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Diffeomorphisms that are Symplectomorphisms

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DIFFEOMORPHISMS THAT ARE SYMPLECTOMORPHISMS

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Abstract. Let (X, ω_X) and (Y, ω_Y) be compact symplectic manifolds of dimension $2n > 2$. Let us fix a number k with $0 < k < n$ and assume that a diffeomorphism $\Phi : X \to Y$ transforms all $2k$-dimensional symplectic submanifolds of X onto symplectic submanifolds of Y. Then Φ is a conformal symplectomorphism, i.e., there is a constant $c \neq 0$ such that $\Phi^* \omega_Y = c\omega_X$.

1. Introduction.

Let (X, ω_0) be a standard symplectic affine space over $\mathbb{R}$ of dimension $2n$, i.e., $X \cong \mathbb{R}^{2n}$ and $\omega_0 = \sum_i dx_i \wedge dy_i$ is the standard non-degenerate skew-symmetric form on X. Linear symplectomorphisms of (X, ω_0) are characterized (cf. [3]) as linear automorphisms of X preserving some minimal, complete data defined by ω_0 on systems of linear subspaces. In this way the linear symplectic group $\mathbf{Sp}(X)$ may be characterized geometrically together with its natural conformal and anti-symplectic extensions.

The purpose of this paper is to put the linear considerations of symplectic invariants into a more general context. Let (X, ω_X) and (Y, ω_Y) be compact symplectic manifolds of dimension $2n$ (all manifolds in this paper are assumed to be connected). We say that a diffeomorphism $F : X \to Y$ is a *conformal symplectomorphism* if there is a non-zero constant $c \in \mathbb{R}$ such that $F^* \omega_Y = c\omega_X$. Recall that a submanifold $Z \subset X$ is a *symplectic submanifold* of X if it is closed and the pair $(Z, \omega_X|_{TZ})$ is itself a symplectic manifold. Our main result is:

Theorem. *Let (X, ω_X) and (Y, ω_Y) be compact symplectic manifolds of dimension $2n > 2$. Fix a number $0 < s < n$. Assume that $\Phi : X \to Y$ is a diffeomorphism which transforms all $2s$-dimensional symplectic (closed) submanifolds of X onto symplectic (closed) submanifolds of Y. Then Φ is a conformal symplectomorphism.*

In other words, for any fixed s as above, the conformal symplectic structure on X is uniquely determined by the family of all $2s$-dimensional (closed) symplectic submanifolds of X.

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2. Generators of the group $Sp(2n)$

Here we recall some basic facts about the linear symplectic group. Let (X, ω) be a symplectic vector space. There exists a basis of X, called a symplectic basis, $u_1, \ldots, u_n, v_1, \ldots, v_n$, such that

$$\omega(u_i, u_j) = \omega(v_i, v_j) = 0, \quad \omega(u_i, v_j) = \delta_{ij}.$$

Let (X, ω_X) and (Y, ω_Y) be symplectic vector spaces. We say that a linear isomorphism $F : X \to Y$ is a *symplectomorphism* (or is *symplectic* on X) if $F^*\omega_Y = \omega_X$, i.e., $\omega_X(x, y) = \omega_Y(F(x), F(y))$ for every $x, y \in X$. The group of automorphisms of (X, ω) is called the symplectic group and is denoted by $\mathbf{Sp}(X, \omega)$. Via a symplectic basis, X can be identified with the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ and $\mathbf{Sp}(X, \omega)$ can be identified with the group of $2n \times 2n$ real matrices A which satisfy $A^T J_0 A = J_0$, where J_0 is the $2n \times 2n$ matrix of ω_0 (in the standard basis), i.e.,

$$J_0 = \begin{bmatrix} 0 & \ldots & 0 & -1 & \ldots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & -1 \\ 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \ldots & 1 & 0 & \ldots & 0 \end{bmatrix}.$$

