Constructing median-unbiased estimators in one-parameter families of distributions *via* optimal nonparametric estimation and stochastic ordering

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ABSTRACT

If $\theta \in \Theta$ is an unknown real parameter of a distribution under consideration, we are interested in constructing an exactly median-unbiased estimator $\hat{\theta}$ of θ, i.e. an estimator $\hat{\theta}$ such that a median $Med(\hat{\theta})$ of the estimator equals θ, uniformly over $\theta \in \Theta$. We shall consider the problem in the case of a fixed sample size n (non-asymptotic approach).

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1. THE MODEL

Let $\mathcal{F}$ be a one-parameter family of distributions $\{F_\theta : \theta \in \Theta\}$, where Θ is an (finite or note) interval on the real line. The family $\mathcal{F}$ is assumed to be a family of distributions with continuous and strictly increasing distribution functions and stochastically ordered by θ so that for every $x \in supp\,\mathcal{F} = \bigcup_{\theta\in\Theta} supp\,F_\theta$ and for every $q \in (0,1)$, the equation $F_\tau(x) = q$ has exactly one solution in $\tau \in \Theta$. It is obvious that the solution depends monotonically both on x and q. Given a sample $X_1, X_2, \ldots, X_n$ from an F_θ, we are interested in a median-unbiased estimation of θ; here n is a fixed integer (non-asymptotic approach).

The model represents a wide range of one-parameter families of distributions.

Example 1. The family of uniform distributions on $(\theta, \theta+1)$, with $-\infty < \theta < +\infty$.

Example 2. The family of power distributions on $(0,1)$ with distribution functions $F_\theta(x) = x^\theta$, $\theta > 0$.

Example 3. The family of gamma distributions with probability distribution functions of the form

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, \quad x > 0$$

with $\alpha > 0$.

Example 4. Consider the family of Cauchy distributions with probability distribution function of the form

$$g_\lambda(y) = \frac{1}{\lambda} \frac{1}{1 + (y/\lambda)^2}, \quad -\infty < y < +\infty$$

and distribution function of the form

$$G_\lambda(y) = \frac{1}{2} + \frac{1}{\pi}\, arctan\, \frac{x}{\lambda}$$

with $\lambda > 0$. The family of distribution functions $\{F_\lambda, \lambda > 0\}$ of $X = |Y|$ with

$$F_\lambda(x) = \frac{2}{\pi}\, arctan\, \frac{x}{\lambda}$$

satisfies the model assumptions so that the problem of estimating λ from a sample $Y_1, Y_2, \ldots, Y_n$ amounts to that from the sample $X_1, X_2, \ldots, X_n$ with $X_i = |Y_i|$, $i = 1, 2, \ldots, n$.

Example 5. Consider the one-parameter family of Weibull distributions with distribution functions of the form

$$G_\alpha(y) = 1 - e^{-y^\alpha}, \quad y > 0,\ \alpha > 0$$

and let $X = max\{Y, 1/Y\}$. The family $\{F_\alpha, \alpha > 0\}$ of distributions of X with distribution functions of the form

$$F_\alpha(x) = e^{-x^{-\alpha}} - e^{-x^\alpha}, \quad x > 1,\ \alpha > 0$$

satisfies the model assumptions.

Example 6. (*Estimating the characteristic exponent of a symmetric α-stable distribution*). Consider the one-parameter family of α-stable distributions with characteristic functions $\exp\{-t^\alpha\}$, $0 < \alpha \le 2$. The problem is to construct a median-unbiased estimator of α. Some related results one can find in Fama and Roll (1971) and Zieliński (2000). We shall not consider the problem in this note because it needs (and it deserves) a special treatment and will be discussed in details elsewhere.

Generally: every family of distributions F_θ with continuous and strictly increasing F_θ and a location parameter θ (i.e. $F_\theta(x) = F_0(x-\theta)$) satisfies the model assumptions. Similarly, every family of continuous and strictly increasing distributions on $(0, +\infty)$ with a scale parameter θ (i.e. $F_\theta(x) = F_1(x/\theta)$) suits the model.

