroup of } K_0(A) \text{ generated by } [A] - [\mathfrak{M}] \text{ for all smooth maximal ideals } \mathfrak{M}.$$

If $V = \operatorname{Spec} A$, then $F^dK_0(A)$ is a quotient of the free abelian group on smooth points of V, modulo a suitable equivalence relation. When V is nonsingular, one can identify this equivalence relation with *rational equivalence*, up to torsion, using the "Riemann-Roch Theorem without denominators", using the d-th Chern class map.

This also suggests a good definition of rational equivalence for 0-cycles for singular V; this was given by Levine and Weibel [11], and Levine (unpublished) has defined a suitable d-th Chern class, for which the Riemann-Roch without denominators is valid.

Now assume $V = \operatorname{Spec} A$ is an affine open subset of a nonsingular projective k-variety X of dimension d. Clearly

$$CH^d(V) = \frac{CH^d(X)}{\text{subgroup generated by points of } X \setminus V}.$$

We saw earlier that *Roitman's Theorem* on torsion 0-cycles (extended by Milne to arbitrary characteristic) gives a description of the torsion in $CH^d(X)$. Using this, it can be shown that $CH^d(V)$ is a torsion free, divisible abelian group (i.e., a vector space over $\mathbb{Q}$). In particular, we see that the map $\psi_d : CH^d(V) \to F^dK_0(V)$ is an *isomorphism*.

Thus, by Murthy's theorem, for nonsingular A, all maximal ideals of A are complete intersections $\Leftrightarrow$ $CH^d(X)$ is generated by points of $X \setminus V$.

We now restate the Bloch and Bloch-Beilinson Conjectures in something resembling their "original" forms.

Conjecture 7.6. *(Bloch Conjecture)* Let X be a projective smooth variety over $\mathbb{C}$. Suppose that, for some integer $r > 0$, X has no nonzero regular (or holomorphic) s-forms for any $s > r$. Then for any "sufficiently large" subvariety $Z \subset X$ of dimension r, we have $CH^d(X \setminus Z) = 0$.

For a smooth projective complex surface X, this conjecture states that if X has no holomorphic 2-forms, then $CH^2(X \setminus C) = 0$ for some curve C in X. This has been verified in several situations, for example, for surfaces of Kodaira dimension ≤ 1 (Bloch, Kas, Lieberman), for general Godeaux surfaces (Voisin), and in some other cases.

In higher dimensions, Roitman proved it for complete intersections in projective space, and there are a few other isolated examples, like the Kummer variety associated to an odd dimensional abelian variety (see [5]).

Conjecture 7.7. *(Bloch-Beilinson Conjecture)* Let X be a smooth projective variety of dimension d over $\overline{\mathbb{Q}}$. Then $CH^d(X)$ is "finite dimensional"; in particular, there is a curve $C \subset X$ so that $CH^d(X \setminus C) = 0$.

As remarked earlier, there is only indirect evidence for this conjecture: *it has not been verified for any smooth projective surface over $\overline{\mathbb{Q}}$ which supports a non-zero* 2*-form* (e.g., any hypersurface in projective 3-space of degree ≥ 4).

To exhibit one such nontrivial example is already an interesting open question.

From the algebraic viewpoint, it seems restrictive to work only with smooth varieties. In any case, it is unknown in characteristic $p > 0$ that a smooth affine variety V can be realized as an open subset of a smooth proper variety X (in characteristic 0, this follows from Hironaka's theorem on *resolution of singularities*).

Inspite of this, it is possible to make a systematic study of the singular case, and to try to extend the above conjectures, using the Levine-Weibel Chow group of 0-cycles; see [28] for further discussion.

