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Completeness of bond market driven by Lévy process

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Completeness of bond market driven by Lévy process*

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Abstract

The completeness problem of the bond market model with noise given by the independent Wiener process and Poisson random measure is studied. Hedging portfolios are assumed to have maturities in a countable, dense subset of a finite time interval. It is shown that under some assumptions the market is not complete unless the support of the Lévy measure consists of a finite number of points. Explicit constructions of contingent claims which can not be replicated are provided.

Key words: bond market, completeness, Lévy term structure

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JEL Classification Numbers: G10,G11

1 Introduction

Tradeable bonds are specified by a set of their maturities, which potentially can consist of infinitely many points - it can be an interval or a half-line for example. Thus we have infinitely many assets and this is a significant difference between a bond and stock market with a finite number of stocks. This is also the reason why the bond market models are not covered by a classical theory of financial markets and thus economic problems, like completeness, have to be studied again.

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The problem of bond market completeness was treated in many different contexts depending on the model settings as well as on the definition of completeness. A classical question of the market completeness is to judge if it is possible to replicate any bounded random variable X, i.e. to find a portfolio which is equal to X at the final time. However, it is sometimes difficult to solve this problem in the set of all bounded random variables and thus another spaces are also considered, for example $L^2(\Omega)$ or even more exotic ones. In Taflin [11] it is shown that the model driven by the infinite dimensional Wiener process is not complete in the class $D_0 := \bigcap_{p>1} L^p(\Omega)$. In Carmona, Tehranchi [5] it is shown that each random variable which is of a special form can be replicated.

Another concept connected with the notion of completeness is the existence of a unique martingale measure. However, contrary to the finite dimensional stock market, this notion is not equivalent to completeness. As it was shown in Björk et. al [3] and [4] in a jump diffusion model uniqueness of the martingale measure is equivalent to the approximate completeness, i.e. for any random variable $X \in L^2(\Omega)$ there exists a sequence of random variables $\{X_n\}$ which converges to X in $L^2(\Omega)$ s.t. each element of the sequence can be replicated.

It was shown in Baran, Jakubowski, Zabczyk [1] that a model driven by the infinite dimensional Wiener process is not complete, i.e. there exists a bounded random variable which can not be replicated. In this paper we focus on a finite dimensional noise with jumps and for simplicity assume that it is given by the one dimensional Wiener process and Poisson random measure. We consider model with a finite time interval $[0, T^*]$. Each bond is specified by its maturity T and usually it is assumed that maturity can by any number from $[0, T^*]$. We adopt the assumptions from Eberlein, Jacod, Raible [6] and consider bonds with maturities in a dense, countable subset of $[0, T^*]$ denoted by J. This set consists of all bonds' maturities which can be involved in the portfolios construction. A bond with maturity T and the price process $P(\cdot, T)$ can be used by a trader if and only if $T \in J$. The completeness problem with the use of bonds with maturities in J can be formulated in two ways:

1) Does there exist a unique equivalent measure Q such that the discounted price of bonds $\hat{P}(\cdot, T)$ is a Q-local martingale for each $T \in J$?

2) Can arbitrary $\mathcal{F}_{T}^{*}$- measurable random variable, satisfying some regularity assumptions, be replicated with the use of bonds with maturities in J?

As far as we consider analogous formulations to (1) and (2) for finite number of stocks, they are equivalent - at least for a wide class of stock market models. However, as it was shown in [3] and [4] they can no longer be equivalent if we examine bond market with infinite number of assets. The problem of completeness with the use of bonds with maturities in J was originally formulated in [6], where it was treated in the sense of the formulation (1). It was shown that under some assumptions there exists exactly one martingale measure. In this paper we study the problem of completeness in the sense of the formulation (2). This approach requires a precise definition of portfolios which can be used by traders, see Section 3. We identify prices of bonds with elements of a Banach space B consisting of all bounded sequences with the supremum norm. The trader's position is identified with an element of l^1 - a subspace of the dual space B^*.

The self-financing condition is expressed by the fact that portfolio is an integral of the l^1-valued strategy with respect to the bond price process.

The general idea in the solution of the completeness problem is to examine the possibility of representing any martingale as a certain stochastic integral with l^1-valued integrand. The key tools used for this purpose are the representation theorem for local martingales, which comes from Kunita [10], and a version of theorem solving the so called problem of moments. The last one provides necessary and sufficient conditions for the existence of a linear, bounded functional satisfying certain conditions. Generally speaking we apply this theorem to the real and vector-valued functions defined on the support of the Lévy measure. Our main result states that every market model with the Lévy measure having a concentration point is incomplete. We provide an explicit construction of a bounded random variable which can not be replicated. If there is no concentration point we prove incompleteness under additional assumptions in the class of square integrable or bounded random variables. In the case when the Lévy measure has a finite support and the model satisfies additional assumptions we prove completeness in the class of integrable random variables. This is result is similar to Theorem 5.6 in [4] but requires weaker assumptions.

The paper is organized as follows: in Section 2 we recall basic facts on stochastic integrals and formulate the representation theorem for local martingales; Section 3 contains a description of the model and definition of portfolios; in Section 4 we present the main results - this section is divided into three parts with respect to the properties of the Lévy measure.

2 Local martingales representation

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t,\ t \in [0, T^*]\}$ generated by the 1-dimensional Wiener process W and Poisson random measure N defined on $\mathbb{R}_+ \times \mathbb{R}\backslash\{0\}$. The processes are assumed to be independent. By $\tilde{N}$ we denote the compensated Poisson random measure, i.e. $\tilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt$, where ν is a Lévy measure corresponding to N. Recall that ν satisfies integrability condition: $\int_{\mathbb{R}} \mid x \mid^2 \wedge 1 \ \nu(dx) < \infty$.

In order to formulate the representation theorem below, we briefly present description of the class of integrable processes with respect to W and $\tilde{N}$. We follow notation used in [10].

The process $\phi = \phi(\omega, t)$ is integrable with respect to the Wiener process if it is predictable and satisfies integrability condition

$$\int_0^{T^*} \mid \phi(s) \mid^2 ds < \infty \qquad P - a.s..$$

This class of processes is denoted by Φ. For any $\phi \in \Phi$ the integral

$$\int_0^t \phi(s) dW(s) := \int_0^{T^*} \phi(s) \mathbf{1}_{[0,t]}(s) dW(s)$$

is well defined and the process $\int_0^{\cdot} \phi(s)dW(s)$ is a continuous locally square integrable martingale.

