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OPTIMAL STOPPING OF A RISK PROCESS WITH DISRUPTION AND INTEREST RATES

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Abstract

A problem of determining the optimal stopping time of the risk process as well as the corresponding payoff in the model with disruption and interest rates is examined. The solution is derived in the model which allows for change in distributions of considered random variables (which represent claim amounts and inter-occurrence times between losses) occurring according to some unobservable process. References to previously examined models as well as numerical examples emphasizing the efficiency of the method are provided.
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1. Introduction

The following model has been often investigated in collective risk theory. An insurance company with a given initial capital u_0 receives premiums, which flow at a constant rate $c > 0$, and has to pay for claims which occur according to some point process at times $T_1 < T_2 < ..., \lim T_n = \infty$. The risk process $(U_t)_{t \in \mathbb{R}_+}$ is defined as the difference between the income and the total amount of claims up to time t.

Many of past articles have been concentrating on solving the problem of optimal stopping of the risk process in such models. In his classical work [6], Jensen provided a

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method of finding an optimal stopping time maximizing the expected net gain $\mathbb{E}(U_{\tau^*}) = \sup\{\mathbb{E}U_\tau : \tau \in \mathcal{C}\}$, where $\mathcal{C}$ was a class of feasible stopping times. The reasoning he applied is based on smooth semi-martingale representation of the risk process.

Such approach turned out to be unprofitable when some utility function of the risk process had to be considered. In [4] the authors solved this problem applying dynamic programming methodology and proposed an effective method of determining the optimal stopping times in such situations. Muciek made an effort to adapt this solution to insurance practice introducing in [8] an extension of this model. He investigated the optimal stopping times under the assumption that the capital of the company can be invested and claims can increase at some given interest rates.

Both introduced models proved to be still quite restrictive as they enforced throughout the entire period of observation only one distribution for random variable describing inter-occurrence times between losses and one for the random variable describing amounts of subsequent losses. Although in [7] a solution to the problem of double optimal stopping was presented, the investigated model still did not take into consideration the possibility of disruption. Allowing for such disruption could make the derived stopping rule more interesting in terms of insurance practice, as based on the observed realization of the process and not only on arbitrary management decisions. Hence, Pasternak-Winiarski in [9] introduced a model in which the mentioned distributions changed according to some unobservable random variable and solved the corresponding optimal stopping problem.

The main motivation for the research described in this article was to combine the findings of the models from [8] and [9] and determine the optimal stopping rule in the model allowing for the widest class of investigated processes making the derived stopping rules more versatile and interesting, also in terms of insurance practice.

2. The model and the optimal stopping problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space. On this space we introduce the following random variables and processes:

1) Unobservable random variable κ with values in $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and having geometrical

distribution with parameters $p, \pi_0 \in [0,1]$:

$$P(\kappa = 0) = \pi_0,$$

$$P(\kappa = n) = (1-\pi_0)p(1-p)^{n-1}, n \in \mathbb{N}.$$

2) Claim counting process with jumps at times $0 < T_1 < T_2 < ....$

3) A sequence of random variables $S_n = T_n - T_{n-1}, n = 1, 2, ..., T_0 = 0$. S_n represents the inter-occurrence time between $n-1$th and nth loss. S_n depends on the unobservable random time κ (as in the disorder problem considered in [13]) and is defined as follows:

$$S_n = W_n' \mathbb{I}_{\{n \leq \kappa\}} + W_n'' \mathbb{I}_{\{n > \kappa\}}.$$

$W_n', n \in \mathbb{N}$ is a sequence of i.i.d. random variables with cumulative distribution function (c.d.f.) F_1 (satisfying the condition $F_1(0) = 0$) and density function f_1. Similarly $W_n'', n \in \mathbb{N}$ forms a sequence of i.i.d. random variables with c.d.f. F_2 ($F_2(0) = 0$) and density function f_2. We assume additionally that f_1 and f_2 are commonly bounded by a constant $C \in \mathbb{R}_+$. Furthermore we impose that W_i' and W_j'' are independent for all $i, j \in \mathbb{N}_0$.

4) A sequence $X_n, n \in \mathbb{N}_0$ of random variables representing successive losses. They also depend on the random variable κ:

$$X_n = X_n' \mathbb{I}_{\{n < \kappa\}} + X_n'' \mathbb{I}_{\{n \geq \kappa\}},$$

where $X_n', n \in \mathbb{N}_0$ is a sequence of i.i.d. random variables with c.d.f. H_1 ($H_1(0) = 0$) and density function h_1 whereas $X_n'', n \in \mathbb{N}_0$ forms a sequence of i.i.d. random variables with c.d.f. H_2 ($H_2(0) = 0$) and density function h_2. X_i' and X_j'' are independent for all $i, j \in \mathbb{N}_0$.

We assume that random variables $W_n', W_n'', X_n', X_n'', \kappa$ are independent.

Let $u_0 > 0$ represent the initial capital and $c > 0$ be a constant rate of income from the insurance premium. We take into account the dynamics of the market situation introducing the interest rate at which we can invest accrued capital (constant $\alpha \in [0,1]$). As a consequence of inflation we can observe the growth of claims. In this model we assume that they increase at rate $\beta \in [0,1]$.

As a capital assets model for the insurance company we take the risk process

$$U_t = u_0 e^{\alpha t} + \int_0^t c e^{\alpha(t-s)} ds + \sum_{i=0}^{N(t)} X_i e^{\beta T_i}, X_0 = 0 \qquad (1)$$

The return at time t is defined by the process

$$Z(t) = g_1(U_t)\mathbb{I}_{\{U_s>0,s<t\}}\mathbb{I}_{\{t<t_0\}},$$

where g_1 is a utility function and the constant t_0 is a fixed time which denotes the end of the investment period. For simplicity we define

$$g(u,t) = g_1(u)\mathbb{I}_{\{t\geq 0\}}.$$

Then

$$Z(t) = g(U_t, t_0 - t)\prod_{i=1}^{N(t)}\mathbb{I}_{\{U_{T_i}>0\}}. \tag{2}$$

We fix the number of claims that may occur in our model, K. We will need the following family of $\sigma-$fields generated by all events up to time $t > 0$:

$$\mathcal{F}(t) = \sigma(U_s, s \leq t) = \sigma(X_1, T_1, ..., X_{N(t)}, T_{N(t)})$$

Let us now denote

$$\mathcal{F}_n = \mathcal{F}(T_n) \ \text{ and } \ \mathcal{G}_n = \mathcal{F}_n \vee \sigma(\kappa).$$

In our calculations we will extensively make use of conditional probabilities $\pi_n = P(\kappa \leq n|\mathcal{F}_n), n \in \mathbb{N}_0$ as well as $\theta_n = P(\kappa = n+1|\mathcal{F}_n), n \in \mathbb{N}_0$.