For our purposes, let me focus on one very special situation. Let

$$Z \subset \mathbb{P}^N_k$$

be a non-singular projective algebraic k-variety, and

$$\begin{aligned} A = &\ \oplus_{n\geq 0} A_n \\ = &\ \textit{homogeneous coordinate ring} \text{ of } Z. \end{aligned}$$

The affine variety $V = \operatorname{Spec} A$ is the *affine cone* over Z with *vertex* corresponding to the unique graded maximal ideal $\mathfrak{M} = \oplus_{n>0} A_n$, and the vertex is the unique singular point of V.

The *projective cone* $C(Z)$ over Z with the same vertex naturally contains V as an open subset, whose complement is a divisor isomorphic to Z, and the vertex is again the only singular point of $C(Z)$.

The following theorem is obtained using results from my paper with Amalendu Krishna [1], in the 2-dimensional case, and a preprint of Krishna's in the higher dimensional case; see [2] for more details.

Theorem 7.8. *(i) Let $k = \overline{\mathbb{Q}}$. Then every smooth maximal ideal of A is a complete intersection.*
(ii) Let $k = \mathbb{C}$. Then every smooth maximal ideal of A is a complete intersection $\Rightarrow H^{d-1}(Z, \mathcal{O}_Z(1)) = 0$ ($\Leftrightarrow H^d(C(Z), \mathcal{O}_{C(Z)}) = 0$). If V is Cohen-Macaulay of dimension ≤ 3, then the converse holds: if $H^{d-1}(Z, \mathcal{O}_Z(1)) = 0$, then every smooth maximal ideal of A is a complete intersection.

Here, (i) is analogous to the Bloch-Beilinson Conjecture, while (ii) is analogous to the Bloch Conjecture.

Here are two examples, which shed some light on the content of the above theorems.

Example 7.9. (Amalendu Krishna + V. S.)

$$A = \frac{\overline{\mathbb{Q}}[x, y, z]}{(x^4 + y^4 + z^4)}.$$

The following properties hold.

(i) All smooth maximal ideals of A are complete intersections.
(ii) "Most" smooth maximal ideals of $A\otimes_{\overline{\mathbb{Q}}}\mathbb{C}$ are *not* complete intersections.
(iii) The complete intersection smooth points on $V_{\mathbb{C}}$ are those lying on rulings of the cone over $\overline{\mathbb{Q}}$-*rational points* of the Fermat Quartic curve.

This is a consequence of Theorem 7.8.

Example 7.10.

$$A=\frac{\overline{\mathbb{Q}}[x,y,z]}{(xyz(1-x-y-z))}.$$

Again, all smooth maximal ideals of A are complete intersections, while "most" smooth maximal ideals of $A\otimes_{\overline{\mathbb{Q}}}\mathbb{C}$ are *not* complete intersections.

In fact, there is an identification

$$F^2K_0(A\otimes_{\overline{\mathbb{Q}}}k)=K_2(k),$$

where K_2 denotes the Milnor K_2 functor.

Now one has the result of Garland (vastly generalized by Borel) that $K_2(\overline{\mathbb{Q}})=0$, while $K_2(\mathbb{C})$ is "very large".

8. 0-cycles on normal surfaces

Here, we give some idea of the proof of Theorem 7.8, for the case of surfaces.

Let X be a normal, quasiprojective surface over a field k. Then $CH^2(X)$ is identified with $F^2K_0(X)$, which in turn is identified with the subgroup of the Grothendieck group $K_0(X)$ of vector bundles consisting of elements (virtual bundles) of trivial rank and determinant.

Let $\pi:Y\to X$ be a resolution of singularities, and let E be the exceptional set, with reduced structure. Let nE denote the subscheme of Y with ideal sheaf $\mathcal{O}_Y(-nE)$; this is in fact an effective Cartier divisor. One can define *relative algebraic K-groups* $K_i(Y,nE)$ for any $n\geq 0$, along the following lines.

