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IM PAN Preprint 713 (2010)

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Published as manuscript

Received 23 February 2010

SHIFTS AND PERIODICITY IN ALGEBRAIC ANALYSiS

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Shifts and periodicity for functional-differential equations and their generalizations have been examined by the author in several papers in various aspects (cf. for instance, PR[4]-PR[7] and following papers). Here we would like to give a comprehensive survey (without proofs) of some of these results in order to recall the most important properties of considered shifts. In particular, there is shown that the so-called true shifts in complete linear metric spaces are hypercyclic and that a necessary and sufficient condition for true shifts in commutative algebras to be multiplicative is that the generating operator D satisfies the Leibniz condition. A consequence of this fact is that in commutative Leibniz algebras with logarithms the operator D is uniquely determined by an isomorphism acting on $\frac{\mathrm{d}}{\mathrm{d}t}$. There are also studied generalized periodic and exponential-periodic solutions of linear and some nonlinear equations with shifts and generalizations of the classical Birkhoff theorem and Floquet theorem. These results are obtained by means of tools given by Algebraic Analysis (cf. PR[4]). A generalization of binomial formula of Umbral Calculus is shown in Section 7 (cf. Roman and Rota RR[1]). Section 11 contains a perturbation theorem for linear differential-difference equations with non-commensurable deviations and some its consequences.

1. Basic notions of Algebraic Analysis.

We recall here the following notions and theorems (without proofs; cf. PR[4], PR[7], PR[10]).

Let X be a linear space (in general, without any topology) over a field $\mathbb{F}$ of scalars of the characteristic zero. Write

- $L(X)$ is the set of all linear operators with domains and ranges in X;
- dom A is the domain of an $A \in L(X)$;
- $\ker A = \{x \in \text{dom } A : Ax = 0\}$ is the kernel of an $A \in L(X)$;
- $L_0(X) = \{A \in L(X) : \text{dom } A = X\}$.

An operator $D \in L(X)$ is said to be *right invertible* if there is an operator $R \in L_0(X)$ such that $RX \subset \text{dom } D$ and $DR = I$, where I denotes the identity operator. The operator R is called a *right inverse* of D. By $R(X)$ we denote the set of all right invertible operators in $L(X)$. Let $D \in R(X)$. Let $\mathcal{R}_D \subset L_0(X)$ be the set of all right inverses for D, i.e. $DR = I$ whenever $R \in \mathcal{R}_D$. We have

$$\text{dom } D = RX \oplus \ker D, \quad \textit{independently of the choice of an } R \in \mathcal{R}_D.$$

Elements of $\ker D$ are said to be *constants*, since by definition, $Dz = 0$ if and only if $z \in \ker D$. The kernel of D is said to be the *space of constants*. We should point out that, in general, constants are different than scalars, since they are elements of the space X. If two right inverses commute each with another, then they are equal. Let

$$\mathcal{F}_D = \{F \in L_0(X) : F^2 = F; FX = \ker D \ \mathit{and}\ \exists_{R\in\mathcal{R}_D}\ FR = 0\}.$$

Any $F \in \mathcal{F}_D$ is said to be an *initial* operator for D corresponding to R. One can prove that **any** projection F' onto $\ker D$ is an initial operator for D corresponding to a right inverse $R' = R - F'R$ independently of the choice of an $R \in \mathcal{R}_D$.

If two initial operators commute each with another, then they are equal. Thus this theory is essentially **noncommutative**.