We will now define the optimization problem, which will be solved in subsequent sections. Let $\mathcal{T}$ be the set of all stopping times with respect to the family $\{\mathcal{F}(t)\}_{t>0}$. For $n = 0, 1, 2, ...k < K$ we denote by $\mathcal{T}_{n,K}$ such subsets of $\mathcal{T}$ that satisfy the condition

$$\tau \in \mathcal{T}_{n,K} \Leftrightarrow T_n \leq \tau \leq T_K \text{ a.s.}$$

We will be seeking the optimal stopping time τ^*_K such that

$$\mathbb{E}(Z(\tau^*_K)) = \sup\{\mathbb{E}(Z(\tau)) : \tau \in \mathcal{T}_{0,K}\}.$$

In order to find τ^*_K we first consider optimal stopping times $\tau^*_{n,K}$ such that

$$\mathbb{E}(Z(\tau^*_{n,K})|\mathcal{F}_n) = \text{ ess sup } \{\mathbb{E}(Z(\tau)|\mathcal{F}_n) : \tau \in \mathcal{T}_{n,K}\}.$$

Then using the standard methods of dynamic programming we will obtain $\tau^*_K = \tau^*_{0,K}$.

After finding τ^*_K for fixed κ we concentrate on solving the optimal stopping problem in the situation when an un unlimited number of claims is attainable i.e. we find such τ^* that

$$\mathbb{E}(Z(\tau^*)) = \sup \{\mathbb{E}(Z(\tau)) : \tau \in \mathcal{T}\}.$$

τ^* will be defined as a limit of finite horizon stopping times τ_K^*.

In order to clarify in more detail the structure of the stopping rule which we will derive in this paper we illustrate it briefly below. First, we find a special set of functions $R_i^*(\cdot,\cdot,\cdot)$, $i = 0, ..., K$. Then, at time $T_0 = 0$, with $U_0 = u_0$, we calculate $R_0^*(U_0, T_0, \pi_0)$. If till time $T_0 + R_0^*(U_0, T_0, \pi_0)$ first claim has not yet been observed, we stop. Otherwise, when the first claim occurs at time $T_1 < T_0 + R_0^*(U_0, T_0, \pi_0)$, we calculate the value $R_1^*(U_1, T_1, \pi_1)$ and wait for the next claim till the time $T_1 + R_1^*(U_1, T_1, \pi_1)$, etc. In other words, the optimal stopping times derived in this model, can be interpreted as constituting a threshold rule.

3. Solution for the finite horizon problem

In this section we will find the form of optimal stopping rules in the finite horizon case, i.e. optimal in the class where K is finite and fixed. First, in Theorem 1, we will derive dynamic programming equations satisfied by

$$\Gamma_{n,K} = \text{ ess sup } \{\mathbb{E}(Z(\tau)|\mathcal{F}_n) : \tau \in \mathcal{T}_{n,K}\}.$$

Then, in Theorem 2, we will find optimal stopping times $\tau_{n,K}^*$ and τ_K^* and corresponding optimal conditional mean rewards and optimal mean rewards, respectively.

For notation simplicity we define

$$\mu_{N(t)} = \prod_{i=1}^{N(t)} \mathbb{I}_{\{U_{T_i}>0\}},\ \mu_0 = 1. \tag{3}$$

A simple consequence of these notations and formula (2) is that

$$\Gamma_{K,K} = Z(T_K) = \mu_K g(U_{T_K}, t_0 - T_K). \tag{4}$$

Crucial role in subsequent reasoning is played by two Lemmas which are given below. The first one defines recursive relation between conditional probabilities π_n as well as θ_n, essential in our further considerations. The second one is a representation theorem for stopping times (to be found in [3]).

Lemma 1. *There exist functions* $\xi_1 : [0,1] \times (0,\infty) \times (0,\infty) \to [0,1]$, $\xi_2 : [0,1] \to [0,1]$ *such that*

$$\pi_n = \xi_1(\pi_{n-1}, X_n, S_n),\ \theta_n = \xi_2(\pi_n),$$

and

$$\xi_1(t,x,s) = \frac{f_2(s)h_2(x)t + pf_1(s)h_2(x)(1-t)}{f_2(s)h_2(x)t + pf_1(s)h_2(x)(1-t) + f_1(s)h_1(x)(1-p)(1-t)},$$

$$\xi_2(t) = \frac{\eta_1(t) + \eta_3(t)t - (1+\eta_1(t))t}{1+\eta_1(t) - \eta_2(t)}$$

for

$$\eta_1(t) = \int_0^\infty \int_0^\infty \xi_1(t,x,w)dH_1(x)dF_1(w),$$

$$\eta_2(t) = \int_0^\infty \int_0^\infty \xi_1(t,x,w)dH_2(x)dF_1(w),$$

$$\eta_3(t) = \int_0^\infty \int_0^\infty \xi_1(t,x,w)dH_2(x)dF_2(w).$$

Proof. The nature of the proof is purely technical so it will be omitted here. The reasoning is based upon the Bayes formula and is similar to the one presented in [13].

Lemma 2. *If $\tau \in \mathcal{T}_{n,K}$, then there exists a positive $\mathcal{F}_n$ - measurable random variable R_n such that* $\min(\tau, T_{n+1}) = \min(T_n + R_n, T_{n+1})$.

For simplicity of further calculations we will also introduce functions

$$d : [0,\infty) \times [0,\infty) \to [0,\infty),$$

$$D : [0,\infty) \times [0,\infty) \times [0,\infty) \to (-\infty,\infty) \text{ and}$$

$$\hat{D} : [0,\infty) \times [0,\infty) \times [-\infty,\infty) \to (-\infty,\infty)$$

defined as follows

$$d(t,r) := \left(\frac{c}{\alpha} + u_0\right)\left(e^{\alpha(t+r)} - e^{\alpha t}\right),$$

$$D(t,r,x) := d(t,r) - xe^{\beta(t+r)},$$

$$\hat{D}(t,r,u) := e^{-\beta(t+r)}(u + d(t,r)).$$

Theorem 1. *(a) For $n = K-1, K-2, ..., 0$ we have*

$$\Gamma_{n,K} = ess\ sup\{\mu_n g(U_{T_n} + d(T_n,R_n)), t_0 - T_n - R_n)(\overline{F_2}(R_n)\pi_n + \overline{F_1}(R_n)(1-\pi_n))$$
$$+\mathbb{E}(\mathbb{I}_{\{R_n \ge S_{n+1}\}}\Gamma_{n+1,K}|\mathcal{F}_n) :\ R_n \ge 0 \textit{ and } R_n \textit{ is } \mathcal{F}_n\textit{-measurable}\}, \tag{5}$$

where $\overline{F} = 1 - F$

(b) For $n = K, K-1, ..., 0$ *we have*

$$\Gamma_{n,K} = \mu_n \gamma_{K-n}(U_{T_n}, T_n, \pi_n) \ a.s. \ , \tag{6}$$

where the sequence of functions $\{\gamma_j : \mathbb{R} \times [0,\infty) \times [0,1] \to \mathbb{R}\}_{j=0,\dots,K}$ *is defined recursively as follows:*

$$\gamma_0(u,t,\pi) = g(u, t_0 - t)$$

$$\gamma_j(u,t,\pi) = \sup_{r \geq 0}\{g(u + d(t,r), t_0 - t - r)(\overline{F_2}(r)\pi + \overline{F_1}(r)(1-\pi))$$