Let $c \in \mathbb{R}$ and $i < j$. We can define following "elementary" symplectomorphisms:

1) $L_i(c)(x_1, ..., x_n, y_1, ..., y_n) = (x_1, ..., x_n, y_1, ..., y_{i-1}, y_i + cx_i, y_{i+1}, ..., y_n)$,
2) $L_{ij}(c)(x_1, ..., x_n, y_1, ..., y_n) = (x_1, ..., x_n, y_1, ..., y_{i-1}, y_i{+}cx_j, y_{i+1}, ..., y_{j-1}, y_j{+}cx_i, y_{j+1}, ..., y_n)$,
3) $R_i(c)(x_1, ..., x_n, y_1, ..., y_n) = (x_1, ..., x_{i-1}, x_i + cy_i, x_{i+1}, ..., x_n, y_1, ..., y_n)$,
4) $R_{ij}(c)(x_1, ..., x_n, y_1, ..., y_n) = (x_1, ..., x_{i-1}, x_i{+}cy_j, x_{i+1}, ..., x_{j-1}, x_j{+}cy_i, x_{j+1}, ..., x_n, y_1, ..., y_n)$.

We have the following basic result:

Theorem 2.1. *Let $X = (\mathbb{R}^{2n}, \omega_0)$ be the standard symplectic vector space. Then the group $\mathbf{Sp}(X)$ is generated by the following family of elementary symplectomorphisms:*

$$\{L_i(c), L_{ij}(c), R_i(c), R_{ij}(c) : \ 0 < i < j \leq n \text{ and } c \in \mathbb{R}\}.$$

Proof. We reason by induction. For $n = 1$ we have $\mathbf{Sp}(\mathbb{R}^2) = \mathbf{SL}(2)$ and the result is well known from linear algebra. Assume $n > 1$.

Let $S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear symplectomorphism. Denote coordinates by $x_1, y_1, ..., x_n, y_n$ (where $\omega_0 = \sum_i dx_i \wedge dy_i$). We have

$$S(x_1, y_1, ..., x_n, y_n) = (\sum_i a_{1,i}x_i + \sum_j b_{1,j}y_j, \ldots, \sum_i a_{2n,i}x_i + \sum_j b_{2n,j}y_j).$$

Observe how the rows of the matrix of S are transformed under composition $S \circ L$ with an elementary symplectomorphism L (for simplicity we consider only the first row and we take the coordinates $x_1, ..., x_n, y_1, ..., y_n$). After composition with $L_i(c)$ we have:

1) $(a_{11}, ..., a_{1n}, b_{11}, ..., b_{1n}) \to (a_{11}, ..., a_{1i} + cb_{1i}, ..., a_{1n}, b_{11}, ..., b_{1n})$,

with $L_{ij}(c)$ we have:

2) $(a_{11}, ..., a_{1n}, b_{11}, ..., b_{1n}) \to (a_{11}, ..., a_{1i} + cb_{1j}, ..., a_{1j} + cb_{1i}, ..., a_{1n}, b_{11}, ..., b_{1n})$,

with $R_i(c)$ we have:

3) $(a_{11}, ..., a_{1n}, b_{11}, ..., b_{1n}) \to (a_{11}, ..., a_{1n}, b_{11}, ..., b_{1i} + ca_{1i}, ..., b_{1n})$,

with $R_{ij}(c)$ we have:

4) $(a_{11}, ..., a_{1n}, b_{11}, ..., b_{1n}) \to (a_{11}, ..., a_{1n}, b_{11}, ..., b_{1i} + ca_{1j}, ..., b_{1j} + ca_{1i}, ..., b_{1n})$.

Transformations 1) - 4) will be called *elementary operations*. Now we show that using only elementary operations we can transform the first row of S to $(1, 0, ..., 0)$ and the second to $(0, ..., 0, 1, 0, ..., 0)$ (here the unit corresponds to b_{1n}).