The process $\psi = \psi(\omega, s, x)$ is called predictable if it is $\mathcal{P} \times \mathcal{B}(\mathbb{R})$ measurable, where $\mathcal{P}$ is a predictable sigma-field. If ψ satisfies condition

$$\int_0^{T^*} \int_{\mathbb{R}} \mid \psi(s,x) \mid \nu(dx)ds < \infty \qquad P-a.s. \tag{2.1}$$

then the integral

$$\int_0^{T^*} \psi(s,x)\tilde{N}(dx,ds) = \int_0^{T^*} \psi(s,x)N(ds,dx) - \int_0^{T^*} \psi(s,x)\nu(dx)ds$$

is well defined and the process $\int_0^{\cdot} \psi(s,x)\tilde{N}(ds,dx) = \int_0^{T^*} \psi(s,x)\mathbf{1}_{(0,\cdot]}(s)\tilde{N}(ds,dx)$ is a local martingale. The class of predictable processes satisfying (2.1) is denoted by Ψ_1.

If a predictable process ψ satisfies condition

$$\int_0^{T^*} \int_{\mathbb{R}} \mid \psi(s,x) \mid^2 \nu(dx)ds < \infty \qquad P-a.s. \tag{2.2}$$

then the integral $\int_0^{T^*} \psi(s,x)\tilde{N}(ds,dx)$ is constructed with the use of simple processes which converge to ψ in L^2. In this case $\int_0^{\cdot} \psi(s,x)\tilde{N}(ds,dx) = \int_0^{T^*} \psi(s,x)\mathbf{1}_{(0,\cdot]}(s)\tilde{N}(ds,dx)$ is a locally square integrable martingale. A class of predictable processes satisfying (2.2) is denoted by Ψ_2.
A class of all predictable processes which satisfy conditions

$$\psi\mathbf{1}_{\{|\psi|>1\}} \in \Psi_1 \quad \text{and} \quad \psi\mathbf{1}_{\{|\psi|\leq 1\}} \in \Psi_2$$

will be denoted by $\Psi_{1,2}$. In other words $\psi \in \Psi_{1,2}$ if and only if

$$\int_0^{T^*} \int_{\mathbb{R}} \mid \psi(s,x) \mid^2 \wedge \mid \psi(s,x) \mid \nu(dx)ds < \infty.$$

For any $\psi \in \Psi_{1,2}$ the integral

$$\begin{aligned}\int_0^{T^*} \psi(s,x)\tilde{N}(ds,dx) &= \int_0^{T^*} \psi(s,x)\mathbf{1}_{\{|\psi(s,x)|>1\}}(s,x)\tilde{N}(ds,dx) \\ &+ \int_0^{T^*} \psi(s,x)\mathbf{1}_{\{|\psi(s,x)|\leq 1\}}(s,x)\tilde{N}(ds,dx)\end{aligned}$$

is well defined and it is a local martingale as a function of the upper integration limit.

The next theorem comes from [10].

Theorem 2.1 *Let M be a local martingale. Then there exist $\phi \in \Phi$ and $\psi \in \Psi_{1,2}$ satisfying*

$$M_t = M_0 + \int_0^t \phi(s)dW(s) + \int_0^t \int_{\mathbb{R}} \psi(s,x)\tilde{N}(dx,ds). \qquad (2.3)$$

Moreover, the pair (ϕ,ψ) is unique i.e., if $(\phi^{'},\psi^{'})$ satisfies (2.3) then

$$\phi = \phi^{'} \ w.r.to \ \ P \otimes \lambda \ a.s. \quad \textit{and} \quad \psi = \psi^{'} \ w.r.to \ \ P \otimes \lambda \otimes \nu \ a.s.,$$

where λ is the Lebesgue measure on $[0,T^*]$.

3 Bond market model

We begin description of the model by specifying the dynamics of the forward rate

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t) + \int_{\mathbb{R}} \gamma(t,x,T)N(dt,dx), \quad t,T \in [0,T^*]. \qquad (3.4)$$

The coefficients are assumed to be predictable and satisfy the following integrability conditions

$$\int_0^{T^*}\int_0^{T^*} \mid \alpha(t,T) \mid dTdt < \infty, \ \int_0^{T^*}\int_0^{T^*} \mid \sigma(t,T) \mid^2 dTdt < \infty,$$

$$\int_0^{T^*}\int_0^{T^*}\int_{\mathbb{R}} \mid \gamma(t,x,T) \mid \nu(dx)dTdt < \infty,$$

where all the inequalities above hold P-a.s.. We put

$$\alpha(t,T) = 0, \quad \sigma(t,T) = 0, \quad \gamma(t,x,T) = 0 \quad \text{for} \quad t > T \quad \forall x \in \mathbb{R}. \qquad (3.5)$$

The value at time t of a bond paying 1 at maturity $T \in [0,T^*]$ is defined by

$$P(t,T) := e^{-\int_t^T f(t,s)ds}. \qquad (3.6)$$

The short rate is defined by $r(t) := f(t,t)$ and thus evolution of the money in the savings account, given by

$$dB(t) = r(t)B(t)dt,$$

is determined by the model. Notice that condition (3.5) implies equality $f(t,T) = f(t,t)$ for $t > T$. This corresponds to the fact that the holder of a bond transfers his money automatically to the bank account after the bond's expiration date. The discounted value of a bond $\hat{P}(t,T) = B(t)^{-1}P(t,T)$ is thus given by

$$\hat{P}(t,T) = P(t,T)\, e^{-\int_0^t r(s)ds} = e^{-\int_t^T f(t,s)ds} \cdot e^{-\int_0^t f(t,s)ds} = e^{-\int_0^T f(t,s)ds}.$$

Putting

$$A(t,T) := -\int_t^T \alpha(t,s)ds$$
$$S(t,T) := -\int_t^T \sigma(t,s)ds$$
$$G(t,x,T) := -\int_t^T \gamma(t,x,s)ds$$

one can check that P satisfies the following equation (see Proposition 2.2. in [4]):

$$dP(t,T) = P(t-,T)\bigg(\Big(r(t) + A(t,T) + \frac{1}{2} \mid S(t,T) \mid^2 \Big)dt + S(t,T)dW(t) + \int_{\mathbb{R}} \Big(e^{G(t,x,T)} - 1\Big)N(dt,dx)\bigg). \tag{3.7}$$

As a consequence of (3.7) and definition of $\hat{P}$ we obtain

$$d\hat{P}(t,T) = \hat{P}(t-,T)\bigg(\Big(A(t,T) + \frac{1}{2} \mid S(t,T) \mid^2 \Big)dt + S(t,T)dW(t) + \int_{\mathbb{R}} \Big(e^{G(t,x,T)} - 1\Big)N(dt,dx)\bigg).$$

As in the case of stock market we are interested in the existence of a martingale measure for the discounted prices. A measure Q is a martingale measure if the process $\hat{P}(\cdot,T)$ is a local martingale with respect to Q for each $T \in [0,T^*]$. The set of all martingale measures is denoted by $\mathcal{Q}$. The set $\mathcal{Q}$ is not empty if the model satisfies the HJM-type conditions, that is if coefficients in (3.4) are related in a special way. For more details see Theorem 3.13 in [4]. Throughout all the paper we assume that the objective measure P is at the same time a martingale one. This assumption allows us to write the following equation for $\hat{P}$, see Proposition 3.14 in [4]:

$$d\hat{P}(t,T) = \hat{P}(t-,T)\bigg(S(t,T)dW(t) + \int_{\mathbb{R}} (e^{G(t,x,T)} - 1)\tilde{N}(dt,dx)\bigg). \tag{3.8}$$

Now, let us fix a set J which is assumed to be a dense, countable subset of $[0,T^*]$. We assume that only bonds with maturities in J are traded, i.e. only they can be used for the portfolio construction. At the beginning we should precise a portfolio definition. Below it is shown a motivation for the form of the portfolio processes used in the sequel.