$$+\xi_2(\pi)\int_0^r \int_0^{\hat{D}(t,w,u)} \gamma_{j-1}(u + D(t,w,x), t+w, \xi_1(\pi,x,w))dH_2(x)dF_1(w)$$

$$+(1-\pi-\xi_2(\pi))\int_0^r \int_0^{\hat{D}(t,w,u)} \gamma_{j-1}(u + D(t,w,x), t+w, \xi_1(\pi,x,w))dH_1(x)dF_1(w)$$

$$+\pi\int_0^r \int_0^{\hat{D}(t,w,u)} \gamma_{j-1}(u + D(t,w,x), t+w, \xi_1(\pi,x,w))dH_2(x)dF_2(w)\}.$$

where ξ_1, ξ_2 *are the functions defined in Lemma 1.*

Proof. (a) Let $\tau \in \mathcal{T}_{n,K}$ and $0 \leq n < K < \infty$. Lemma 2 implies that

$$A_n := \{\tau < T_{n+1}\} = \{T_n + R_n < T_{n+1}\} = \{R_n < S_{n+1}\} =$$

$$\left(\left\{R_n < W'_{n+1}\right\} \cap \{\kappa > n\}\right) \cup \left(\left\{R_n < W''_{n+1}\right\} \cap \{\kappa \leq n\}\right) = A_n^1 \cup A_n^2.$$

Thus,

$$\mathbb{E}(Z(\tau)|\mathcal{F}_n) = \mathbb{E}(Z(\tau)\mathbb{I}_{A_n^1}|\mathcal{F}_n) + \mathbb{E}(Z(\tau)\mathbb{I}_{A_n^2}|\mathcal{F}_n) + \mathbb{E}(Z(\tau)\mathbb{I}_{\overline{A}_n}|\mathcal{F}_n) = a_n^1 + a_n^2 + b_n. \tag{7}$$

We will now calculate a_n^1. First, we transform the given form of $Z(\tau)$ using (2) and (3)

$$a_n^1 = \mu_n \mathbb{E}(\mathbb{I}_{\left\{R_n < W'_{n+1}\right\}}\mathbb{I}_{\{\kappa > n\}} g(U_\tau, t_0 - \tau)|\mathcal{F}_n).$$

As $R_n < W'_{n+1} = S_{n+1}$ it is obvious that since T_n till τ no loss had been observed. Therefore, as $U_{t+R_n} - U(t) = d(t, R_n)$ we can rewrite U_τ as follows

$$a_n^1 = \mu_n \mathbb{E}(\mathbb{I}_{\left\{R_n < W'_{n+1}\right\}}\mathbb{I}_{\{n < \kappa\}} g(U_{T_n} + d(T_n, R_n), t_0 - T_n - R_n)|\mathcal{F}_n).$$

$$= \mu_n g(U_{T_n} + d(T_n, R_n), t_0 - T_n - R_n)\mathbb{E}(\mathbb{I}_{\{R_n < W'_{n+1}\}})\mathbb{E}(\mathbb{I}_{\{\kappa > n\}}|\mathcal{F}_n)$$

$$= \mu_n g(U_{T_n} + d(T_n, R_n), t_0 - T_n - R_n)\overline{F_1}(R_n)(1 - \pi_n). \quad (8)$$

Similarly one can show that

$$a_n^2 = \mu_n g(U_{T_n} + d(T_n, R_n), t_0 - T_n - R_n)\overline{F_2}(R_n)\pi_n. \quad (9)$$

If we additionally define $\tau' := \max(\tau, T_{n+1})$, then it is easy to see that $\tau' \in \mathcal{T}_{n+1,K}$ and:

$$b_n = \mathbb{E}(Z(\tau)\mathbb{I}_{\overline{A}_n}|\mathcal{F}_n) = \mathbb{E}(\mathbb{E}(Z(\tau')\mathbb{I}_{\{S_{n+1} \le R_n\}}|\mathcal{F}_{n+1})|\mathcal{F}_n) = \mathbb{E}(\mathbb{I}_{\{S_{n+1} \le R_n\}}\mathbb{E}(Z(\tau')|\mathcal{F}_{n+1})|\mathcal{F}_n). \quad (10)$$

The formulas (7)-(10) imply that

$$\mathbb{E}(Z(\tau)|\mathcal{F}_n)$$

$$= \mu_n g(U_{T_n} + d(T_n, R_n), t_0 - T_n - R_n)(\overline{F_2}(R_n)\pi_n + \overline{F_1}(R_n)(1 - \pi_n))$$

$$+\mathbb{E}(\mathbb{I}_{\{S_{n+1} \le R_n\}}\mathbb{E}(Z(\tau')|\mathcal{F}_{n+1})|\mathcal{F}_n).$$

Now, following the standard reasoning of optimal stopping theory, we get the dynamic programming equation for $\Gamma_{n,K}, n = K, K-1, ..., 0$, given in (a), with $\Gamma_{K,K} = \mu_K g(U_{T_K}, t_0 - T_K)$.

(b) We will prove (b) using the backward induction method. First, one should note that (6) is satisfied for $n = K$, as (4) gives

$$\Gamma_{K,K} = \mu_K g(U_{T_K}, t_0 - T_K) = \mu_K \gamma_0(U_{T_K}, T_K, \pi_K). \quad (11)$$

Let $n = K - 1$. It is easy to observe that

$$\{R_{K-1} \ge S_K\} = (\{R_{K-1} \ge W'_K\} \cap \{\kappa \le K\}) \cup (\{R_{K-1} \ge W''_K\} \cap \{\kappa > K\})$$

$$= (\{R_{K-1} \ge W'_K\} \cap \{\kappa = K\}) \cup (\{R_{K-1} \ge W''_K\} \cap (\{\kappa > K\}))$$

$$\cup (\{R_{K-1} \ge W'_K\} \cap \{\kappa \le K-1\}) := B_K^1 \cup B_K^2 \cup B_K^3.$$

The above equality implies that:

$$\Gamma_{K-1,K} = \mu_{K-1} \text{ ess sup}\Big\{g(U_{T_{K-1}} + d(T_{K-1}, R_{K-1}), t_0 - T_{K-1} - R_{K-1})\overline{F_2}(R_{K-1})\pi_{K-1}$$

$$+g(U_{T_{K-1}}+d(T_{K-1},R_{K-1}),t_0-T_{K-1}-R_{K-1})\overline{F_1}(R_{K-1})(1-\pi_{K-1})$$

$$+\mathbb{E}\left(\left(\mathbb{I}_{B_K^1}+\mathbb{I}_{B_K^2}+\mathbb{I}_{B_K^3}\right)\Gamma_{K,K}|\mathcal{F}_{K-1}\right):\ R_{K-1}\geq 0 \text{ and } R_K \text{ is } \mathcal{F}_K\text{-measurable}\}.$$