Indeed, consider the first row. Of course it has a non-zero element, say b_{1s}. Using $L_s(c)$ we can assume that also $a_{1s} \neq 0$. Now using $L_{is}(c)$ and $R_{js}(d)$ for sufficiently general c and d we can assume that all elements of the first row are non-zero. Again applying $R_i(c)$ for $i > 1$ we can now transform the first row to $(a_{11}, ..., a_{1n}, 1, 0, ..., 0)$. Using $L_{1j}(c)$ we can transform this row to $(1, 0, ..., 0, 1, 0, ..., 0)$ and finally using $R_1(-1)$ we obtain $(1, 0, ..., 0)$. Now consider the second row (after these transformations): $(a_{21}, ..., a_{2n}, b_{21}, ..., b_{2n})$. We can apply our method to the subrow $(a_{22}, ..., a_{2n}, b_{22}, ..., b_{2n})$ (if it is non-zero) and obtain finally the row $(a_{21}, 1, 0, ..., 0, b_{21}, 0, ..., 0)$ (or $(a_{21}, 0, ..., 0, b_{21}, 0, ..., 0)$). Since the value of ω_0 on these two rows is 1 we conclude that $b_{21} = 1$. Now (in the first case) we can use $L_{12}(-1)$ to obtain a row of the form $(a_{21}, 0, ..., 0, 1, 0, ..., 0)$. Finally applying $L_1(-a_{12})$ we get $(0, ..., 0, 1, 0, ..., 0)$.

Thus under all these compositions the matrix of S in the coordinates $x_1, y_1, ..., x_n, y_n$ has the form

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ * & * & a_{33} & \dots & b_{3n} \\ * & * & a_{43} & \dots & b_{4n} \\ \vdots & \vdots & \vdots & & \vdots \\ * & * & a_{n3} & \dots & b_{n1} \end{bmatrix}.$$

Let $\mathbf{r}_i$ denote the i^{th} row of the matrix of S. For $j > 2$ we have $\omega_0(\mathbf{r}_1, \mathbf{r}_j) = 0$ and $\omega_0(\mathbf{r}_2, \mathbf{r}_j) = 0$. We can easily conclude that all the $*$ in the matrix of S are 0. Since

$$\begin{bmatrix} a_{33} & \dots & b_{3n} \\ a_{43} & \dots & b_{4n} \\ \vdots & & \vdots \\ a_{n3} & \dots & b_{n1} \end{bmatrix}$$

is a symplectic matrix we can apply the induction hypothesis. □

We conclude this section by recalling (and extending) some result from [3].

Definition 2.2. Let $\mathcal{A}_{l,2r} \subset G(l,2n)$ denote the set of all l-dimensional linear subspaces of X on which the form ω has rank $\leq 2r$.

Of course $\mathcal{A}_{l,2r} \subset \mathcal{A}_{l,2r+2}$ if $2r+2 \leq l$. We have the following (see [3], Theorem 6.2):

Proposition 2.3. *Let (X,ω) be a symplectic vector space of dimension $2n$ and let $F : X \to X$ be a linear automorphism. Let $0 < 2r < 2n$. Assume F transforms $\mathcal{A}_{2r,2r-2}$ into $\mathcal{A}_{2r,2r-2}$. Then there is a non-zero constant c such that $F^*\omega = c\omega$.*

From Proposition 2.3 we can deduce the following interesting fact:

Proposition 2.4. *Let (X,ω_X) and (Y,ω_Y) be symplectic vector spaces of dimension $2n$ and let $F : X \to Y$ be a linear isomorphism. Fix a number $s :\ 0 < s < n$ and assume that F transforms all $2s$-dimensional symplectic subspaces of X onto symplectic subspaces of Y. Then there is a non-zero constant c such that $F^*\omega_Y = c\omega_X$.*