Notice that $P(t) = P(t,\cdot)$, given by (3.6), is a continuous function, so restricted to J it is a bounded sequence. The space

$$B = \Big\{z = (z_1, z_2, ...) : \sup_i \mid z_i \mid < \infty\Big\}$$

with the norm $\|z\|_B = \sup_i \mid z_i \mid$ is thus the state space for the bond prices. In the classical case of stock markets with the price process in $\mathbb{R}^d$, where $d < \infty$, it is clear that the space of portfolios can be identified with the dual space $(\mathbb{R}^d)^* = \mathbb{R}^d$. This approach is being generalized in the context of bond markets with infinite dimensional price process. For example in [4] and [3] the price process takes values in $C_0[0,\infty)$ - the space of continuous functions converging to zero in infinity. The space of portfolios is thus $C_0^*[0,\infty)$ - a space of measures with finite total variation. In our model treating B^* as a state space for portfolios does not seem to be justified. The reason is that the dual space is to large and contains abstract elements with a doubtful financial interpretation, for example generalized Banach limits. The portfolio space should be chosen in such a way to be closer to practical aspects of trading. In practice the trader's portfolio can consists of finite number of bonds only, so the portfolio can be of the form

$$\varphi = (\varphi(T_{i_1}), \varphi(T_{i_2}), ..., \varphi(T_{i_n})); \qquad T_{i_j} \in J,\ j = 1, 2, ..., n;\ n \in \mathbb{N}.$$

Since the number of bonds held by a trader can be arbitrary large, we also allow the portfolio to contain infinite number of bonds but such that the value of the investment is finite. Since the bond prices are bounded it is thus natural to assume that the portfolio satisfies

$$\varphi = \{\varphi(T_j)\}_{j=1}^{\infty}; \quad \sum_{j=1}^{\infty} \mid \varphi(T_j) \mid < \infty.$$

Concluding, we choose $l^1 \subset B^*$ as the portfolio space. The value of the investment is a value of the functional φ on the element $P \in B$ and is denoted by

$$< \varphi, P >_{B^*,B} := \sum_{i=1}^{\infty} \varphi(T_i) P(T_i).$$

By trading strategy we mean any predictable process $\{\varphi(t);\ t \in [0, T^*]\}$ taking values in l^1. Besides investing in bonds one can also save money in a savings account. The wealth process at time t is thus given by

$$X(t) = b(t) \cdot B(t) + < \varphi(t), P(t) >_{B^*,B} \qquad t \in [0, T^*], \tag{3.9}$$

where $b(t)$, $\varphi(t)$ correspond to money saved in a bank and investing in bonds respectively. Here we use the notation $P(t)$ for $\{P(t,T); T \in J\}$ since the latter is treated as an element of the Banach space B. This notation will be used with respect to other processes too.

As usual, the wealth process should be self-financing, so the additional requirement is supposed to hold

$$dX(t) = b(t) dB(t) + < \varphi(t), dP(t) >_{B^*,B} \qquad t \in [0, T^*]. \tag{3.10}$$

Notice that applying the integration by parts formula to the process $\hat{X}(t) = B(t)^{-1} X(t)$ and using (3.9), (3.10) we obtain

$$\begin{aligned} d\hat{X}(t) =& B(t)^{-1}\Big(b(t)dB(t) + < \varphi(t), dP(t) > \Big) - \Big(b(t)B(t) + < \varphi(t), P(t) > \Big) B(t)^{-2} dB(t) \\ =& < \varphi(t), B(t)^{-1} dP(t) - P(t) B(t)^{-2} dB(t) > \\ =& < \varphi(t), d\hat{P}(t) >_{B^*,B} . \end{aligned}$$

Taking (3.8) into account we can give a precise meaning for the the integral $\int < \varphi(t), d\hat{P}(t) >_{B^*,B}$.

Definition 3.1 *A process φ taking values in l^1 is $\hat{P}$ integrable if it is predictable and satisfies the following conditions*

$$< \varphi(s), \hat{P}(s-)S(s) >_{B^*,B} \in \Phi, \qquad < \varphi(s), \hat{P}(s-)(e^{G(s,x)} - 1) >_{B^*,B} \in \Psi_{1,2}. \tag{3.11}$$

If (3.11) *holds we set:*

$$\begin{aligned} \int_0^t < \varphi(s), d\hat{P}(s) >_{B^*,B} := & \int_0^t < \varphi(s), \hat{P}(s-)S(s) >_{B^*,B} dW(s) \\ & + \int_0^t \int_{\mathbb{R}} < \varphi(s), \hat{P}(s-)(e^{G(s,x)} - 1) >_{B^*,B} \tilde{N}(ds, dx); \quad t \in [0, T^*]. \end{aligned} \tag{3.12}$$

Let us notice that integrands on the right hand side of (3.12) are well defined since $\hat{P}(s-) = \hat{P}(s-, \cdot)$ is a continuous function on $[0, T^*]$. Indeed, let L be the Lévy process corresponding to the jump measure N. Due to (3.8) we obtain $\Delta\hat{P}(t,T) = \hat{P}(t-,T)(e^{G(t,\Delta L(t),T)} - 1)$ and putting this value to the equality $\hat{P}(t,T) = \hat{P}(t-,T) + \Delta\hat{P}(t,T)$ we obtain $\hat{P}(t-,T) = \frac{\hat{P}(t,T)}{e^{G(t,\Delta L(t),T)}}$. The last function is continuous with respect to T. As a consequence, we have

$$\hat{P}(t-)S(t) \in B, \qquad \hat{P}(t-)(e^{G(t,x)} - 1) \in B \qquad \forall t \in [0, T^*] \quad \forall x \in \mathbb{R}.$$

Summarizing, the wealth process can be identified with its discounted value through a pair (x, φ) s.t.

$$\hat{X}(t) = x + \int_0^t < \varphi(s), d\hat{P}(s) >_{B^*,B} .$$

4 Completeness

We start this section with a definition of admissible strategies - a class of strategies involved in the definition of the market completeness.

Definition 4.1 *Assume that a process φ taking values in l^1 is $\hat{P}$ integrable. Then φ is an admissible strategy if the (discounted) wealth process*

$$\int_0^{\cdot} < \varphi(s), d\hat{P}(s) >_{B^*,B}$$

is a martingale. The class of all admissible strategies will be denoted by $\mathcal{A}$.