Let us present the calculations only for the set B_K^1 - the remaining summands under the conditional expectation can be transformed in a similar way. Taking (11) into consideration, rewriting μ_K as $\mu_{K-1}\mathbb{I}_{\{U_{T_{K-1}}+d(T_{K-1},S_K)-X_K e^{\beta(T_{K-1}+S_K)}>0\}}$ and applying the definitions of random variables S_K, X_K and the process U_{T_K} we get that

$$\mathbb{E}\left(\mathbb{I}_{B_K^1}\Gamma_{K,K}|\mathcal{F}_{K-1}\right)=\mu_{K-1}\times$$

$$\times\mathbb{E}(\mathbb{I}_{B_K^1}\mathbb{I}_{\left\{U_{T_{K-1}}+D(T_{K-1},W_K',X_K'')>0\right\}}g(U_{T_{K-1}}+D(T_{K-1},W_K',X_K''),t_0-T_{K-1}-W_K')|\mathcal{F}_{K-1})$$

The independence of random variables W_K' and X_K'' from $\mathcal{F}_{K-1}$ along with the definition of conditional probability θ_K implies that

$$\mathbb{E}\left(\mathbb{I}_{B_K^1}\Gamma_{K,K}|\mathcal{F}_{K-1}\right)=\mu_{K-1}\xi_2(\pi_{K-1})\times$$

$$\times\int_0^{R_{K-1}}\int_0^{\hat{D}\left(T_{K-1},w,U_{T_{K-1}}\right)} g(U_{T_{K-1}}+D(T_{K-1},w,x),t_0-T_{K-1}-w)dH_2(x)dF_1(w)$$

Analogical calculations for B_K^2 and B_K^3 complete the backward induction step for $n=K-1$.

Let $1\leq n<K-1$ and suppose that $\Gamma_{n,K}=\mu_n\gamma_{K-n}(U_{T_n},T_n,\pi_n)$.
From (a) we have

$$\Gamma_{n-1,K}=\text{ess sup}\Big\{\mu_{n-1}g(U_{T_{n-1}}+d(T_{n-1},R_{n-1}),t_0-T_{n-1}-R_{n-1})\overline{F_2}(R_{n-1})\pi_{n-1}$$

$$+\mu_{n-1}g(U_{T_{n-1}}+d(T_{n-1},R_{n-1}),t_0-T_{n-1}-R_{n-1})\overline{F_1}(R_{n-1})(1-\pi_{n-1})$$

$$+\mathbb{E}\left(\left(\mathbb{I}_{B_n^1\cup B_n^2\cup B_n^3}\right)\mu_n\gamma_{K-n}(U_{T_n},T_n,\pi_n)|\mathcal{F}_{n-1}\right):\ R_{n-1}\geq 0,\ R_{n-1} \text{ is } \mathcal{F}_{n-1}\text{-measurabe}\Big\}. \tag{12}$$

Since $\mu_n=\mu_{n-1}\mathbb{I}_{\left\{U_{T_{n-1}}+D(T_{n-1},S_n,X_n)>0\right\}}$, similarly to the calculations presented above we derive the formula for conditional expectation from (12) related to $\mathbb{I}_{B_n^1}$. We get

$$\mathbb{E}\left(\mathbb{I}_{B_n^1}\mu_n\gamma_{K-n}(U_{T_n},T_n,\pi_n)|\mathcal{F}_{n-1}\right)$$

$$=\mu_{n-1}\mathbb{E}(\mathbb{I}_{B_n^1}\mathbb{I}_{\left\{U_{T_{n-1}}+D(T_{n-1},W_n',X_n'')>0\right\}}\gamma_{K-n}(U_{T_n},T_n,\pi_n)|\mathcal{F}_{n-1})$$

$$= \mu_{n-1}\mathbb{E}(\mathbb{I}_{\{\kappa=n\}}\mathbb{E}(\mathbb{I}_{\{R_{n-1}\geq W_n'\}}\mathbb{I}_{\{U_{T_{n-1}}+D(T_{n-1},W_n',X_n'')>0\}}\times$$

$$\times\gamma_{K-n}(U_{T_{n-1}}+D(T_{n-1},W_n',X_n''),T_{n-1}+W_n',\xi_1(\pi_{n-1},X_n'',W_n'))|\mathcal{G}_{n-1})|\mathcal{F}_{n-1}) = \xi_2(\pi_{n-1})\times$$

$$\times\int_0^{R_{n-1}}\int_0^{\hat{D}(T_{n-1},w,U_{T_{n-1}})}\gamma_{K-n}(U_{T_{n-1}}+D(T_{n-1},w,x),T_{n-1}+w,\xi_1(\pi_{n-1},x,w))dH_2(x)dF_1(w).$$

Analogical calculations for B_n^2 and B_n^3 complete the proof of the theorem.

We will now concentrate on the problem of finding optimal stopping time τ_K^*. To this end, as it was proved in [4], we have to analyze properties of the sequence of functions γ_n.

Let $B = B[(-\infty,\infty)\times[0,\infty)\times[0,1]]$ be the space of all bounded continuous functions on $(-\infty,\infty)\times[0,\infty)\times[0,1]$, $B^0 = \{\delta : \delta(u,t,\pi) = \delta_1(u,t,\pi)\mathbb{I}_{\{t\leq t_0\}}, \delta_1\in B\}$. On B^0 we define a norm:

$$\|\delta\|_\alpha = \sup_{u,0\leq t\leq t_0,\pi}\{\left(\frac{t}{t_0}\right)^\alpha|\delta(u,t,\pi)|\},$$

where $\alpha > 1$ is an arbitrary constant, such that $\chi = \frac{Ct_0}{\alpha-1}\in(0,1)$ (the properties of similar norms were considered in [5]).

For any $\delta\in B^0$, $u\in\mathbb{R}, t,r\geq 0$ and $\pi\in[0,1]$ we define:

$$\phi_\delta(r,u,t,\pi) = g(u+d(t,r),t_0-t-r)(\overline{F_2}(r)\pi+\overline{F_1}(r)(1-\pi))$$
$$+\xi_2(\pi)\int_0^r\int_0^{\hat{D}(t,w,u)}\delta(u+D(t,w,x),t+w,\xi_1(\pi,x,w))dH_2(x)dF_1(w)$$
$$+(1-\pi-\xi_2(\pi))\int_0^r\int_0^{\hat{D}(t,w,u)}\delta(u+D(t,w,x),t+w,\xi_1(\pi,x,w))dH_1(x)dF_1(w)$$
$$+\pi\int_0^r\int_0^{\hat{D}(t,w,u)}\delta(u+D(t,w,x),t+w,\xi_1(\pi,x,w))dH_2(x)dF_2(w). \quad (13)$$

We will now make use of the properties of the c.d.f.'s F_1 and F_2. They imply that if g_1 is continuous and $t\neq t_0-r$, then $\phi_\delta(r,u,t,\pi)$ is continuous with respect to (r,u,t,π). Therefore we make the following

Assumption 1. *The function $g_1(\cdot)$ is bounded and continuous.*

For any $\delta \in B^0$ we define an operator Φ as follows:

$$(\Phi\delta)(u,t,\pi) = \sup_{r\geq 0}\{\phi_\delta(r,u,t,\pi)\}. \tag{14}$$

Lemma 3. *For $\pi \in [0,1]$ and for every $\delta \in B^0$ we have:*

$$(\Phi\delta)(u,t,\pi) = \max_{0\leq r\leq t_0-t}\{\phi_\delta(r,u,t,\pi)\}.$$

Furthermore, there exists a function r_δ satisfying $(\Phi\delta)(u,t,\pi) = \phi_\delta(r_\delta(u,t,\pi),u,t,\pi)$.