2. THE METHOD

The method consists in

1) for a given $q \in (0,1)$, estimating the q-th quantile (the quantile of order q) of the underlying distribution in a non-parametric setup; denote the estimator by $\hat{x}_q$. A restriction is that for a fixed n a median-unbiased estimator of the q-th quantile exists iff $max\{q^n, (1-q)^n\} \leq \frac{1}{2}$; in our approach the restriction does not play any role.

2) solving the equation $F_\tau(\hat{x}_q) = q$ with respect to τ. The solution, to be denoted by $\hat{\theta}_q$, is considered as an estimator of θ. The solution of the equation $F_\tau(x) = q$ with respect to τ will be denoted by $\hat{\theta}_q(x)$ so that $\hat{\theta}_q = \hat{\theta}_q(\hat{x}_q)$.

In the model, if $\hat{x}_q$ is a median-unbiased estimator of x_q then, due to monotonicity of $\hat{\theta}_q(x)$ with respect to x, $\hat{\theta}_q$ is a median-unbiased estimator of θ. What is more, if $\hat{x}_q$ is the median-unbiased estimator of x_q the most concentrated around x_q in the class of all median-unbiased estimators which are equivariant with respect to monotone transformations of data (shortly: the best estimator) then, due to monotonicity again, $\hat{\theta}_q$ is the most concentrated around θ median-unbiased estimator of θ in the class of all median-unbiased estimators which are equivariant with respect to monotone transformations of data (shortly: the best estimator).

Given $q \in (0,1)$, the best estimator $\hat{x}_q$ of x_q is given by the formula

$$[E] \qquad \hat{x}_q = X_{k:n}\mathbf{1}_{(0,\lambda]}(U) + X_{k+1:n}\mathbf{1}_{(\lambda,1)}(U),$$

where $X_{k:n}$ is the the k-th order statistic $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ from the sample $X_1, X_2, \ldots, X_n$ and

$$\begin{aligned} k &= k(q) \\ &= \textit{the unique integer such that } Q(k;n,q) \geq \frac{1}{2} \geq Q(k+1;n,q), \end{aligned}$$

$$\lambda = \lambda(q) = \frac{\frac{1}{2} - Q(k+1;n,q)}{Q(k;n,q) - Q(k+1;n,q)},$$

$$Q(k;n,q) = \sum_{j=k}^{n} \binom{n}{j} q^j (1-q)^{n-j};$$

here U is a random variable uniformly distributed on $(0,1)$ and independent of the sample $X_1, X_2, \ldots, X_n$ (Zieliński 1988).

When estimating θ in a parametric model $\{F_\theta : \theta \in \Theta\}$, the problem is to choose an "optimal" q. To define a criterion of optimality (or "an ordering in the class $\hat{\theta}_q$, $0 < q < 1$, of estimators"), let us recall (e.g. Lehmann 1983, Sec. 3.1) that a median-unbiased estimator $\hat{\theta}$ of a parameter θ is that for which

$$[K] \qquad E_\theta|\hat{\theta} - \theta| \leq E_\theta|\hat{\theta} - \theta'| \quad \textit{for all } \theta, \theta' \in \Theta$$

(the estimator is closer to the "true" value $\theta \in \Theta$ than to any other value $\theta' \in \Theta$ of the parameter). According to the property, we shall choose q_{opt} as that with minimal risk under the loss function $|\hat{\theta} - \theta|$, i.e. such that

$$E_\theta|\hat{\theta}_{q_{opt}} - \theta| \le E_\theta|\hat{\theta}_q - \theta|, \quad 0 < q < 1$$

for all $\theta \in \Theta$, if possible.

Using the fact that $\theta \in \Theta$ generates the stochastic ordering of the family $\{F_\theta : \theta \in \Theta\}$, we shall restrict our attention to finding q_{opt} which satisfies criterion [K] for a fixed θ, for example $\theta = 1$ (if θ is a scale or a shape parameter) or $\theta = 0$ if θ is a location parameter; then the problem reduces to minimizing

$$R(q) = E|\hat{\theta}_q - 1| \quad \mathit{or} \quad R(q) = E|\hat{\theta}_q|$$

with respect to $q \in (0,1)$, where $E = E_1$ or $E = E_0$, respectively. Formulas below are given for the case $\theta = 1$; the case of $\theta = 0$ can be treated in full analogy.