An operator F such that $FX \subset \ker D$ is initial for D if and only if there is an $R \in \mathcal{R}_D$ such that

$$F = I - RD \quad \mathit{on}\ \mathrm{dom}\ D. \tag{1.1}$$

Even more. Write $\mathcal{R}_D = \{R_\gamma\}_{\gamma\in\Gamma}$. Then, by (1.1), we conclude that $\mathcal{R}_D$ induces in a unique way the family $\mathcal{F}_D = \{F_\gamma\}_{\gamma\in\Gamma}$ of the corresponding initial operators defined by means of the equality $F_\gamma = I - R_\gamma D$ on dom $D \quad (\gamma \in \Gamma)$. Formula (1.1) yields (by a two-lines induction) the *Taylor Formula*:

$$I = \sum_{k=0}^{n} R^k F D^k + R^n D^n \ \mathit{on}\ \mathrm{dom}\ D^n \ (n \in \mathbb{N}). \tag{1.2}$$

It is enough to know one right inverse in order to determine all right inverses and all initial operators. Note that a superposition (if exists) of a finite number of right invertible operators is again a right invertible operator.

The equation $Dx = y$ $(y \in X)$ has the general solution $x = Ry + z$, where $R \in \mathcal{R}_D$ is arbitrarily fixed and $z \in \ker D$ is arbitrary. However, if we put an *initial condition*: $Fx = x_0$, where $F \in \mathcal{F}_D$ and $x_0 \in \ker D$, then this equation has a unique solution $x = Ry + x_0$.

If $T \in L(X)$ belongs to the set $\Lambda(X)$ of all left invertible operators, then $\ker T = \{0\}$. If $D \in \mathcal{I}(X) = \Lambda(X) \cap R(X)$ then $\mathcal{F}_D = \{0\}$ and $\mathcal{R}_D = \{D^{-1}\}$.

If $P(t) \in \mathbb{F}[t]$ (i.e. $P(t)$ is a polynomial with scalar coefficients, where $\mathbb{F}$ is the field of scalars under consideration) then all solutions of the equation

$$P(D)x = y, \qquad y \in X, \tag{1.3}$$

can be obtained by a decomposition of a rational function induced by $P(t)$ into vulgar fractions. One can distinguish subspaces of X with the property that all solutions of Equation (1.3) belong to a subspace Y whenever $y \in Y$ (cf. von Trotha T[1], PR[6]).

If X is an algebra over $\mathbb{F}$ with a $D \in L(X)$ such that $x, y \in \text{dom } D$ implies $xy, yx \in \text{dom } D$, then we shall write $D \in \mathbf{A}(X)$. If X is *commutative* then $\mathbf{A}(X)$ will be denoted by $\mathsf{A}(X)$. If $D \in \mathbf{A}(X)$ then we can write

$$(1.4) \qquad f_D(x, y) = D(xy) - c_D[xDy + (Dx)y] \quad \textit{for} \quad x, y \in \text{dom } D,$$

where c_D is a scalar dependent on D only. Clearly, f_D is a bilinear (i.e. linear in each variable) form which is symmetric when X is commutative, i.e. when $D \in \mathsf{A}(X)$. This form is called a *non-Leibniz component*. Non-Leibniz components have been introduced for right invertible operators $D \in \mathsf{A}(X)$ (cf. PR[1]). If $D \in \mathbf{A}(X)$ then the product rule in X can be written as follows:

$$D(xy) = c_D[xDy + (Dx)y] + f_D(x, y) \quad \textit{for} \quad x, y \in \text{dom } D.$$

If $D \in \mathbf{A}(X)$ and if D satisfies the *Leibniz condition*:

$$(1.5) \qquad D(xy) = xDy + (Dx)y \quad \textit{for } x, y \in \text{dom } D,$$

then X is said to be a *Leibniz algebra*. It means that in Leibniz algebras $c_D = 1$ and $f_D = 0$. The Leibniz condition implies that $xy \in \text{dom } D$ whenever $x, y \in \text{dom } D$. If X is a Leibniz algebra with unit e then $e \in \ker D$, i.e. D is not left invertible.