Proof. When $r > t_0 - t$ and $\delta \in B^0$ then $g(u+cr, t_0-t-r) = 0$ and the equality (13) can be rewritten in the following way

$$\phi_\delta(r,u,t,\pi) = \xi_2(\pi)\int_0^{t_0-t}\int_0^{\hat{D}(t,w,u)} \delta(u+D(t,w,x), t+w, \xi_1(\pi,x,w))dH_2(x)dF_1(w)$$

$$+(1-\pi-\xi_2(\pi))\int_0^{t_0-t}\int_0^{\hat{D}(t,w,u)} \delta(u+D(t,w,x), t+w, \xi_1(\pi,x,w))dH_1(x)dF_1(w)$$

$$+\pi\int_0^{t_0-t}\int_0^{\hat{D}(t,w,u)} \delta(u+D(t,w,x), t+w, \xi_1(\pi,x,w))dH_2(x)dF_2(w).$$

Hence, Assumption 1 and the fact that F_1 and F_2 are continuous functions in the compact interval $[0,t_0]$ imply the form of Φ.

It is easy to note that for $i = 0,1,2,...,K-1$, $u \in \mathbb{R}$, $t \geq 0$, $\pi \in [0,1]$ the sequence $\gamma_i(u,t,\pi)$ can be rewritten according to a following pattern

$$\gamma_{i+1}(u,t,\pi) = \begin{cases} (\Phi\gamma_i)(u,t,\pi) & \text{if } u \geq 0,\ t \leq t_0, \\ 0 & \text{otherwise.} \end{cases}$$

From Lemma 3 we know that there exist functions $r_{K-1-i} := r_{\gamma_i}$, such that

$$\gamma_{i+1}(u,t,\pi) = \begin{cases} \phi_{\gamma_i}(r_{K-1-i}(u,t,\pi),u,t,\pi) & \text{if } u \geq 0,\ t \leq t_0, \\ 0 & \text{otherwise.} \end{cases}$$

To determine the form of optimal stopping times $\tau^*_{n,K}$ we define following random variables:

$$R^*_i = r_i(U_{T_i}, T_i, \pi_i)$$

and

$$\sigma_{n,K} = \min\{K, \inf\{i \geq n : R^*_i < S_{i+1}\}\}.$$

Theorem 2. *Let*

$$\tau^*_{n,K} = \begin{cases} T_{\sigma_{n,K}} + R^*_{\sigma_{n,K}} & \text{if } \sigma_{n,K} < K, \\ T_K & \text{if } \sigma_{n,K} = K, \end{cases} \quad \text{and } \tau^*_K = \tau^*_{0,K}.$$

Then for any $0 \le n \le K$ *we have*

$$\Gamma_{n,K} = \mathbb{E}(Z(\tau^*_{n,K})|\mathcal{F}_n) \ \text{a.s. and } \Gamma_{0,K} = \mathbb{E}(Z(\tau^*_K)).$$

Proof. This is a straightforward consequence of the definition of random variables R^*_i, $\sigma_{n,K}$ and Theorem 1.

4. Solution for the infinite horizon problem

Lemma 4. *The operator* $\Phi : B^0 \to B^0$ *defined by (14) is a contraction in the norm* $\|\cdot\|_\alpha$

Proof. Let $\delta_1, \delta_2 \in B^0$. Then by Lemma 3 there exist $\varrho_i := r_{\delta_i}(u,t,\pi) \le t_0 - t, i = 1,2$, such that $(\Phi\delta_i) = \phi_{\delta_i}(\varrho_i, u, t, \pi)$. It is obvious that $\phi_{\delta_2}(\varrho_2, u, t, \pi) \ge \phi_{\delta_2}(\varrho_1, u, t, \pi)$. Hence,

$$(\Phi\delta_1)(u,t,\pi) - (\Phi\delta_2)(u,t,\pi) \le \phi_{\delta_1}(\varrho_1, u, t, \pi) - \phi_{\delta_2}(\varrho_1, u, t, \pi)$$

$$= \xi_2(\pi) \int_0^{\varrho_1} \int_0^{\hat{D}(t,w,u)} (\delta_1 - \delta_2)(u + D(t,w,x), t+w, \xi_1(\pi,x,w)) \left(\frac{t+w}{t_0}\right)^\alpha \left(\frac{t_0}{t+w}\right)^\alpha dH_2(x) dF_1(w)$$

$$+ (1-\pi-\xi_2(\pi)) \int_0^{\varrho_1} \int_0^{\hat{D}(t,w,u)} (\delta_1 - \delta_2)(u + D(t,w,x), t+w, \xi_1(\pi,x,w)) \left(\frac{t+w}{t_0}\right)^\alpha \left(\frac{t_0}{t+w}\right)^\alpha dH_1(x) dF_1(w)$$

$$+ \pi \int_0^{\varrho_1} \int_0^{\hat{D}(t,w,u)} (\delta_1 - \delta_2)(u + D(t,w,x), t+w, \xi_1(\pi,x,w)) \left(\frac{t+w}{t_0}\right)^\alpha \left(\frac{t_0}{t+w}\right)^\alpha dH_2(x) dF_2(w)$$

$$\le (t_0)^\alpha \xi_2(\pi) \int_0^{t_0-t} \|\delta_1 - \delta_2\|_\alpha \left(\frac{1}{t+w}\right)^\alpha dF_1(w)$$

$$+ (t_0)^\alpha (1 - \pi - \xi_2(\pi)) \int_0^{t_0-t} \|\delta_1 - \delta_2\|_\alpha \left(\frac{1}{t+w}\right)^\alpha dF_1(w)$$

$$+ (t_0)^\alpha \pi \int_0^{t_0-t} \|\delta_1 - \delta_2\|_\alpha \left(\frac{1}{t+w}\right)^\alpha dF_2(w).$$

As analogical estimation can be carried through for $(\Phi\delta_2)(u,t,\pi)-(\Phi\delta_1)(u,t,\pi)$ and we have assumed that the density functions f_1 and f_2 are commonly bounded by a constant C we get:

$$|(\Phi\delta_2)(u,t,\pi)-(\Phi\delta_1)(u,t,\pi)| \leq$$

$$\leq (t_0)^\alpha C\|\delta_1-\delta_2\|_\alpha(\xi_2(\pi)+1-\pi-\xi_2(\pi)+\pi)\int_0^{t_0-t}\left(\frac{1}{t+w}\right)^\alpha dw < (t_0)^\alpha\frac{C}{\alpha-1}||\delta_1-\delta_2||_\alpha\left(\frac{t_0}{t^\alpha}\right)$$

As a straightforward consequence

$$\|(\Phi\delta_2)(u,t,\pi)-(\Phi\delta_1)(u,t,\pi)\|_\alpha < \frac{Ct_0}{\alpha-1}||\delta_1-\delta_2||_\alpha \leq \chi||\delta_1-\delta_2||_\alpha,$$

where $\chi < 1$.