Proof. Via a symplectic basis we can assume that $(X,\omega_X) \cong (\mathbb{R}^{2n},\omega_0) \cong (Y,\omega_Y)$. By assumption the mapping F^* induced by F transforms the set $A = \mathcal{A}_{2s,2s} \setminus \mathcal{A}_{2s,2s-2}$ into the same set A. Of course $F^* : A \to A$ is an injection. Since A is a smooth algebraic variety and F^* is regular, the Borel Theorem (see [1]) implies that F^* is a bijection. This means that F transforms $\mathcal{A}_{2s,2s-2}$ into the same set, and we conclude the proof by applying Proposition 2.3. □

We end this section by:

Proposition 2.5. *Let X be a vector space of dimension $2n$ and let $\omega_1,\ \omega_2$ be two symplectic forms on X. If $\mathbf{Sp}(X,\omega_1) \subset \mathbf{Sp}(X,\omega_2)$, then there exists a non-zero constant c such that $\omega_2 = c\omega_1$.*

Proof. If $n = 1$, then theorem is obvious. Assume that $n > 1$. Let $\mathcal{A}_1$ $(\mathcal{A}_2)$ be a set of all ω_1 (ω_2) symplectic 2 dimensional subspaces of X. These sets are open and dense in the Grassmannian $G(2,2n)$. Hence $\mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset$. Take $H \in \mathcal{A}_1 \cap \mathcal{A}_2$. We have $\mathcal{A}_1 = \mathbf{Sp}(X,\omega_1)H \subset \mathbf{Sp}(X,\omega_2)H = \mathcal{A}_2$. Now apply Proposition 2.4 to $X = (X,\omega_1),\ Y = (X,\omega_2)$ and $F = identity$. □

3. Technical Results

Let $X = (\mathbb{R}^{2n},\omega_0)$ be the standard symplectic vector space. In X we consider the norm $\|(a_1,\ldots,a_{2n})\| = \max_{i=1}^{2n}|a_i|$. Take a smooth function $H : X \times \mathbb{R} \ni (z,t) \to \mathbb{R}$ and consider a system of differential equations

$$\phi'(t,x) = J_0(\nabla_z H)(\phi(t),t),\ \phi(0,x) = x.$$

Assume that this system has a solution $\phi(t,x)$ for every x and every t (this is satisfied, e.g., if supports of all functions $H_t,\ t \in \mathbb{R}$ are contained in a compact set). Then we can define the diffeomorphism

$$\Phi(x) = \phi(1, x) \tag{3.1}$$

It is not difficult to check that Φ is a symplectomorphism.

Definition 3.1. Let $\Phi : X \to X$ be a symplectomorphism. We say that Φ is a *hamiltonian symplectomorphism* if it is given by the formula (3.1) for some smooth function H. We also say that H is a Hamiltonian of Φ.

Lemma 3.2. *All elementary linear symplectomorphisms are hamiltonian symplectomorphisms.*

Proof. Indeed, we have:

1) $L_i(c)$ is given by the Hamiltonian $H(x, y) = (c/2)x_i^2$,

2) $L_{ij}(c)$ is given by the Hamiltonian $H(x, y) = cx_ix_j$,

3) $R_i(c)$ is given by the Hamiltonian $H(x, y) = -(c/2)y_i^2$,

4) $R_{ij}(c)$ is given by the Hamiltonian $H(x, y) = -cy_iy_j$. □

Now we show how to compute a Hamiltonian of a linear symplectomorphism:

Theorem 3.3. *Let $L : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear symplectomorphism. Then L has a polynomial Hamiltonian*

$$H_L(z, t) = \sum_{i,j=1}^{2n} a_{i,j}(t)z_iz_j, \tag{3.2}$$

where $a_{i,j}(t) \in \mathbb{R}[t]$ are polynomials of one variable t. Moreover, we can compute H_L effectively.