Non-Leibniz components for powers of $D \in \mathbf{A}(X)$ are determined by recurrence (equivalent) formulae. Namely, for all $k \in \mathbb{N}$, $x, y \in \text{dom } D^k$ such that $f_D(x, y) \in \text{dom } D^k$ we have $xy \in \text{dom } D^k$ and

$$D^k(xy) = c_D^k[xD^k y + (D^k x)y] + f_D^{(k)}(x, y), \quad \textit{where } f_D^{(1)} = f_D \textit{ and for } k = 2, 3, ...$$

$$(1.7) \qquad f_D^{(k)}(x, y) = c_D^k[(Dx)D^{k-1}y + (D^{k-1}x)Dy] +$$

$$+ c_D^{k-1}[f_D(x, D^{k-1}y) + f_D(D^{k-1}x, y)] + Df_D^{(k-1)}(x, y),$$

i.e.

$$c_{D^k} = c_D^k \quad \textit{and} \quad f_{D^k} = f_D^{(k)} \quad \textit{for } k \in \mathbb{N}.$$

Moreover, $f_D^{(k)}$ are bilinear mappings of $\text{dom } D^k \times \text{dom } D^k$ into $\text{dom } D^k$ ($k \in \mathbb{N}$).

Observe that, by definition, $f_D^{(k)}$ is a bilinear mapping of $\text{dom } D^k \times \text{dom } D^k$ into $\text{dom } D^k$ ($k = 2, 3, ...$).

Suppose that $D \in \mathbf{A}(X)$ and $p \neq 0$ is an arbitrarily fixed scalar. Then $pD \in \mathbf{A}(X)$ and

$$(1.8) \qquad c_{pD} = c_D, \; f_{pD}^{(k)} = p^k f_D^{(k)} \quad \textit{for } k \in \mathbb{N}.$$

If $D_1, D_2 \in \mathbf{A}(X)$, the superposition $D = D_1 D_2$ exists and $D_1 D_2 \in \mathbf{A}(X)$, then

$$(1.9) \qquad c_{D_1 D_2} = c_{D_1} c_{D_2} \quad \textit{and for } x, y \in \text{dom } D = \text{dom } D_1 \cap D_2$$

$$f_{D_1D_2}(x,y) = f_{D_1}(x,y) + D_1 f_{D_2}(x,y) + +c_{D_1}c_{D_2}[(D_1x)D_2y + (D_2x)D_1y].$$

For higher powers of D in Leibniz algebras, by an easy induction from Formulae (1.6) and the Leibniz condition, we obtain *the Leibniz formula*:

$$D^n(xy) = \sum_{k=0}^{n} \binom{n}{k} (D^k x)D^{n-k}y \quad for\ x,y \in \mathrm{dom}\ D^n \quad (n \in \mathbb{N}), \tag{1.10}$$

i.e.

$$f_{D^n}(x,y) = f_D^{(n)}(x,y) + \sum_{k=1}^{n-1} \binom{n}{k}(D^k x)D^{n-k}y, \quad x,y \in \mathrm{dom}\ D^n\ (n \in \mathbb{N}). \tag{1.11}$$

By $M(X)$ we shall denote the set of all multiplicative mappings in X, i.e.

$$M(X) = \{A : X \to X : A(xy) = (Ax)(Ay)\ for\ x,y \in X\}. \tag{1.12}$$

Let X be an algebra with unit e. Then A is an *algebra isomorphism* if it is a structure preserving invertible mapping, i.e. $A \in L_0(X) \cap \mathcal{I}(X) \cap M(X)$. If it is the case then A^{-1} is also an algebra isomorphism. Moreover, $Ae = e$. Write

$$v_{\mathbb{F}}A = \{0 \neq \lambda \in \mathbb{F} : I - \lambda A \quad is \quad invertible\} \quad for\ A \in L(X). \tag{1.13}$$

It means that $0 \neq \lambda \in v_{\mathbb{F}}A$ if and only if $\frac{1}{\lambda}$ is a regular value of A.

By $V(X)$ we denote the set of all *Volterra operators* belonging to $L(X)$, i.e. the set of all operators $A \in L(X)$ such that $I - \lambda A$ is invertible for all scalars λ. Clearly, $A \in V(X)$ if and only if $v_{\mathbb{F}}A = \mathbb{F} \setminus \{0\}$ (cf. Formula (1.13)).