Applying Banach's Fixed Point Theorem we get following Lemma:

Lemma 5. *There exists $\gamma \in B^0$ such that*

$$\gamma = \Phi\gamma \text{ and } \lim_{K\to\infty}||\gamma_K-\gamma||_\alpha = 0.$$

The thesis of Lemma 5 will turn out to be useful in the crucial part of the proof of Theorem 3, describing the optimal stopping rule in the case of infinite horizon.

Theorem 3. *Assume that the utility function g_1 is differentiable and nondecreasing and F_i have commonly bounded density functions f_i for $i=1,2$. Then:*
(a) for $n=0,1,...$ the limit $\hat{\tau_n} := \lim_{K\to\infty}\tau^*_{n,K}$ exists and $\hat{\tau_n}$ is an optimal stopping rule in $\mathcal{T}\cap\{\tau\geq T_n\}$.
(b) $\mathbb{E}(Z(\hat{\tau}_n)|\mathcal{F}_n) = \mu_n\gamma(U_{T_n},T_n,\pi_n)$ a.s.

Proof. (a) It is obvious that $\tau^*_{n,K}\leq\tau^*_{n,K+1}$ a.s for $n\geq 0$. Hence, the stopping rule $\hat{\tau}_n$: $T_n\leq\hat{\tau}_n = \lim_{K\to\infty}\tau^*_{n,K}\leq t_0$ exists. We only have to prove the optimality of $\hat{\tau}_n$, to which aim we will apply arguments similar to those used in [1], [2] and [4].

Let $\xi(t) = (t,U_t,Y_t,V_t,\pi_{N(t)},N(t))$, where $Y_t = t-T_{N(t)}$ and $V_t = \mu_{N(t)} = \prod_{i=1}^{N(t)}\mathbb{I}_{\{U_{T_i}>0\}}$, $t\geq 0$. Then, one can show that $\xi = \{\xi(t) : t\geq 0\}$ is a Markov process with the state space $\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+\times\{0,1\}\times[0,1]\times\mathbb{N}_0$. One can see that the return $Z(t)$ can be described as a function, say $\tilde{g}$, of $\xi(t)$. Then, we can calculate a strong generator of ξ in the form

$$(A\tilde{g})(t,u,y,v,\pi,n) = v\left\{\pi_n\left(e^{\alpha t}(u_0\alpha+c)g'(u) - \frac{f_2(y)}{\overline{F_2}(y)}\left(g_1(u) - \int_0^{ue^{-\beta t}} g_1(u-e^{\beta t}x)dH_2(x)\right)\right)\right.$$

$$+(1-\pi_n)\left(e^{\alpha t}(u_0\alpha+c)g'(u) - \frac{f_1(y)}{\overline{F_1}(y)}\left(g_1(u) - \int_0^{ue^{-\beta t}} g_1(u-e^{\beta t}x)dH_1(x)\right)\right)$$

$$\left.+\,\xi_2(\pi_n)\frac{f_1(y)}{\overline{F_1}(y)}\left(\int_0^{ue^{-\beta t}} g_1(u-e^{\beta t}x)dH_2(x) - \int_0^{ue^{-\beta t}} g_1(u-e^{\beta t}x)dH_1(x)\right)\right\}, \quad (15)$$

where the expression above is well defined as we can assume $f_i(\tilde{t}_0^i) = 0$ for $\tilde{t}_0^i := \sup_{t\le t_0}\{F_i(t) < 1\}$.

Obviously $M_t := \tilde{g}(\xi(t)) - \tilde{g}(0) - \int_0^t (A\tilde{g})(\xi(s))ds,\ t \ge 0$, is a martingale with respect to the filtration $\sigma(\xi(s), s \le t)$, which is the same as $\mathcal{F}(t)$. As T_n and $\tau^*_{n,K}$ are stopping times satisfying the condition $T_n \le \tau^*_{n,K}, a.s.$, we can apply optional sampling theorem and get

$$E(M_{\tau^*_{n,K}}|\xi(T_n)) = M_{T_n} \text{ a.s.},$$

$$\mathbb{E}\left(\tilde{g}(\xi(\tau^*_{n,K})) - \tilde{g}(0) - \int_0^{\tau^*_{n,K}} (A\tilde{g})(\xi(s))ds|\xi(T_n)\right) = \tilde{g}(\xi(T_n)) - \tilde{g}(0) - \int_0^{T_n}(A\tilde{g})(\xi(s))ds, \text{a.s.},$$

and finally

$$\mathbb{E}\left(\tilde{g}(\xi(\tau^*_{n,K}))|\xi(T_n)\right) - \tilde{g}(\xi(T_n)) = \mathbb{E}\left(\int_{T_n}^{\tau^*_{n,K}} (A\tilde{g})(\xi(s))ds|\mathcal{F}_n\right) \text{ a.s.} \quad (16)$$

We will now calculate the limit of the expression from the right hand side of the equality (16) with $K \to \infty$. Firstly, applying the form of the generator from (15) we get

$$(A\tilde{g})(\xi(s)) = \mu_{N(s)}\left\{\pi_{N(s)}\left(e^{\alpha s}(u_0\alpha+c)g'(U(s)) + \frac{f_2(s-T_{N(s)})}{\overline{F_2}(s-T_{N(s)})}\left(\int_0^{U(s)e^{-\beta s}} g_1(U(s)-e^{\beta s}x)dH_2(x) - g_1(U(s))\right)\right)\right.$$

$$+(1-\pi_{N(s)})\left(e^{\alpha s}(u_0\alpha+c)g'(U(s)) + \frac{f_1(s-T_{N(s)})}{\overline{F_1}(s-T_{N(s)})}\left(\int_0^{U(s)e^{-\beta s}} g_1(U(s)-e^{\beta s}x)dH_1(x) - g_1(U(s))\right)\right)$$

$$+ \xi_2(\pi_{N(s)}) \frac{f_1(s - T_{N(s)})}{\overline{F_1}(s - T_{N(s)})} \left(\int_0^{U(s)e^{-\beta s}} g_1(U(s) - e^{\beta s}x) dH_2(x) - \int_0^{U(s)e^{-\beta s}} g_1(U(s) - e^{\beta s}x) dH_1(x) \right) \Bigg\} \tag{17}$$