The algebraic K-groups of a scheme T are defined (by Quillen) by

$$K_i(T)=\pi_{i+1}(\mathbb{K}(T)),$$

where $\mathbb{K}(T)$ is a certain connected CW complex associated to the category of vector bundles on T, and π_n denotes the n-th homotopy group. Quillen shows that $K_0(T)$ defined in this way coincides with the usual Grothendieck group, and if $T=\operatorname{Spec}A$ is affine, then $K_1(T)$, $K_2(T)$ coincide with the groups $K_1(A)$ of Bass, and $K_2(A)$ of Milnor, respectively. For an introduction to these ideas, see [30].

Given a morphism $f:S\to T$ of schemes, there is an induced continuous map $f^*:\mathbb{K}(T)\to\mathbb{K}(S)$, and hence induced homomorphisms $K_i(T)\to K_i(S)$, for all i. Let $\mathbb{K}(f)$ denote the *homotopy fibre* of the continuous map $f^*:\mathbb{K}(T)\to\mathbb{K}(S)$. Then there is an associated long exact sequence of homotopy groups

$$\cdots\to\pi_{i+1}(\mathbb{K}(S))\to\pi_i(\mathbb{K}(f))\to\pi_i(\mathbb{K}(T))\to\pi_i(\mathbb{K}(S))\to\cdots$$

If we define $K_i(f)=\pi_{i+1}(\mathbb{K}(f))$, this exact sequence may be rewritten as

$$\cdots\to K_{i+1}(S)\to K_i(f)\to K_i(T)\to K_i(S)\to\cdots$$

In particular, if T is a scheme, and $f : S \to T$ is the inclusion of a closed subscheme, we write $K_i(T,S)$ instead of $K_i(f)$, and call it the i-th relative K group of the pair (T,S). We also write $\mathbb{K}(T,S)$ to mean $\mathbb{K}(f)$, in this situation, so that $K_i(T,S) = \pi_{i+1}(\mathbb{K}(T,S))$.

The relative K-groups have certain functoriality properties. In particular, if S is the singular locus (with reduced scheme structure) of our normal surface X, and nS is the subscheme of X defined by the n-th power of the ideal sheaf of S, then there is a commutative diagram of schemes and morphisms

$$\begin{array}{ccc} nE & \to & Y \\ \pi\mid_{nE}\downarrow & & \downarrow\pi \\ nS & \to & X \end{array}$$

which gives rise to a commutative diagram of relative K-groups

$$\begin{array}{ccccc} \to K_1(Y) \to & K_1(nE) & \to K_0(Y,nE) \to & K_0(Y) & \to K_0(nE) \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \to K_1(X) \to & K_1(nS) & \to K_0(X,nS) \to & K_0(X) & \to K_0(nS) \end{array}$$

Another functoriality property of relative K-groups implies the following. Let (T,S) be a pair consisting of a scheme and a closed subscheme, and $i : T' \hookrightarrow T$ another closed subscheme such that (a) $T' \cap S = \emptyset$, and (b) all vector bundles on T' have finite $\mathcal{O}_T$-homological dimension (e.g., $T \setminus S$ is a Noetherian regular scheme). There is a well-defined homotopy class of maps $i_* : \mathbb{K}(T') \to \mathbb{K}(T)$, together with a lifting $\mathbb{K}(T') \to \mathbb{K}(T,S)$, which is again well-defined up to homotopy. Thus the natural "Gysin" maps $K_i(T') \to K_i(T)$ lift to maps $K_i(T') \to K_i(T,S)$.

In our context, in particular, any point $x \in X \setminus S = Y \setminus E$ defines homomorphisms $K_0(k(x)) \to K_0(X,nS)$ and $K_0(x) \to K_0(Y,nE)$, giving a commutative triangle. Identifying $K_0(k(x)) = \mathbb{Z}$, we see that any point in $X \setminus S$ has a class in $K_0(X,nS)$ as well as in $K_0(Y,nE)$, compatibly with the map $\pi^* : K_0(X,nS) \to K_0(Y,nE)$. Define

$$F^2K_0(Y,nE) \subset K_0(Y,nE), \quad F^2K_0(X,nS) \subset K_0(X,nS)$$

to be the subgroups generated by the classes of points of $X \setminus S$. We then have a commutative square

$$\begin{array}{ccc} F^2K_0(Y,nE) & \to & F^2K_0(Y) \\ \uparrow & & \uparrow \\ F^2K_0(X,nS) & \to & F^2K_0(X) \end{array}$$