The definition of admissible strategies which imposes martingale property on the wealth process is often considered in literature, see for example [9].

Definition 4.2 *Let A be a subset in the set of all $\mathcal{F}_{T^*}$ measurable random variables. The market is A-complete if for each $X \in A$ there exists a strategy $\varphi \in \mathcal{A}$ which satisfies condition*

$$X = x + \int_0^{T^*} < \varphi(t), d\hat{P}(t) >_{B^*,B}, \tag{4.13}$$

for some $x \in \mathbb{R}$. If there exists $X \in A$ s.t. condition (4.13) *does not hold, then the market is not A-complete. If the random variable X satisfies* (4.13) *then we say that X can be replicated.*

Lemma 4.3 *Let $\varphi \in \mathcal{A}$, $\phi \in \Phi$, $\psi \in \Psi_{1,2}$. Assume that the proces*

$$\int_0^{\cdot} \phi(s)dW(s) + \int_0^{\cdot} \int_{\mathbb{R}} \psi(s,x)\tilde{N}(ds,dx) \tag{4.14}$$

is a martingale. If the equality

$$x + \int_0^{T^*} < \varphi(s), d\hat{P}(s) >_{B^*,B} = y + \int_0^{T^*} \phi(s)dW(s) + \int_0^{T^*} \int_{\mathbb{R}} \psi(s,x)\tilde{N}(ds,dx) \tag{4.15}$$

holds for some $x, y \in \mathbb{R}$ then $x = y$ and

$$\phi(s) =< \varphi(s), \hat{P}(s-)S(s) >_{B^*,B} \qquad P \otimes \lambda \quad a.s., \tag{4.16}$$

$$\psi(s,x) =< \varphi(s), \hat{P}(s-)(e^{G(s,x)} - 1) >_{B^*,B} \qquad P \otimes \lambda \otimes \nu \quad a.s.. \tag{4.17}$$

Proof. Taking expectations in (4.15) we obtain $x = y$. The process

$$\begin{aligned} M_t :&= \int_0^t < \varphi(s), d\hat{P}(s) >_{B^*,B} - \int_0^t \phi(s)dW(s) - \int_0^t \int_{\mathbb{R}} \psi(s,x)\tilde{N}(ds,dx) \\ &= \int_0^t \Big(< \varphi(s), \hat{P}(s-)S(s) >_{B^*,B} - \phi(s) \Big) dW(s) \\ &+ \int_0^t \int_{\mathbb{R}} \Big(< \varphi(s), \hat{P}(s-)(e^{G(s,x)} - 1) >_{B^*,B} - \psi(s,x) \Big) \tilde{N}(ds,dx) \end{aligned}$$

is thus a martingale equal to zero. With the use of Theorem (2.1) we obtain (4.16) and (4.17). □

The fact of considering a specific class of admissible strategies in the completeness problem is crucial in our approach. If we are looking for a replicating strategy for a given integrable random variable X in the class $\mathcal{A}$ then we can identify X with a martingale $\mathbf{E}[X \mid \mathcal{F}_t]$. On the other hand, in view of the decomposition

$$\mathbf{E}[X \mid \mathcal{F}_t] = \mathbf{E}X + \int_0^t \phi_X(s)dW(s) + \int_0^t \int_{\mathbb{R}} \psi_X(s,x)\tilde{N}(ds,dx), \tag{4.18}$$

and Theorem (2.1) this martingale is uniquely determined by the processes ϕ_X, ψ_X. Thus X itself can be identified with the integrands ϕ_X, ψ_X. In virtue of Lemma (4.3) if there exists $\varphi_X \in \mathcal{A}$ satisfying (4.16) and (4.17) with $\phi = \phi_X, \psi = \psi_X$ then φ_X is a replicating strategy for X. As a consequence, if (4.16) and (4.17) are not satisfied for any $\varphi \in \mathcal{A}$ then X can not be replicated.

Remark 4.4 *If we do not impose any restrictions on the class of strategies or only forbid the wealth process to take negative values then X can not be uniquely identified with the integrands ϕ_X, ψ_X given by* (4.18). *An example of two different integrands such that after integrating with respect to the Wiener process give the same bounded random variable can be found in [1], ex.3.10.*

Our method of examining conditions (4.16), (4.17) is based on the following lemma which is an extension of the moment problem solution, see Yosida [12].

Lemma 4.5 *Let $\mathbf{E}$ be a normed linear space and $\mathbf{U}$ an arbitrary set. Let $g : \mathbf{U} \longrightarrow \mathbb{R}$ and $h : \mathbf{U} \longrightarrow \mathbf{E}$. Then there exists $e^* \in E^*$ such that*

$$g(u) =< e^*, h(u) >_{E^*,E} \quad \forall u \in \mathbf{U} \tag{4.19}$$

if and only if

$$\exists\, \gamma > 0 \quad \forall\, n \in \mathbb{N} \quad \forall\, \{\beta_i\}_{i=1}^n,\ \beta_i \in \mathbb{R} \quad \forall\, \{u_i\}_{i=1}^n,\ u_i \in \mathbf{U} \quad holds:$$

$$\Big| \sum_{i=1}^n \beta_i g(u_i) \Big| \leq \gamma \, \Big\| \sum_{i=1}^n \beta_i h(u_i) \Big\|_E. \tag{4.20}$$

Proof. Necessity is obvious, (4.20) holds with $\gamma = \|e^*\|_{E^*}$. To prove sufficiency let us define a linear subspace of $\mathbf{E}$ as follows

$$\mathbf{M} = \Big\{ e \in \mathbf{E} : e = \sum_{i=1}^n \beta_i h(u_i); \quad n \in \mathbb{N},\ \beta_i \in \mathbb{R},\ u_i \in \mathbf{U} \Big\}$$

and a linear transformation $\tilde{e}^* : \mathbf{M} \longrightarrow \mathbb{R}$ by the formula

$$\tilde{e}^* \Big(\sum_{i=1}^n \beta_i h(u_i) \Big) = \sum_{i=1}^n \beta_i g(u_i).$$

Notice, that for $e_1 = \sum_{i=1}^n \beta_i h(u_i)$ and $e_2 = \sum_{j=1}^m \beta_j' h(u_j)$ by (4.20) we obtain

$$\begin{aligned} \Big| \tilde{e}^*(e_1) - \tilde{e}^*(e_2) \Big| &= \Big| \sum_{i=1}^n \beta_i g(u_i) - \sum_{j=1}^m \beta_j' g(u_j) \Big| \\ &\leq \gamma \Big\| \sum_{i=1}^n \beta_i h(u_i) - \sum_{j=1}^m \beta_j' h(u_j) \Big\|_E = \gamma \|e_1 - e_2\|. \end{aligned}$$

If $e_1 = e_2$ then $\tilde{e}^*(e_1) = \tilde{e}^*(e_2)$, so this transformation is well defined, because its value does not depend on the representation. It is also continuous and thus by the Hahn-Banach theorem it can be extended to the functional $e^* \in E^*$ which clearly satisfies (4.19). □

In the sequel we use the following proposition which simplifies examining conditions (4.16) and (4.17).