Inserting to (16) the formula for infinitesimal generator given in (17) we get

$$\mathbb{E}\left(\tilde{g}(\xi(\tau^*_{n,K}))|\xi(T_n)\right) - \tilde{g}(\xi(T_n)) = \mathbb{E}\left(I^1_{n,K}|\mathcal{F}_n\right) - \mathbb{E}\left(I^2_{n,K}|\mathcal{F}_n\right) \quad a.s.$$

where

$$I^1_{n,K} = \int_{T_n}^{\tau^*_{n,K}} \Bigg(\Bigg(e^{\alpha s}(u_0\alpha + c)g'(U(s)) + \left(\frac{f_2(s - T_{N(s)})}{\overline{F_2}(s - T_{N(s)})} + \frac{\xi_2(\pi_{N(s)})}{\pi_{N(s)}} \frac{f_1(s - T_{N(s)})}{\overline{F_1}(s - T_{N(s)})} \right) \times$$

$$\times \int_0^{U(s)e^{-\beta s}} g_1(U(s) - e^{\beta s}x) dH_2(x) \Bigg) \pi_{N(s)} +$$

$$\left(e^{\alpha s}(u_0\alpha + c)g'(U(s)) + \frac{f_1(s - T_{N(s)})}{\overline{F_1}(s - T_{N(s)})} \left(1 - \frac{\xi_2(\pi_{N(s)})}{1 - \pi_{N(s)}} \right) \int_0^{U(s)e^{-\beta s}} g_1(U(s) - e^{\beta s}x) dH_1(x) \right) \left(1 - \pi_{N(s)}\right) \Bigg) \mu_{N(s)} ds,$$

$$I^2_{n,K} = \int_{T_n}^{\tau^*_{n,K}} \left(\pi_{N(s)} \frac{f_2(s - T_{N(s)})}{\overline{F_2}(s - T_{N(s)})} + (1 - \pi_{N(s)}) \frac{f_1(s - T_{N(s)})}{\overline{F_1}(s - T_{N(s)})} \right) g_1(U(s)) \mu_{N(s)} ds.$$

Note that $I^2_{n,K}$ is a nonnegative random variable. As $\pi_{N(s)} + \xi_2(\pi_{N(s)}) \leq 1$, $I^1_{n,K}$ is also a nonnegative random variable. Moreover, it can be shown that $I^2_{n,K}$ is bounded. Applying Monotone Convergence Theorem, the properties of conditional expectation and the fact that $\lim_{K\to\infty} \tau^*_{n,K} = \hat{\tau}_n$ we get

$$\lim_{K\to\infty} \mathbb{E}\left(\int_{T_n}^{\tau^*_{n,K}} (A\tilde{g})(\xi(s)) ds | \mathcal{F}_n \right) = \mathbb{E}\left(\int_{T_n}^{\hat{\tau}_n} (A\tilde{g})(\xi(s)) ds | \mathcal{F}_n \right) \text{ a.s.}$$

On the other hand, Dynkin formula implies also that

$$\mathbb{E}\left(\int_{T_n}^{\hat{\tau}_n} (A\tilde{g})(\xi(s)) ds | \mathcal{F}_n \right) = \mathbb{E}\left(\tilde{g}(\xi(\hat{\tau}_n))|\mathcal{F}_n)\right) - \tilde{g}(\xi(T_n) \text{ a.s.}$$

Hence,

$$\lim_{K\to\infty} \mathbb{E}\left(\tilde{g}(\xi(\tau^*_{n,K}))|\mathcal{F}_n\right) = \mathbb{E}\left(\tilde{g}(\xi(\hat{\tau}_n))|\mathcal{F}_n\right) \text{ a.s.} \tag{18}$$

To complete this part of the proof we only have to show that $\hat{\tau}_n$ is an optimal stopping time in the class $\mathcal{T} \cap \{\tau : \tau \geq T_n\}$. To that end let us assume that τ is some other stopping rule in $\mathcal{T} \cap \{\tau : \tau \geq T_n\}$. Then, as $\tau^*_{n,K}$ is optimal in $\mathcal{T}_{n,K}$, we have for any K,

$$\mathbb{E}\left(\tilde{g}(\xi(\tau^*_{n,K}))|\mathcal{F}_n\right) \geq \mathbb{E}\left(\tilde{g}(\xi(\tau \wedge T_K))|\mathcal{F}_n\right) \text{ a.s.}$$

A similar argumentation to one that led to formula (18) reveals that

$$\mathbb{E}\left(\tilde{g}(\xi(\hat{\tau}_n))|\mathcal{F}_n\right) \geq \mathbb{E}\left(\tilde{g}(\xi(\tau))|\mathcal{F}_n\right) \text{ a.s.}$$

The proof of (a) is now complete.
(b) As it was shown in part (b) of Theorem 1 a following equality stands

$$\mathbb{E}\left(\tilde{g}(\xi(\tau^*_{n,K}))|\mathcal{F}_n\right) = \mu_n \gamma_{K-n}(U_{T_n}, T_n, \pi_n).$$

Then Lemma 5 and (18) imply

$$\mathbb{E}\left(Z(\hat{\tau}_n)|\mathcal{F}_n\right) = \lim_{K\to\infty} \mathbb{E}\left(\tilde{g}(\xi(\tau^*_{n,K}))|\mathcal{F}_n\right) = \lim_{K\to\infty} \mu_n \gamma_{K-n}(U_{T_n}, T_n, \pi_n) = \mu_n \gamma(U_{T_n}, T_n, \pi_n) \text{ a.s.}$$

what completes the proof of the Theorem.

5. Numerical examples

In order to provide the reader with a complete overview of the stopping rules in the suggested model a numerical example will be given. For the sake of simplicity, to focus the attention of the reader on the method itself rather than on strenuous calculations, we will assume that the rates of inflation α as well as claim severity growth β are equal to zero. The application of the example below to a model with non-zero rates is straightforward.

Let us assume that the observation period is equal to 1 year ($t_0 = 1$), $c = 1$ is the constant rate of income from the insurance premium and the fixed number of claims that may occur in our model, K, is 1. Moreover, let us assume that the probability p defining the distribution of κ is equal to $\frac{1}{2}$.

We impose that W'_1 and W''_1 are uniformly distributed over the interval $[0, t_0]$. The claim severity distribution changes from $H_1(x) = \frac{e^x-1}{4}\mathbb{I}_{\{x\in[0,ln5]\}}$ for X'_1 to $H_2(x) = \frac{e^x-1}{8}\mathbb{I}_{\{x\in[0,ln9]\}}$ for X''_1. As $\mathbb{E}X' < \mathbb{E}X''$ it is obvious that the examples reflects a situation of deteriorating market conditions. Three models will be investigated:

Model I. In this scenario the insurance company does not apply any optimal stopping rules. Hence, the observation ends along with first claim or when the time t_0 is reached, whichever happens first.

Model II. In this scenario the company applies the optimal stopping rule suggested in the articles [4] and [8]. Hence, having no possibility to anticipate potential distribution changes the company assumes that $S_1 = W_1'$ and $X_1 = X_1'$. In such case

$$\Phi_\delta(r,u,t) = e^{u+r}(1-r)\mathbb{I}_{\{t+r<1\}} + \frac{1}{4}\int_0^r \int_0^{u+s} \delta(u+s-x, t+s)e^x dxds$$

As $K = 1$ we execute only one step of iterative procedure and we get that

$$\gamma_1(u,t) = \Phi_{\gamma_0}(u,t) = \max_{0\le r\le 1}\{e^{u+r}(1-r)\mathbb{I}_{\{t+r<1\}} + \frac{1}{4}\int_0^r e^{u+s}(u+s)\mathbb{I}_{\{t+s<1\}}ds\}$$

Standard calculations reveal that in such case

$$\tau^*_{0,1} = R^*_0\mathbb{I}_{\{R^*_0<T_1\}} + T_1\mathbb{I}_{\{R^*_0\ge T_1\}}, \text{ where } R^*_0 = \min\{\frac{1}{3}u_0, 1\}.$$