In fact, it is easy to see that all the four maps in this square are *surjective*. Indeed, this is clear for the maps with domain $F^2K_0(X,nS)$, since both the target groups are also generated by points of $X \setminus S$, by definition. This same set of points also generates $F^2K_0(Y)$, from an easy moving lemma, since the class group of any curve is generated by the classes of points in any nonempty Zariski open subset.

For any scheme T, one has a decomposition

$$K_1(T) = \Gamma(T, \mathcal{O}_T^*) \oplus SK_1(T),$$

which is functorial for arbitrary morphisms. For affine T, this is defined using the determinant. In fact for any T there is a functorial morphism $T \to \operatorname{Spec}\Gamma(T,\mathcal{O}_T)$, inducing a map on $K_1(\Gamma(T,\mathcal{O}_T)) \to K_1(T)$, as well as a map $K_1(T) \to \Gamma(T,\mathcal{O}_T^*)$, where we identify the sheaf of units $\mathcal{O}_T^*$ with the sheaf (for the Zariski topology) associated to the presheaf $V \mapsto K_1(V)$.

Next, one sees that

$$\ker\left(F^2K_0(Y, nE) \to F^2K_0(Y)\right) \subset \text{image} SK_1(nE),$$

and similarly

$$\ker\left(F^2K_0(X, nS) \to F^2K_0(X)\right) \subset \text{image} SK_1(nS).$$

This follows easily from the fact that any invertible function on the complement of a finite subset of X extends to one on all of X, and similarly for Y. However, $SK_1(nS) = 0$ since nS is a 0-dimensional affine scheme. Hence we see that $F^2K_0(X, nS) = F^2K_0(X)$. Thus we obtain an induced surjective map

$$CH^2(X) = F^2K_0(X) \to F^2K_0(Y, n),$$

for each n, compatible with the natural restriction maps

$$F^2K_0(Y, nE) \to F^2K_0(Y, (n-1)E)) \to F^2K_0(Y).$$

The following theorem, which is the main new ingredient in the proof of theorem 7.8, proves a conjecture of Bloch and myself, first stated in my Chicago thesis (1982) (see also [24], page 6).

Theorem 8.1. *Let $\pi : Y \to X$ be a resolution of singularities of a normal, quasi-projective surface, and let E be the exceptional locus, with its reduced structure. Then for all large $n > 0$, the maps*

$$F^2K_0(X) \to F^2K_0(Y, nE), \quad F^2K_0(Y, nE) \to F^2K_0(Y, (n-1)E)$$

are isomorphisms.

The proof of this theorem is in two steps, and is motivated by a paper [34] of Weibel, which studied negative K-groups of surfaces, and proved two old conjectures of mine from the paper [25].

First, one shows that the resolution $\pi : Y \to X$ can be factorized as a composition of two maps $f : Y \to Z$, $g : Z \to X$, where $g : Z \to X$ is the blow up of a local complete intersection subscheme supported on S (the singular locus of X), and $f : Y \to Z$ is the normalization map.

Next, one has that the maps on algebraic k-groups $g^* : K_i(X) \to K_i(Z)$ are split inclusions for all $i \geq 0$, since g is a proper, birational morphism of finite Tor dimension: indeed, these conditions imply that there is a well-defined *push-forward map* $g_* : K_i(Z) \to K_i(X)$, satisfying the projection formula, which implies that $g_* \circ g^*$ equals multiplictation by the class of $g_*[\mathcal{O}_Z] \in K_0(X)$. But this element of the ring $K_0(X)$ is invertible, since on an open dense subset U of X, it restricts to the unit element of $K_0(U)$; now one remarks that $\ker K_0(X) \to K_0(U))$ is a nilpotent ideal.