Proposition 4.6 *Let $(E_1, \mathcal{E}_1, \mu_1)$, $(E_2, \mathcal{E}_2, \mu_2)$ be measurable spaces with sigma-finite measures μ_1, μ_2 and $(E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2, \mu_1 \otimes \mu_2)$ be their product space. If two measurable functions $f_1 : E_1 \times E_2 \longrightarrow \mathbb{R}$, $f_2 : E_1 \times E_2 \longrightarrow \mathbb{R}$ satisfy condition*

$$f_1 = f_2 \qquad \mu_1 \otimes \mu_2 \quad a.s. \tag{4.21}$$

then there exists a set $\hat{E}_1 \in \mathcal{E}_1$ such that

$$\hat{E}_1 \quad is\ of\ full\ \mu_1\ measure \tag{4.22}$$

$$\forall x \in \hat{E}_1 \quad the\ set \quad \{y : f_1(x,y) = f_2(x,y)\} \quad is\ of\ full\ \mu_2\ measure. \tag{4.23}$$

Proof. The assertion follows from the Fubini theorem applied to the function $h = \mathbf{1}_A$ where $A := \{(x,y) \in E_1 \times E_2 : f_1(x,y) \neq f_2(x,y)\}$.

□

4.1 Lévy measure with a finite support

In this section we assume that the support of the Lévy measure consists of finite number of points: $x_1, x_2, ..., x_n$.
We start with an auxiliary lemma on linear independence of infinite sequences. For the convenience of the reader we provide its proof.

Lemma 4.7 *Let M be an infinite matrix of the form*

$$M = \begin{pmatrix} z^1 \\ z^2 \\ \vdots \\ z^n \end{pmatrix} = \begin{bmatrix} z_1^1 & z_2^1 & z_3^1 & \cdots \\ z_1^2 & z_2^2 & z_3^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ z_1^n & z_2^n & z_3^n & \cdots \end{bmatrix},$$

with linearly independent rows $z^1, z^2, ..., z^n$. Then there exists a set of n linearly independent columns of the matrix M.

Proof. We will show that for some natural number m the following finite vectors

$$z^k(m) := z_1^k, z_2^k, ..., z_m^k; \qquad k = 1, 2, ..., n$$

are linearly independent. Assume, to the contrary, that for each m there exist numbers $\alpha^1(m)$, $\alpha^2(m)$,..., $\alpha^n(m)$ such that $\sum_{k=1}^n \mid \alpha^k(m) \mid > 0$ and

$$\sum_{k=1}^n \alpha^k(m) z^k(m) = 0. \tag{4.24}$$

Without a loss of generality we can assume that

$$\sum_{k=1}^n \mid \alpha^k(m) \mid = 1, \qquad \forall m = 1, 2, ...$$

Then there exists a subsequence $m_l \to \infty$ such that

$$\alpha^k(m_l) \longrightarrow \bar{\alpha}^k, \qquad k = 1, 2, ..., n$$

and $\sum_{k=1}^n \mid \bar{\alpha}^k \mid = 1$. From 4.24, for each l, we have

$$\sum_{k=1}^n \alpha^k(m_l) z^k(m_l) = 0.$$

Thus, for each $\bar{m} \leq m_l$,

$$\sum_{k=1}^n \alpha^k(m_l) z^k(\bar{m}) = 0.$$

Consequently

$$\sum_{k=1}^n \bar{\alpha}^k z^k(\bar{m}) = 0, \qquad \forall \bar{m} = 1, 2, ... \,.$$

Therefore we arrive at a contradiction. $\square$

Theorem 4.8 *Let us assume that the following vectors in the space B:*

$$S(t),\ e^{G(t,x_1)} - 1,\ e^{G(t,x_2)} - 1,\ ...\ , e^{G(t,x_n)} - 1 \tag{4.25}$$

are linearly independent $P \otimes \lambda$ a.s.. Then the market is L^1-complete. Moreover, for each $X \in L^1$ there exists a replicating strategy consisting of $n+1$ bonds with different maturities.

Proof. In virtue of Lemma 4.7 one can find maturities $T_{i_1}, T_{i_2}, ..., T_{i_{n+1}} \in J$ such that vectors

$$\begin{pmatrix} S(t, T_{i_j}) \\ e^{G(t,x_1,T_{i_j})} - 1 \\ \vdots \\ e^{G(t,x_n,T_{i_j})} - 1 \end{pmatrix}, j = 1, 2, ..., n+1; \tag{4.26}$$

form a set of linearly independent vectors in $\mathbb{R}^{n+1}$. Consider any $X \in L^1$ and the representation of the process $\mathbf{E}[X \mid \mathcal{F}_t]$ given by Theorem 2.1

$$\mathbf{E}[X \mid \mathcal{F}_t] = \mathbf{E}X + \int_0^t \phi_X(s) dW(s) + \int_0^t \int_{\mathbb{R}} \psi_X(s,x) \tilde{N}(ds, dx). \tag{4.27}$$

Let us define a strategy $\varphi_X(t, T_{i_j}); j = 1, 2, ..., n+1$ involving only bonds with maturities $T_{i_1}, T_{i_2}, ..., T_{i_{n+1}}$ as a solution of the following system of linear equations

$$\begin{bmatrix} S(t,T_{i_1}) & ... & S(t,T_{i_{n+1}}) \\ e^{G(t,x_1,T_{i_1})} - 1 & ... & e^{G(t,x_1,T_{i_{n+1}})} - 1 \\ \vdots & & \vdots \\ e^{G(t,x_n,T_{i_1})} - 1 & ... & e^{G(t,x_n,T_{i_{n+1}})} - 1 \end{bmatrix} \begin{bmatrix} \hat{P}(t-, T_{i_1}) \cdot \varphi_X(t, T_{i_1}) \\ \hat{P}(t-, T_{i_2}) \cdot \varphi_X(t, T_{i_2}) \\ \vdots \\ \hat{P}(t-, T_{i_{n+1}}) \cdot \varphi_X(t, T_{i_{n+1}}) \end{bmatrix} = \begin{bmatrix} \phi_X(t) \\ \psi_X(t, x_1) \\ \vdots \\ \psi_X(t, x_n) \end{bmatrix}$$

The strategy is well defined because the matrix above is nonsingular. Moreover, φ_X is a replicating strategy for X. Indeed, we have

$$\begin{aligned}
X &= x_0 + \int_0^{T^*} \sum_{j=1}^{n+1} \hat{P}(t-,T_{i_j})S(t,T_{i_j})\varphi_X(t,T_{i_j})dW(t) \\
&\quad + \int_0^{T^*} \int_{\mathbb{R}} \sum_{j=1}^{n+1} \hat{P}(t-,T_{i_j})(e^{G(t,x,T_{i_j})}-1)\varphi_X(t,T_{i_j})\tilde{N}(dt,dx) \\
&= x_0 + \int_0^{T^*} < \varphi_X(t), d\hat{P}(t) >_{B^*,B} .
\end{aligned}$$

□

Remark 4.9 *Theorem 4.8 shows that the assumptions of Theorem 5.6. in [4] can be weakened. Indeed, due to Lemma 4.7 the problem is reduced to the system of linear equations with nonsingular matrix and thus additional assumption imposed on coefficients $\sigma(t,\cdot)$, $\gamma(t,\cdot)$ to be analytic functions can be relaxed.*

4.2 Lévy measure with a concentration point

We start examining the completeness problem in a more general setting by introducing the following property of the Lévy measure.