Proof. Let $L = L_m \circ \cdots \circ L_1$ where the L_i are elementary symplectomorphisms. We proceed by induction with respect to m. If $m = 1$ then we can use Lemma 3.2. In this case the flow $L_1(t)$ depends linearly on t.

Now consider $L' = L_{m-1} \circ \cdots \circ L_1$. By the induction hypothesis $L'(t) = L_{m-1}(t) \circ \cdots \circ L_1(t)$ is given by the Hamiltonian H' of the form 3.2. Let H'' be the Hamiltonian of L_m (as in Lemma 3.2). Now the flow $L(t) = L_m(t) \circ L'(t)$ is given by the Hamiltonian

$$H(z, t) = H''(z) + H'(L_m(t)^{-1}(z), t).$$

Of course it has also the form 3.2. Since we can decompose L into the product $L = L_m \circ \cdots \circ L_1$ effectively (see the proof of Theorem 2.1), we can also compute H in effective way. □

Proposition 3.4. *Let $L : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a hamiltonian symplectomorphism given by the flow $x \to \phi(t, x);\ t \in \mathbb{R}$. Assume that $\phi(t, 0) = 0$ for $t \in [0, 1]$. For every $\eta > 0$ there is an $\epsilon > 0$ and a hamiltonian symplectomorphism $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that*

1) $\Phi(x) = L(x)$ for all x with $\|x\| \leq \epsilon$,

2) $\Phi(x) = x$ for all x with $\|x\| \geq \eta$.

Proof. We know that $L(x) = \phi(1, x)$, where $\phi(t, x)$ is the solution of some differential equation

$$\phi'(t) = J_0(\nabla_z H)(\phi(t), t);\ \phi(0) = x.$$

Since $\phi(t, 0) = 0$ for every $t \in [0, 1]$, we can find $\epsilon > 0$ so small, that all trajectories $\{\phi(t, x),\ 0 \leq t \leq 1\}$, which start from the ball $B(0, \epsilon)$ are contained in the ball $B(0, \eta/2)$. Let $\sigma : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function such that

$$\sigma(z) = \begin{cases} 1 & \text{if } \|z\| \leq \eta/2, \\ 0 & \text{if } \|z\| \geq \eta. \end{cases}$$

Take $S = \sigma H$. The hamiltonian symplectomorphism Φ given by the differential equation

$$\phi'(t) = J_0(\nabla_z S)(\phi(t), t),\ \phi(0) = x,$$

is well defined on the whole of $\mathbb{R}^{2n}$ and

$$\Phi(x) = \begin{cases} L(x) & \text{if } ||x|| \leq \epsilon, \\ x & \text{if } \|x\| \geq \eta. \end{cases}$$

□

Now Theorem 3.3 easily yields the following important:

Corollary 3.5. *Let $L : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear symplectomorphism. For every $\eta > 0$ there is an $\epsilon > 0$ and a hamiltonian symplectomorphism $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that*

1) $\Phi(x) = L(x)$ for all x with $\|x\| \leq \epsilon$,

2) $\Phi(x) = x$ for all x with $\|x\| \geq \eta$.

Before we formulate our next result we need the following:

Lemma 3.6. *Let $X = (\mathbb{R}^{2n}, \omega_0)$ be the standard symplectic vector space. Fix $\eta > 0$ and let $a, b \in B(0, \eta)$. Then there exists a symplectomorphism $\Phi : X \to X$ such that*

$$\Phi(a) = b \text{ and } \Phi(x) = x \text{ for } \|x\| \geq 2\eta.$$

Proof. Let $c = (c_1, \ldots, c_{2n}) = b - a$. Define a sequence of points as follows:

1) $a_0 = a$,

2) $a_i = a_{i-1} + (0, \ldots, 0, c_i, 0, \ldots, 0)$.