In particular, we see that $F^2K_0(X) \to F^2K_0(Z)$ is an isomorphism. Hence, for any closed subscheme T supported in $g^{-1}(S)$, we see that the two maps

$$F^2K_0(X) \to F^2K_0(Z, T), \ \ F^2K_0(Z, T) \to F^2K_0(Z)$$

are isomorphism as well (since both are surjective, and their composition is an isomorphism. We also get that if $T \subset T' \subset Z$ are two such subschemes of Z, then $F^2K_0(Z,T') \to F^2K_0(Z,T)$ is an isomorphism.

Next, one applies a suitable Mayer-Vietoris technique to study the relation between K-groups of Z and its normalization Y. Let T be a conductor subscheme for $f : Y \to Z$, and let $\widetilde{T}$ be the corresponding subscheme of Y. There are natural maps $K_i(Z,T) \to K_i(Y,\widetilde{T})$, inducing in particular a surjection

$$F^2K_0(Z,T) \to F^2K_0(Y,\widetilde{T}).$$

Now, using a fundamental localization theorem of Thomason-Trobaugh [33], it is shown in [21] (Cor. A.6) that there is an exact sequence

$$H^1(\widetilde{T}, \mathcal{I}/\mathcal{I}^2 \otimes \Omega_{\widetilde{T}/T}) \to K_0(Z,T) \to K_0(Y,\widetilde{T}),$$

which is functorial in T. Here $\mathcal{I}$ is the ideal sheaf of $\widetilde{T}$ on Y (whose direct image on Z equals the ideal sheaf of T). Let $2\widetilde{T}$ be the subscheme of Z defined by the ideal sheaf $\mathcal{I}^2$, and let $2T$ denote the subscheme of Z defined similarly. There is then a commutative diagram with exact rows

$$\begin{array}{ccccc} H^1(\mathcal{I}^2/\mathcal{I}^4 \otimes \Omega_{2\widetilde{T}/2T}) & \to & K_0(Z,2T) & \to & K_0(Y,2\widetilde{T}) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(\mathcal{I}/\mathcal{I}^2 \otimes \Omega_{\widetilde{T}/T}) & \to & K_0(Z,T) & \to & K_0(Y,2\widetilde{T}) \end{array}$$

But the left hand vertical arrow is 0, since the sheaf map $\mathcal{I}^2/\mathcal{I}^4 \to \mathcal{I}/\mathcal{I}^2$ is 0! Hence

$$\ker\left(F^2K_0(Z,2T) \to F^2K_0(Y,2\widetilde{T})\right) \subset \ker\left(F^2K_0(Z,2T) \to F^2K_0(Z,T)\right).$$

But we've seen already that $F^2K_0(Z,2T) \to F^2K_0(Z,T)$ is an isomorphism, and in fact both of these groups are isomorphic to $F^2K_0(X)$ (as well as to $F^2K_0(Z)$). Hence the surjective map

$$F^2K_0(Z,2T) \to F^2K_0(Y,2\widetilde{T})$$

must in fact be an isomorphism, and so $F^2K_0(X) \to F^2K_0(Y,2\widetilde{T})$ is an isomorphism. Finally, if $n > 0$ is large enough so that $2\widetilde{T}$ is a subscheme of nE, then we have that the two maps

$$F^2K_0(X) \to F^2K_0(Y,nE), \quad F^2K_0(Y,E) \to F^2K_0(Y,2\widetilde{T})$$

are isomorphisms, since both maps are surjective, and their composition is an isomorphism. This proves Theorem 8.1.