Definition 4.10 *The point $x_0 \in \mathbb{R}$ is a concentration point of the measure ν if there exists a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ s.t. $\varepsilon_n \searrow 0$ satisfying*

$$\nu\Big\{B(x_0,\varepsilon_n)\backslash B(x_0,\varepsilon_{n+1})\Big\} > 0 \quad \forall\, n = 1,2,... \tag{4.28}$$

where $B(x_0,\varepsilon) = \{x \in \mathbb{R} : \mid x - x_0 \mid \leq \varepsilon\}$.

Let us notice that the condition formulated in Definition 4.10 is very often satisfied. For example, every Lévy measure with a density has a concentration point. Thus the following theorem covers a large class of models.

Theorem 4.11 *Assume that the Lévy measure ν has a concentration point $x_0 \neq 0$. If $\gamma(t,\cdot,T)$ is differentiable for each $t \in [0,T^*]$, $T \in [0,T^*]$ and the following condition is satisfied*

$$\forall t \in [0,T^*] \quad \exists \delta = \delta(t) > 0 \quad s.t. \quad \int_t^{T^*} \sup_{x \in B(x_0,\delta)} \mid \gamma_x^{'}(t,x,s) \mid ds < \infty \tag{4.29}$$

then the bond market is not L^{∞}-complete.

Proof. We will construct a bounded random variable X which can not be represented in the form (4.13) for any strategy $\varphi \in \mathcal{A}$. At the beginning we construct an auxiliary function ψ such

that there is no $\hat{P}$ integrable process φ satisfying condition (4.17).
Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence satisfying (4.28) and define a deterministic function ψ by the formula

$$\psi(x) = \begin{cases} \mid x \mid \wedge 1 & \text{for} \quad x \in \{B(x_0, \varepsilon_{2k+1}) \backslash B(x_0, \varepsilon_{2k+2})\} \quad k = 0, 1, ... \\ \\ -(\mid x \mid \wedge 1) & \text{for} \quad x \in \{B(x_0, \varepsilon_{2k}) \backslash B(x_0, \varepsilon_{2k+1})\} \quad k = 1, 2, ... \\ \\ \mid x \mid \wedge 1 & \text{for} \quad x \in (-\infty, x_0 - \varepsilon_1) \cup (x_0 + \varepsilon_1) \cup \{x_0\}. \end{cases}$$

We will show that condition (4.17) is not satisfied by any $\hat{P}$ integrable process φ. Let us fix any pair $(\omega, t) \in \Omega \times [0, T^*]$ and assume that equality

$$< \varphi(t), \hat{P}(t-)(e^{G(t,x)} - 1) >_{B^*,B} = \psi(x) \tag{4.30}$$

holds ν a.s.. Thus there exists a set $A_\nu(\omega, t)$ of a full ν measure s.t. equality (4.30) is satisfied for each $x \in A_\nu(\omega, t)$. Due to Lemma 4.5 there exists $\gamma = \gamma(\omega, t) > 0$ such that

$$\forall\, n \in \mathbb{N} \quad \forall\, \{\beta_i\}_{i=1}^{n},\ \beta_i \in \mathbb{R} \quad \forall\, \{x_i\}_{i=1}^{n},\ x_i \in A_\nu(\omega, t)$$

$$\Big| \sum_{i=1}^{n} \beta_i \psi(x_i) \Big| \leq \gamma \Big\| \sum_{i=1}^{n} \beta_i \hat{P}(t-)(e^{G(t,x_i)} - 1) \Big\|_B. \tag{4.31}$$

Let us notice that due to (4.28) we have

$$\nu\Big\{A_\nu(\omega, t) \cap \big\{B(x_0, \varepsilon_n) \backslash B(x_0, \varepsilon_{n+1})\big\}\Big\} > 0$$

so we can choose a sequence $\{a_k\}_{k=1}^{\infty}$ s.t.

$$a_k \in A_\nu(\omega, t) \cap \big\{B(x_0, \varepsilon_k) \backslash B(x_0, \varepsilon_{k+1})\big\} \quad \forall\, k = 1, 2, ...$$

Let us examine the condition (4.31) with $n = 2$, $\beta_1 = 1, \beta_2 = -1$ and $x_1 = a_{2k+1}$, $x_2 = a_{2k+2}$ for $k = 0, 1, ...$ Then the left hand side of (4.31) is of the form

$$\frac{1}{\gamma}\Big|\beta_1 \psi(a_{2k+1}) + \beta_2 \psi(a_{2k+2})\Big| = \frac{1}{\gamma}\Big((\mid a_{2k+1} \mid \wedge 1) + (\mid a_{2k+2} \mid \wedge 1)\Big)$$

and thus satisfies

$$\lim_{k \longrightarrow \infty} \frac{1}{\gamma}\Big|\beta_1 \psi(a_{2k+1}) + \beta_2 \psi(a_{2k+2})\Big| = \frac{2(\mid x_0 \mid \wedge 1)}{\gamma} \neq 0.$$

In estimating of the right hand side of (4.31) we will use the inequality (4.32) and (4.33) below.
In view of (4.29) we have

$$\begin{aligned} \sup_{T \in J} \sup_{x \in B(x_0,\delta)} \mid G(t, x, T) \mid &\leq \sup_{T \in J} \sup_{x \in B(x_0,\delta)} \int_t^T \mid \gamma(t, x, s) \mid ds \\ &\leq \sup_{T \in J} \int_t^T \sup_{x \in B(x_0,\delta)} \mid \gamma(t, x, s) \mid ds \\ &\leq \sup_{T \in J} \int_t^T \Big\{ \mid \gamma(t, x_0, s) \mid + \sup_{x \in B(x_0,\delta)} \mid \gamma_x^{'}(t, x, s) \mid 2\delta \Big\} ds \\ &\leq \int_t^{T^*} \mid \gamma(t, x_0, s) \mid ds + 2\delta \int_t^{T^*} \sup_{x \in B(x_0,\delta)} \mid \gamma_x^{'}(t, x, s) \mid ds < \infty. \end{aligned} \tag{4.32}$$