Of course $a_i \in B(0, \eta)$ and $a_{2n} = b$. Now consider the translation

$$T_i : \mathbb{R}^{2n} \ni (x, y) \mapsto (x, y) + (0, \ldots, 0, c_i, 0, \ldots, 0) \in \mathbb{R}^{2n}.$$

We have $T_i(a_{i-1}) = a_i$ for $i = 1, ..., 2n$.

The translation T_i is a hamiltonian symplectomorphism given by the Hamiltonian

$$H_i(x, y) = \begin{cases} -c_i y_i & \text{if } i \leq n, \\ c_i x_{i-n} & \text{if } i > n. \end{cases}$$

Let V_i be the symplectic vector field which is determined by the Hamiltonian H_i. Since the ball $B(0,r)$ is a convex set, all trajectories $\phi(t),\ 0 \le t \le 1$, of the symplectic vector fields V_i, which begin at a_i lie in the ball $B(0,\eta)$. Let $\sigma : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function such that

$$\sigma(x) = \begin{cases} 1 & \text{if } \|x\| \le \eta, \\ 0 & \text{if } \|x\| \ge 2\eta. \end{cases}$$

Now let $F_i : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the hamiltonian symplectomorphism given by the Hamiltonian $G_i = \sigma H_i$. Then

$$G_i(a_{i-1}) = a_i \text{ and } G_i(x) = x \text{ if } \|x\| \ge 2\eta.$$

Now it is enough to take $\Phi = G_{2n} \circ G_{2n-1} \circ \cdots \circ G_1$. □

We apply Proposition 3.5 to the general case:

Theorem 3.7. *Let (X,ω) be a symplectic manifold. Let $a_1, ..., a_m$ and $b_1, ..., b_m$ be two families of points of X. For every $i = 1, ..., n$ choose a linear symplectomorphism $L_i : T_{a_i}X \to T_{b_i}X$. Then there is a symplectomorphism $\Phi : X \to X$ such that*

1) $\Phi(a_i) = b_i$,

2) $d_{a_i}\Phi = L_i$.

Proof. By the Darboux Theorem every point $x \in X$ has an open neighborhood V_x which is symplectically isomorphic to the ball $B(0, r_x)$ in the standard vector space $(\mathbb{R}^{2n}, \omega_0)$. Denote by $U_x \subset V_x$ the open set which corresponds to the ball $B(0, r_x/3)$.

Since $\dim X \ge 2$ the manifold $X \setminus \{a_2, ..., a_m\}$ is also connected. Hence there exists a smooth path $\gamma : I \to X$ such that $\gamma(0) = a_1,\ \gamma(1) = b_1$ and $\{a_2, ..., a_m\} \cap \gamma(I) = \emptyset$. Additionally we can assume that the sets V_x which cover $\gamma(I)$ are also disjoint from $\{a_2, ..., a_m\}$.

Let ϵ be a Lebesgue number for the function $\gamma : I \to X$ with respect to the cover $\{U_x\}_{x\in X}$ and choose an integer N with $1/N < \epsilon$. If $I_k := [k/N, (k+1)/N]$, then $\gamma(I_k)$ is contained in some $\{U_x\}$; denote it by U_k, the set V_x by V_k, and r_x by r_k. Let $A_k := \gamma(k/N)$, in particular $A_0 = a_1, A_N = b_1$.

Since $V_k \cong B(0, r_k)$ and $A_k, A_{k+1} \in B(0, r_k/3)$ we can apply Lemma 3.6 to obtain a symplectomorphism $\Phi : B(0, r_k) \to B(0, r_k)$ such that

$$\Phi(A_k) = A_{k+1} \text{ and } \ \Phi(x) = x \quad \text{for } \|x\| \ge (2/3)r_k.$$

We can extend Φ to the whole of X (we glue it with the identity); denote this extension by Φ_k. Put

$$\Psi = \Phi_N \circ \Phi_{N-1} \circ \cdots \circ \Phi_0.$$

Then $\Psi(a_1) = b_1$ and $\Psi(a_i) = a_i$ for $i > 1$. Repeating this process, we finally arrive at a symplectomorphism $\Sigma : X \to X$ such that $\Sigma(a_i) = b_i$ for $i = 1, \ldots, m$. In a similar way using Proposition 3.5 we can construct a symplectomorphism $\Pi : X \to X$ such that

1) $\Pi(b_i) = b_i$,

2) $d_{b_i}\Pi = L_i \circ (d_{a_i}\Sigma)^{-1}$.