As a consequence of this theorem, we see that $\ker(F^2K_0(X) \to F^2K_0(Y))$ is identified with a subgroup of coker $(SK_1(Y) \to SK_1(nE))$, for sufficiently large n. In fact, it is shown in [1] that equality holds, i.e., that there is an exact sequence (for any large enough n)

$$SK_1(Y) \to SK_1(nE) \to F^2K_0(X) \to F^2K_0(Y) \to 0.$$

Assume now that k has characteristic 0, and E is a divisor with simple normal crossings. The groups $SK_1(nE)$ is then shown to fit into an exact sequence

$$H^1(Y, \mathcal{I}_E/\mathcal{I}_E^n) \otimes_k \Omega_{k/\mathbb{Z}} \to SK_1(nE) \to SK_1(E) \to 0. \tag{8.1}$$

Without going into technical details, let me say that this follows from a combination of several ingredients.

The first is a formula $SK_1(W) = H^1(W, \mathcal{K}_{2,W})$ for any scheme W of dimension ≤ 1, where $\mathcal{K}_{2,W}$ is the sheaf associated to the presheaf $U \mapsto K_2(\Gamma(\mathcal{O}_U))$ of Milnor K-groups.

In our case, since E is a reduced divisor on a smooth surface, the local ring of nE at a smooth point $x \in E$ has the form $\mathcal{O}_{x,E}[t]/(t^n)$, where t is a local generator for the ideal sheaf $\mathcal{I}_E$ in the local ring $\mathcal{O}_{x,Y}$. A result of Bloch gives a formula for any local $\mathbb{Q}$-algebra of the form $A = R[t_1, \ldots, t_r]$, where the ideal I in A generated by $t_1, \ldots, t_r$ is nilpotent, with quotient ring $A/I = R$. His result is that

$$K_2(A) = K_2(R) \oplus \frac{\ker \Omega_{A/\mathbb{Z}} \to \Omega_{R/\mathbb{Z}}}{d(I)}.$$

In particular, this is applicable when $A = R[t]/(t^n)$ is a truncated polynomial algebra, and thus gives a local description of $\mathcal{K}_{2,nE}$ at smooth points $x \in E$.

Since E has simple normal crossings, at a singular point of E, the local ring of nE has the form $\mathcal{O}_{x,Y}/(s^n t^n)$ where s, t are a regular system of parameters. This local ring is not of the type covered by the Bloch formula, but one has a Mayer-Vietoris sequence relating the K_2 of such a ring to the K_2 groups of the two (non-reduced) branches, and to K_2 of their intersection, and the Bloch formula is applicable to compute these K_2 groups. The local identifications with truncated polynomial algebras depend on choices of local generators for the ideal sheaves in Y of components of E, so when the above descriptions of the stalks of the $\mathcal{K}_2$ sheaf are globalized, one has appropriate terms involving the ideal sheaves of components of E.

Finally, one manages to reduce to the above sequence (8.1), by showing that certain additional terms appearing at the sheaf level have no contribution to the cohomology of the $\mathcal{K}_2$ sheaf, using the Grauert-Riemenschneider vanishing theorem for $\pi : Y \to X$, that $R^1\pi_*\omega_Y = 0$.

Now if $k = \overline{\mathbb{Q}}$, we see that since $\Omega_{k/\mathbb{Z}} = 0$ in this case, we get that for a resolution $\pi : Y \to X$ of a normal surface X over $\overline{\mathbb{Q}}$, with a normal crossing exceptional locus E, we have a formula

$$CH^2(X) = F^2K_0(X) \cong F^2K_0(Y, E).$$

If X is the affine cone over a smooth projective curve C over $\overline{\mathbb{Q}}$, then Y may be taken to be the blow up of the vertex of the cone. Then Y is a geometric line bundle over the original curve C, and E is its 0-section. Hence $K_i(Y) \to K_i(E)$ is an isomorphism for all i, by the homotopy invariance of algebraic K-theory for regular schemes, and so $K_i(Y, E) = 0$ for all i. This gives in particular, from Theorem 8.1, that $CH^2(X) = 0$. This was the conclusion of Theorem 7.8 for $k = \overline{\mathbb{Q}}$.