The condition (4.29) implies differentiability of $G(t,\cdot,T)$ and the following estimation

$$\sup_{T\in J}\sup_{x\in B(x_0,\delta)} \mid G_x^{'}(t,x,T)\mid = \sup_{T\in J}\sup_{x\in B(x_0,\delta)} \mid \int_t^T \gamma_x^{'}(t,x,s)ds \mid$$
$$\leq \int_t^{T^*} \sup_{x\in B(x_0,\delta)} \mid \gamma_x^{'}(t,x,s)\mid ds < \infty. \tag{4.33}$$

The right hand side of (4.31) can be estimated as follows

$$\Big\|\hat{P}(t-)(e^{G(t,a_{2k+1})}-1) - \hat{P}(t-)(e^{G(t,a_{2k+2})}-1)\Big\|_B$$
$$= \sup_{T\in J}\Big|\hat{P}(t-,T)(e^{G(t,a_{2k+1},T)}-1) - \hat{P}(t-,T)(e^{G(t,a_{2k+2},T)}-1)\Big|$$
$$\leq \sup_{T\in J}|\hat{P}(t-,T)| \sup_{T\in J}\Big|e^{G(t,a_{2k+1},T)} - e^{G(t,a_{2k+2},T)}\Big|.$$

The first supremum is finite since $\hat{P}(t-,\cdot)$ is a continuous function. To deal with the second supremum let us notice that for sufficiently large k the points a_{2k+1}, a_{2k+2} are in $B(x_0,\delta)$ and thus we have

$$\sup_{T\in J}\Big|e^{G(t,a_{2k+1},T)} - e^{G(t,a_{2k+2},T)}\Big| \leq \sup_{T\in J}\sup_{x\in B(x_0,\delta)}\Big|\frac{d}{dx}e^{G(t,x,T)}\Big|\cdot \mid a_{2k+1}-a_{2k+2}\mid$$
$$\leq \sup_{T\in J}\sup_{x\in B(x_0,\delta)} e^{|G(t,x,T)|}\cdot \sup_{T\in J}\sup_{x\in B(x_0,\delta)} \mid G_x^{'}(t,x,T)\mid \cdot \mid a_{2k+1}-a_{2k+2}\mid. \tag{4.34}$$

In view of (4.32) and (4.33) we see that the last product in (4.34) goes to 0 when $k\to\infty$.

Thus we conclude that condition (4.31) is not satisfied for any $(\omega,t)\in\Omega\times[0,T^*]$ and thus (4.30) does not hold $\nu - a.s.$ for any $(\omega,t)\in\Omega\times[0,T^*]$. As a consequence of Proposition 4.6 there is no $\hat{P}$ integrable process satisfying (4.17).

Now, with the use of the function ψ, we construct a bounded random variable X which can not be replicated.

It is clear that $\psi\in\Psi_{1,2}$. Let us define the stopping time τ_k by

$$\tau_k = \inf\{t : \Big|\int_0^t\int_{\mathbb{R}}\psi(x)\tilde{N}(ds,dx)\Big| \geq k\}\wedge T^*$$

and choose a number k_0 s.t. the set $\{(\omega,\tau_{k_0}(\omega));\omega\in\Omega\}\subseteq\Omega\times[0,T^*]$ is of positive $P\otimes\lambda$ measure. Then the process $\psi(x)\mathbf{1}_{(0,\tau_{k_0}]}(s)$ is predictable and bounded. The random variable

$$X = \int_0^{T^*}\int_{\mathbb{R}}\psi(x)\mathbf{1}_{(0,\tau_{k_0}]}(s)\tilde{N}(ds,dx) \tag{4.35}$$

is thus well defined and it is also bounded because $|\Delta \int_0^{\cdot} \int_{\mathbb{R}} \psi(x)\tilde{N}(ds,dx)| \leq 1$. For any $(\omega,t) \in \{(\omega,\tau_{k_0}(\omega)); \omega \in \Omega\}$ condition (4.31) is not satisfied ν a.s.. As a consequence of Proposition 4.6 condition (4.17) is not satisfied by any $\hat{P}$ integrable process. Moreover, $\int_0^{\cdot} \int_{\mathbb{R}} \psi(s,x)\tilde{N}(ds,dx)$ is a martingale. As a consequence of Lemma 4.3 there is no admissible strategy which replicates X. □

4.3 Lévy measure with a discrete support

In this section we consider the Lévy measure with a support consisting of infinite number of discrete points denoted by $\{x_i\}_{i=1}^{\infty}$. To exclude the case studied in Section 4.2 we assume that the support has no concentration point, so the sequence satisfies

$$\lim_{i\to\infty} \mid x_i \mid = \infty. \tag{4.36}$$

Let us notice, that in this case the Lévy measure is a sequence of positive numbers $\{\nu(x_i)\}_{i=1}^{\infty}$ which, due to relation $\int_{\mathbb{R}} \mid x \mid \wedge 1 \; \nu(dx) < \infty$, satisfies condition

$$\sum_{i=1}^{\infty} \nu(\{x_i\}) < \infty. \tag{4.37}$$

In the following theorem we show that under additional condition imposed on the coefficient γ we obtain a result on incompleteness.

Theorem 4.12 *Assume that the following set*

$$A = \Big\{(\omega,t) \in \Omega \times [0,T^*] \quad s.t. \quad G(t,x_i,T) \leq 0 \quad \forall T \in [0,T^*] \quad \forall i = 1,2,...\Big\}$$

is of positive $P \otimes \lambda$ measure. Then the market is not L^2-complete.

Proof. We construct a random variable $X \in L^2$ which can not be represented in the form (4.13). At the beginning, using condition (4.37), let us define a sequence $\{\psi(x_i)\}_{i=1}^{\infty}$ which depends neither on ω nor t in the following way

$$\psi(x_i) = \begin{cases} \sqrt{k} & \text{for} \quad i = i_k \\ 0 & \text{for} \quad i \neq i_k, \end{cases} \tag{4.38}$$

where $i_k := \inf \Big\{ i : \nu(x_i) \leq \frac{1}{k^3} \Big\}$. This sequence satisfies the following two conditions

$$\limsup_{i\to\infty} \mid \psi(x_i) \mid = \infty, \tag{4.39}$$

$$\sum_{i=1}^{\infty} \mid \psi(x_i) \mid^2 \nu(\{x_i\}) \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty. \tag{4.40}$$

We show that the representation (4.17) which we write in the form

$$< \varphi(t), \hat{P}(t-)(e^{G(t)x_i} - 1) >_{B^*,B} = \psi(x_i) \quad \forall i = 1,2,... \tag{4.41}$$

does not hold $P \otimes \lambda \otimes \nu$ a.s. for any $\hat{P}$ integrable process φ. Let us fix $(\omega, t) \in A$ and assume to the contrary that (4.41) is satisfied for some $\varphi(t)$. Then by Lemma 4.5 there exists $\gamma = \gamma(\omega, t) > 0$ such that

$$\forall\, n \in \mathbb{N} \quad \forall\, \{\beta_k\}_{k=1}^{n},\ \beta_k \in \mathbb{R} \quad \forall\, \{x_{i_k}\}_{k=1}^{n}$$

$$\Big|\sum_{k=1}^{n} \beta_k \psi(x_{i_k})\Big| \leq \gamma \Big\| \sum_{k=1}^{n} \beta_k \hat{P}(t-)(e^{G(t,x_{i_k})} - 1)\Big\|_B. \tag{4.42}$$