Now it is enough to take $\Phi = \Pi \circ \Sigma$. □

Now we need the following result which is due to S.K. Donaldson (see [2]):

Theorem 3.8. *Let (X, ω_X) be a compact symplectic manifold of dimension $2n > 2$. Fix a number $0 < s < n$. There exists a closed $2s-$dimensional symplectic submanifold $Z \subset X$.*

Using Theorem 3.7 we can restate this result as follows:

Proposition 3.9. *Let (X, ω) be a compact symplectic manifold of dimension $2n > 2$. Let $a_1, ..., a_m$ be a family of points of X. Take $0 < s < n$. For every $i = 1, ..., m$ choose a linear $2s$-dimensional symplectic subspace $H_i \subset T_{a_i}X$. Then there is a closed symplectic $2s$-dimensional submanifold $Y \subset X$ such that*

1) $a_i \in Y$,

2) $T_{a_i}Y = H_i$.

Proof. Let $Z \subset X$ be as in Theorem 3.8. Take points $b_1, \ldots, b_m \in Z$. Let $S_i = T_{b_i}Z$. There are linear symplectomorphisms $L_i : T_{b_i}X \to T_{a_i}X$ such that $L_i(S_i) = H_i$ for $i = 1, \ldots, m$. By Theorem 3.7 there is a symplectomorphism $\Phi : X \to X$ such that

1) $\Phi(b_i) = a_i$,

2) $d_{b_i}\Phi = L_i$.

Now it is enough to take $Y = \Phi(Z)$. □

4. Main result

Finally we show that a symplectomorphism can be described as a diffeomorphism which preserves symplectic submanifolds.

Theorem 4.1. *Let (X, ω_X) and (Y, ω_Y) be compact symplectic manifolds of dimension $2n > 2$. Fix a number $0 < s < n$. Assume that $\Phi : X \to Y$ is a diffeomorphism which transforms all $2s$-dimensional symplectic submanifolds of X onto symplectic submanifolds of Y. Then Φ is a conformal symplectomorphism, i.e., there exists a non-zero number $c \in \mathbb{R}$ such that*

$$\Phi^*\omega_Y = c\omega_X.$$

Proof. Fix $x \in X$ and let $H \subset T_xX$ be a $2s$-dimensional symplectic subspace of T_xX. By Proposition 3.9 (applied for $m = 1$, $a_1 = x$ and $H_1 = H$) there exists a $2s$-dimensional symplectic submanifold M of X such that $x \in M$ and $T_xM = H$.

Let $\Phi(M) = M'$, $x' = \Phi(x)$. By assumption the submanifold $M' \subset Y$ is symplectic. This means that the space $d_x\Phi(H) = T_{x'}M'$ is symplectic. Hence the mapping $d_x\Phi$ transforms all linear $2s$-dimensional symplectic subspaces of T_xX onto subspaces of the same type. By Proposition 2.4

this implies that $d_x\Phi$ is a conformal symplectomorphism, i.e.,

$$(d_x\Phi)^*\omega_Y = \lambda(x)\omega_X,$$

where $\lambda(x) \neq 0$. This means that there is a smooth function $\lambda : X \to \mathbb{R}^*$ $(= \mathbb{R} \setminus \{0\})$ such that

$$\Phi^*\omega_Y = \lambda\omega_X.$$

But since the form ω_X is closed, so is $\Phi^*\omega_Y$. Since $n > 1$ this implies that the derivative $d\lambda$ vanishes, i.e., the function λ is constant. □