On the other hand, suppose $k = \mathbb{C}$, the complex numbers, and X is the cone over a smooth projective complex curve C. Again using the blow-up $Y \to X$ of the vertex, and the line bundle structure on Y with 0-section E, we first see that (8.1) simplifies in this case to a formula

$$SK_1(nE) = SK_1(E) \oplus \oplus_{j=1}^{n-1} H^1(C, \mathcal{O}_C(j)) \otimes_{\mathbb{C}} \Omega_{\mathbb{C}/\mathbb{Z}}.$$

This uses that the inclusion $E \hookrightarrow nE$ has a retraction, so (8.1) becomes a split exact sequence; then one uses that $\mathcal{I}/\mathcal{I}^n \cong \oplus_{j=1}^{n-1}\mathcal{O}_C(j))$ as sheaves of $\mathbb{C}$-vector spaces, using the underlying graded structure.

Theorem 8.1 now implies that

$$CH^2(X) = \oplus_{j=1}^{\infty} H^1(C, \mathcal{O}_C(1)) \otimes_{\mathbb{C}} \Omega_{\mathbb{C}/\mathbb{Z}},$$

where the direct sum is of course finite, since $H^1(C, \mathcal{O}_C(j))$ vanishes for all large j. Since for the curve C,

$$H^1(C, \mathcal{O}_C(j)) = 0 \implies H^1(C, \mathcal{O}_C(j+1)) = 0 \quad \forall\, j,$$

since multiplication by a suitable section of $\mathcal{O}_C(1)$ gives an inclusion $\mathcal{O}_C(j) \to \mathcal{O}_C(j+1)$ with a cokernel supported at points, and so this induces a surjection on H^1. Hence, we see that

$$CH^2(X) = 0 \Leftrightarrow H^1(C, \mathcal{O}_C(1)) = 0.$$

This is the other conclusion of Theorem 7.8 for the case $k = \mathbb{C}$.

It was known earlier that Theorem7.2 follows from the special case of normal affine surfaces, as discussed earlier. In this case, it can be deduced using Theorem 8.1. Again, one first takes a resolution $Y \to X$ with normal crossing exceptional locus E, and makes the analysis of $SK_1(nE)$ as in characteristic 0. Then, instead of using the Bloch formula, which is in fact not valid in characteristic p, one shows that $\ker(SK_1(nE) \to SK_1(E))$ is p^N torsion, for some N, using certain local descriptions of the $\mathcal{K}_2$ sheaves, found for example in work of Bloch and Kato.

On the other hand, one sees also that $\ker(F^2K_0(X) \to F^2K_0(Y))$ is a divisible abelian group, and further, that

$$\operatorname{coker} SK_1(Y) \to SK_1(E)$$

is a torsion-free divisible group (one ingredient in the proof of the latter is the negative definiteness of the intersection pairing on components of E). This implies that $F^2K_0(Y, E) \to F^2K_0(Y)$ is an isomorphism on torsion, and so $\ker F^2K_0(X) \to F^2K_0(Y, E)$ is also divisible, by a diagram chase. But we have also seen that it is p^N-torsion for some large enough N, so it is 0, i.e., $F^2K_0(X) \cong F^2K_0(Y, E)$, and also $F^2K_0(X) \to F^2K_0(Y)$ is an isomorphism on torsion subgroups.

This analysis is valid over an arbitrary algebraically closed ground field of characteristic p. Further, if X is affine, one sees that $CH^2(Y)$ is in fact torsion-free, from the theorems of Roitman and Milne, cited earlier. Hence $CH^2(X)$ is torsion-free. But if now $k = \overline{\mathbb{F}}_p$, clearly $CH^2(X)$ is torsion, since the Picard group of any affine curve over $\overline{\mathbb{F}}_p$ is torsion. Hence we get that $CH^2(X) = 0$ in this case, as claimed in Theorem 7.2.

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai-400005, India.

E-mail address: `srinivasmath.tifr.res.in`