Let us check (4.42) with $n = 1$, $\beta_1 = 1$ and for $i_1 = 1, 2, ...$ successively, that is

$$\Big|\psi(x_i)\Big| \leq \gamma \sup_{T \in J} \Big|\hat{P}(t-, T)(e^{G(t,x_i,T)} - 1)\Big| \qquad \forall i = 1, 2, ... \tag{4.43}$$

By the definition of the set A for any $i = 1, 2, ...$ we have

$$\mid e^{G(t,x_i,T)} - 1 \mid \leq 1 \quad \forall\, T \in J.$$

Using inequality

$$\sup_{T \in J} \Big|\hat{P}(t-, T)(e^{G(t,x_i,T)} - 1)\Big| \leq \sup_{T \in J} \Big|\hat{P}(t, T)\Big| \cdot \sup_{T \in J} \Big|e^{G(t,x_i,T)} - 1\Big|$$

and the fact that $\hat{P}(t, \cdot)$ is continuous we see that

$$\limsup_{i \to \infty} \sup_{T \in J} \Big|\hat{P}(t-, T)(e^{G(t,x_i,T)} - 1)\Big| < \infty.$$

However, recall that the left hand side of (4.43) satisfies (4.39), so the required constant γ does not exist. We have shown that for any $(\omega, t) \in A$ the representation (4.41) does not hold. But $P \otimes \lambda(A) > 0$, so in view of Proposition 4.6, the representation (4.41) does not hold $P \otimes \lambda \otimes \nu$ a.s. for any $\hat{P}$ integrable process.

In view of (4.40) we see that $\psi \in \Psi_{1,2}$ and that the process $\int_0^{\cdot} \int_{\mathbb{R}} \psi(x) \tilde{N}(ds, dx)$ is a martingale. Thus with the use of Lemma 4.3 we conclude that the following random variable

$$X := \int_0^{T^*} \int_{\mathbb{R}} \psi(x) \tilde{N}(ds, dx) \tag{4.44}$$

can not be replicated by strategies from the class $\mathcal{A}$. By application isometric formula to X we obtain that X is square integrable. □

The next theorems are based on the behavior of the expression $\| \; G(t, x_i) \; \|_B$ for large i. Since their proofs are similar to those presented earlier, we provide the sketches only.

Theorem 4.13 *If the following condition holds*

$$\liminf_{|x_i| \to \infty} \| \; G(t, x_i) \; \|_B = 0 \qquad P \otimes \lambda - a.s. \tag{4.45}$$

then the market in not L^∞-complete.

Proof. The condition (4.45) implies

$$\liminf_{|x_i|\to\infty} \| e^{G(t,x_i)} - 1 \|_B \leq \lim_{|x_i|\to\infty} e^{\|G(t,x_i)\|_B} - 1 = 0$$

For $\psi(x_i) \equiv 1$ condition (4.43) is thus not satisfied what we can check by calculating $\liminf_i$ for both sides.
The bounded random variable which can not be replicated is constructed in the same way as in the proof of Theorem (4.11), see formula (4.35). $\square$

Theorem 4.14 *If the set*

$$A = \Big\{ (\omega, t) \in \Omega \times [0, T^*] : \exists \alpha = \alpha(\omega, t); 0 < \alpha < \infty \; s.t.$$
$$\lim_{|x_i|\to\infty} \| G(t, x_i) \|_B = \alpha \Big\} \tag{4.46}$$

is of positive $P \otimes \lambda$ measure then the market in not L^2-complete.

Proof. We use ψ constructed in the proof of Theorem 4.12, given by the formula (4.38). Then (4.46) implies that

$$\limsup_{|x_i|\to\infty} \frac{| \psi(x_i) |}{\| e^{G(t,x_i)} - 1 \|_B} = \infty$$

and thus condition (4.43) does not hold. A square integrable random variable which can not be replicated is given by (4.44). $\square$

To study the case when $\| G(t, x_i) \|_B$ tends to infinity we restrict ourselves to the linear form of the coefficient γ, i.e. $\gamma(t, x, T) = \gamma(t, T)x$. This is done to simplify a formulation of the next theorem. Notice that in this case we have $G(t, x, T) = G(t, T)x$.

Theorem 4.15 *Assume that $\gamma(t, x, T) = \gamma(t, T)x$. If there exists a constant $\tilde{G} > 0$ such that the set*

$$A = \Big\{ (\omega, t) \in \Omega \times [0, T^*] : \| G(t, T) \|_B \leq \tilde{G} \Big\}$$

is of positive $P \otimes \lambda$ measure and the Lévy measure has exponential moment of order $2(\tilde{G} + \varepsilon)$ for some $\varepsilon > 0$, i.e.

$$\sum_{i=1}^{\infty} e^{2(\tilde{G}+\varepsilon)|x_i|} \nu(\{x_i\}) < \infty$$

than the market is not L^2-complete.

Proof. Define

$$\psi(x_i) = e^{(\tilde{G}+\varepsilon)|x_i|}, \qquad i = 1, 2, ...$$

For any $(\omega, t) \in A$ condition (4.43) is not satisfied because we have

$$\lim_{i\to\infty} \frac{|\ \psi(x_i)\ |}{\|\ e^{G(t)x_i} - 1\ \|_B} \geq \lim_{i\to\infty} \frac{|\ \psi(x_i)\ |}{|\ e^{\tilde{G}|x_i|} - 1\ |} = \infty.$$

As a consequence the following random variable

$$X := \int_0^{T^*} \int_{\mathbb{R}} \psi(x) \tilde{N}(ds, dx)$$

can not be replicated and it is square integrable because

$$\mathbf{E}(X^2) = \mathbf{E} \int_0^{T^*} \sum_{i=1}^{\infty} e^{2(\tilde{G}+\varepsilon)|x_i|} \nu(\{x_i\}) ds < \infty.$$

$\square$

Remark 4.16 *In this paper we assume that only bonds with maturities in J can be traded and thus we accepted B for the state space. However, if we admit for the portfolio construction all bonds with maturities in $[0, T^*]$ and the state space $C([0, T^*])$ - a space of continuous functions with the supremum norm, then all the results remain true. This is because for any continuous function $h : [0, T^*] \longrightarrow \mathbb{R}$ we have*

$$\|h\|_B = \sup_{T\in J} |\ h(T)\ | = \sup_{T\in[0,T^*]} |\ h(T)\ | = \|h\|_{C([0,T^*])}$$

and thus all the arguments based on the norm in B can be automatically replaced by the norm in $C([0, T^*])$.

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