Model III. In the third scenario, the company can utilize the suggested model with disruption. It is easy to see that under the assumptions stated above we have

$$\xi_1(\pi,x,s) = \frac{\pi+1}{3-\pi}, \xi_2(\pi) = \frac{(1-\pi)^2}{3-\pi},$$

$$\Phi_\delta(r,u,t,\pi) = e^{u+r}(1-r)\mathbb{I}_{\{t+r<1\}} + \frac{5-3\pi}{24-8\pi}\int_0^r \int_0^{u+s} \delta(u+s-x, t+s, \frac{\pi+1}{3-\pi})e^x dxds,$$

$$\gamma_1(u,t,\pi) = \Phi_{\gamma_0}(u,t,\pi) = \max_{0\le r\le 1}\{e^{u+r}(1-r)\mathbb{I}_{\{t+r<1\}} + \frac{5-3\pi}{24-8\pi}\int_0^r e^{u+s}(u+s)\mathbb{I}_{\{t+s<1\}}ds\}.$$

It is easy to notice that

$$\tau^*_{0,1} = R^*_0\mathbb{I}_{\{R^*_0<T_1\}} + T_1\mathbb{I}_{\{R^*_0\ge T_1\}}, \text{ where } R^*_0 = \min\{\frac{5-3\pi_0}{19-5\pi_0}u_0, 1\}.$$

Below we enclose the results of simulations for different values of initial capital u and probability π_0. The presented numbers are averaged returns for the company over 10000 trajectories generated in each scenario.

Returns for $\pi_0 = 0.5$ and $p = 0.5$			
Initial capital	Model I	Model II	Model III
0,2	0,311	1,222	1,222
0,4	0,474	1,496	1,497
0,6	0,687	1,831	1,836
0,8	0,978	2,252	2,258
1	1,323	2,759	2,777
1,2	1,741	3,408	3,437
1,4	2,248	4,176	4,243
1,6	2,808	5,055	5,219
1,8	3,486	6,041	6,387
2	4,295	7,151	7,817
2,2	5,26	8,335	9,51
2,4	6,41	9,484	11,483
2,6	7,832	10,595	13,83
2,8	9,539	11,353	16,622
3	11,645	11,645	19,858
3,2	14,254	14,254	23,596
3,4	17,431	17,431	27,857
3,6	21,212	21,212	32,588
3,8	26,084	26,084	38,373
4	31,658	31,658	43,858
4,2	38,81	38,81	50,297
4,4	47,349	47,349	56,278
4,6	57,844	57,844	62
4,8	71,016	71,016	71,016

Returns for $u_0 = 1$ and $p = 0.5$			
π_0	Model I	Model II	Model III
0,04	1,699	2,842	2,837
0,08	1,664	2,838	2,831
0,12	1,619	2,82	2,819
0,16	1,598	2,823	2,82
0,2	1,561	2,806	2,809
0,24	1,532	2,811	2,812
0,28	1,506	2,801	2,806
0,32	1,464	2,794	2,8
0,36	1,434	2,783	2,792
0,4	1,407	2,784	2,791
0,44	1,374	2,777	2,79
0,48	1,34	2,768	2,784
0,52	1,306	2,764	2,785
0,56	1,271	2,758	2,772
0,6	1,238	2,754	2,77
0,64	1,208	2,748	2,772
0,68	1,178	2,732	2,762
0,72	1,142	2,727	2,76
0,76	1,119	2,723	2,764
0,8	1,078	2,715	2,76
0,84	1,054	2,712	2,751
0,88	1,019	2,706	2,753
0,92	0,988	2,694	2,746
0,96	0,965	2,702	2,745
1	0,919	2,687	2,742

It is easy to see that the scenarios where the optimal stopping rules have been applied offer significantly higher returns regardless of the initial probability π_0. The model with disruption is better in the situations when the probability of distribution change at $t = 0$ is higher than 0.2, whereas it somewhat fails when π_0 is low. However, this can easily be justified. As the model III is bound to deal with more complex market situations and covers a brighter spectrum of processes, the simplicity of model II prevails in the environment for which it was originally designed (as low probability of disruption in the first step in a model with only one claim observed in fact reflects the environment with no disruption). We believe however that with greater values of K this effect would not be observed.

The analysis of company's returns for different values of initial capital confirms the considerable advantage of the models with optimal stopping rules over "passive" models. As the stopping moment in models II and III is defined by a minimum of observation period and an increasing function of initial capital u_0 it is obvious that from some initial capital forth ($u_0 = 3$ in case of model II and $u_0 = 4\frac{5}{7}$ in case of model III) all the models will give the same returns. This fact underlines the advantages of optimal stopping models especially in the situation when company's initial capital is low and the insolvency probability is substantial.

6. Final remarks

A diligent reader will take notice of the fact that this work does not only introduce a new model and solves the problem of optimal stopping of the risk process within this scheme. It also widens the spectrum of the processes for which we can apply stopping rules derived in the models from [4], [7] and [8]. Authors assumed there that distribution of the random variable describing inter-occurrence times between losses, F, fulfills the condition $F(t_0) < 1$. As it was shown in this article, such assumption is irrelevant, provided that $F'(t)$ is bounded. A simple change of norm allows to apply the Fixed Point Theorem.

References

[1] Boshuizen, F.A. and Gouweleeuw, J.M.(1993). General Optimal Stopping Theorems for Semi-Markov Processes. *Adv.Appl.Prob.* **25,** 825–846.

[2] Boshuizen, F.A. and Gouweleeuw, J.M.(1995). A Continuous-time Job Search

Model: General Renewal Process. *Commun.Statist.Stoch.Models* **11,** 349–369.

[3] Davis M.H.A.(1993). *Markov Models and Optimization.* Chappman and Hall.

[4] Ferenstein, E. and Sierocinski,A.(1997). Optimal Stopping of a Risk Process. *Appl.Math.* **24,** 335–342.

[5] Goebel, K. and Kirk, W.A.(1990). *Topics in Metric Fixed Point Theory* Cambridge Univ. Press.

[6] Jensen, U.(1997). An Optimal Stopping Problem in Risk Theory. *Scand. Actuarial J.* **1997,** 149–159.

[7] Karpowicz, A. and Szajowski, K.(2007). Double Optimal Stopping of a Risk Process. *Stochastics: An International Journal of Probability and Stochastics Processes* **79,** 155–167.

[8] Muciek, B.K.(2002). Optimal Stopping of a Risk Process: Model with Interest Rates. *J.Appl.Prob.* **39,** 261–270.

[9] Pasternak-Winiarski, A.(2007). *Optimal Stopping of a Risk Process.* Masterthesis,

[10] Peskir, G. and Shiryayev, A.(2006). *Optimal Stopping and Free-Boundary Problems.* Birkhäuser Verlag.

[11] Rolski, T. and Schmidli, H. and Schmidt, V. and Teugels, J.(1999). *Stochastic Processes for Insurance and Finance.* John Wiley and Sons.

[12] Schöttl, A.(1998). Optimal Stopping of a Risk Reserve Process with Interest and Cost Rates. *J.Appl.Prob* **35,** 115–123.

[13] Shiryayev, A.(1978). *Optimal Stopping Rules.* Springer Verlag.