By [E] we obtain

$$R(q) = \lambda(q)\, E|\hat{\theta}_q(X_{k(q):n}) - 1| + (1 - \lambda(q))\, E|\hat{\theta}_q(X_{k(q)+1:n}) - 1|.$$

By the fact that $R(q)$ is a convex combination of two quantities, it is obvious that q_{opt} satisfies

$$\lambda(q_{opt}) = 1$$

and

$$E|\hat{\theta}_{q_{opt}}\left(X_{k(q_{opt}):n}\right) - 1| \le E|\hat{\theta}_q\left(X_{k(q):n}\right) - 1|, \quad 0 < q < 1.$$

By the very definition of λ, $\lambda(q) = 1$ iff $q \in \{q_1, q_2, \ldots, q_n\}$ where q_i satisfies $Q(i; n, q_i) = \frac{1}{2}$, and the problem reduces to finding the smallest element of the finite set

$$\{E|\hat{\theta}_{q_i}(X_{i:n}) - 1|, \quad i = 1, 2, \ldots, n\}$$

If $X_{k:n}$ is the k-th order statistic from the sample $X_1, X_2, \ldots, X_n$ from a distribution function F, then $U_{k:n} = F(X_{k:n})$ is the k-th order statistic from the sample $U_1, U_2, \ldots, U_n$ from the uniform distribution on $(0,1)$ which gives us

$$E|\hat{\theta}_{q_i}(X_{i:n}) - 1| = E|\hat{\theta}_{q_i}(F^{-1}(U_{i:n})) - 1|$$
$$= \frac{\Gamma(n)}{\Gamma(i)\Gamma(n-i+1)} \int_0^1 \left|\hat{\theta}_{q_i}\left(F^{-1}(t)\right) - 1\right| t^{i-1}(1-t)^{n-i} dt$$

The latter can be easily calculated numerically. For numerical integration it should be taken into account that $\hat{\theta}_{q_i}(F^{-1}(U_{i:n})) \geq 1$ iff $0 < t \leq q_i$ or $q_i \leq t < 1$.

3. APPLICATIONS

Example 1A. In the case of uniform distributions on $(\theta, \theta+1)$, the solution τ of the equation $F_\tau(\hat{x}_q) = q$ takes on the form $\tau = \hat{x}_q - q$. For example for $n=10$ the best estimator is $X_{1:10} - 0.067$ or $X_{10:10} - 0.933$.

Example 2A. In the case of power distributions, the best estimator is given as the (unique) solution, with respect to τ, of the equation $\hat{x}_q^\theta = q_{opt}$ which, for example in the case with $n = 10$ takes on the form $-1.81854/Log[X_{2:10}]$.

Example 3A. In the case of gamma distributions, the best estimator is given as the (unique) solution, with respect to τ, of the equation $F_\tau(\hat{x}_q) = q_{opt}$ which, for example in the case with $n = 10$ takes on the form $F_\tau(X_{3:10}) = 0.2586$; we are not able to give an explicite formula for the estimator here.

Example 4A. In the case of Cauchy distributions the solution τ of the equation $F_\tau(\hat{x}_q) = q$ can be written in the form

$$\tau = \frac{\hat{x}_q}{tg(\frac{\pi}{2}q)}$$

and, for example for $n = 10$, the best estimator is of the form $1.16456 \cdot X_{5:10}$.

Example 5A. In the case of Weibull distributions, the best estimator is given as the (unique) solution, with respect to τ, of the equation $F_\tau(\hat{x}_q) = q_{opt}$ which, for example in the case with $n = 10$ takes on the form $F_\tau(X_{8:10}) = 0.7414$ which gives us the optimal estimator of the form $0.302/Log(X_{8:10})$.

3. ACKNOWLEDGEMENT

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