Corollary 4.2. *Let X be a compact manifold of dimension $2n > 2$. Let ω_1 and ω_2 be two symplectic forms on X. Fix a number $0 < k < n$. Assume that the family of all $2k$-dimensional ω_1-symplectic submanifolds of X is contained in the family of all $2k$-dimensional ω_2-symplectic submanifolds of X. Then there exists a non-zero number $c \in \mathbb{R}$ such that*

$$\omega_1 = c\omega_2.$$

Proof. It is enough to apply Theorem 4.1 to $X = (X, \omega_1)$, $Y = (X, \omega_2)$ and $\Phi =$ *identity*. □

Corollary 4.3. *Let (X, ω) be a compact symplectic manifold of dimension $2n > 2$. Fix a number $0 < s < n$. Assume that $\Phi : X \to X$ is a diffeomorphism which transforms all $2s$-dimensional symplectic submanifolds of X onto symplectic submanifolds. Then Φ is a symplectomorphism or antisimplectomorphism, i.e., $\Phi^*\omega = \pm\omega$. If Φ preserves an orientation and n is odd, then Φ is a symplectomorphism. Moreover, if n is even, then Φ has to preserve the orientation.*

Proof. Indeed, we have $\Phi^*\omega = c\omega$. We have

$$vol(X) = \int_X \omega^n = \pm \int_X \Phi^*\omega^n = \pm c^n \int_X \omega^n \tag{4.1}$$

hence $c = \pm 1$. Moreover, if Φ preserves an orientation and n is odd, then we get that $c = 1$. If n is even then $(-\omega)^n = \omega^n$ and Φ has to preserve the orientation. □

Example 4.4. We show that in the general case Φ do not need be a symplectomorphism. Let $Y = (S^2, \omega)$ (where ω is a standard volume form on the sphere) and let $(X_n, \omega_n) = \prod_{i=1}^n Y$ be a standard symplectic product. Further let $\sigma : S^2 \ni (x, y, z) \to (x, y, -z) \in S^2$ be a miror symmetry. Of course $\sigma^*\omega = -\omega$. More general if $\Sigma = \prod_{i=1}^n \sigma : X_n \to X_n$, then $\Sigma^*\omega_n = -\omega_n$. Hence it is possible that Φ from Corollary 4.3 is an antisimplectomorphism.

However, in any case either Φ or $\Phi \circ \Phi$ is a symplectomorphism.

Now let (X, ω) be a symplectic manifold and let us denote by $\mathbf{Symp}(X, \omega)$ the group of symplectomorphisms of X. At the end of this note we show that this group also determine a conformal symplectic structure on X:

Theorem 4.5. *Let X be a smooth manifold of dimension $2n > 2$ and let $\omega_1,\ \omega_2$ be two symplectic forms on X. If $\mathbf{Symp}(X, \omega_1) \subset \mathbf{Symp}(X, \omega_2)$, then there exists a non-zero constant c such that $\omega_2 = c\omega_1$.*

Proof. Take $x \in X$ and consider symplectic vector spaces $V_1 = (T_x X, \omega_1)$ and $V_2 = (T_x X, \omega_2)$. By Theorem 3.7 we have that for every linear symplectomorphism S of V_1, there is a symplectomorphism $\Phi_S \in \mathbf{Symp}(X, \omega_1)$, such that

a) $\Phi_S(x) = x$,

b) $d_x \Phi_S = S$.

Since $\mathbf{Symp}(X, \omega_1) \subset \mathbf{Symp}(X, \omega_2)$ we easily obtain that $\mathbf{Sp}(V_1) \subset \mathbf{Sp}(V_2)$. Consequently by Proposition 2.5 there exist a non-zero number $\lambda(x)$ such that $\omega_2(x) = \lambda(x)\omega_1(x)$. Now we finish the proof as in the proof of Theorem 4.1. □

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