Note 1.1. Nguyen Van Mau (cf. N[1]) has shown that there is a right invertible singular integral operator which has **no** Volterra right inverses. □

Let X be a Banach space. Denote by $QN(X)$ the set of all quasinilpotent operators belonging to $L(X)$, i.e. the set of all bounded operators $A \in L_0(X)$ such that

$$\lim_{n\to\infty} \sqrt[n]{\|A^n x\|} \ = \ 0 \qquad for \quad x \in X.$$

It is well-known that $QN(X) \subset V(X)$. If $\mathbb{F} = \mathbb{C}$ then $QN(X) = V(X) \cap B(X)$, where $B(X)$ is the set of all bounded operators belonging to $L(X)$.

Definition 1.1. (cf. BPR[1], PR[5], also PR[7]). Let X be a complete linear metric space over a field $\mathbb{F}$ of scalars. Let $A \in L(X)$ be continuous. Let $E \subset \mathrm{dom}\ A \subset X$ be a subspace. Let ω be a non-empty subset of $v_{\mathbb{F}}A$. The operator $A \in L(X)$ is said to be *ω-almost quasinilpotent on E* if

$$\lim_{n\to\infty} \lambda^n A^n x = 0 \quad for\ all\ \lambda \in \omega,\ x \in E. \tag{1.14}$$

The set of all operators ω-almost quasinilpotent on the set E will be denoted by $AQN(E;\omega)$. If $\omega = v_{\mathbb{F}}A$ then we say that A is *almost quasinilpotent on* E. The set of all almost quasinilpotent operators on E will be denoted by $AQN(E)$. □

Theorem 1.1. (cf. PR[5], also PR[10]). *Let E be a subspace of a complete linear metric space X over $\mathbb{F}$. If $A \in L(X)$, $E \subset \mathrm{dom}\ A$ and $\emptyset \neq \omega \subset v_{\mathbb{F}}A$, then the following conditions are equivalent:*

(i) A is ω-almost quasinilpotent on E;

(ii) for every $\lambda \in \omega$, $x \in E$ the series $\sum_{n=0}^{\infty} \lambda^n A^n x$ is convergent and

$$(I - \lambda A)^{-1} x = \sum_{n=0}^{\infty} \lambda^n A^n x \qquad (\lambda \in \omega,\ x \in E); \tag{1.15}$$

(iii) for every $\lambda \in \omega$, $x \in E$, $m \in \mathbb{N}$ the series $\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} \lambda^n A^n x$ is convergent and

$$(I - \lambda A)^{-m} x = \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} \lambda^n A^n x \qquad (\lambda \in \omega,\ x \in E,\ m \in \mathbb{N}). \tag{1.16}$$

For given $D \in R(X)$, $R \in \mathcal{R}_D$ we shall consider (cf. T[1], PR[6]) the following subspaces

- the space of *smooth* elements

$$D_\infty = \bigcap_{k \in \mathbb{N}_0} \mathrm{dom}\ D^k, \quad \textit{where}\ \mathrm{dom}\ D^0 = X;$$

- the space of *D-polynomials*

$$\mathbf{S} = \bigcup_{n \in \mathbb{N}} \ker\ D^n; \quad \mathbf{S} = P(R) = \mathrm{lin}\ \{R^k z : z \in \ker\ D,\ k \in \mathbb{N}_0\} \subset D_\infty,$$

which, by definition, is independent of the choice of an $R \in \mathcal{R}_D$;

- the space of *exponentials*

$$E(R) = \bigcup_{\lambda \in v_{\mathbb{F}}R} \ker\ (D - \lambda I) =$$

$$= \mathrm{lin}\ \{(I - \lambda R)^{-1} z : z \in \ker\ D,\ \lambda \in v_{\mathbb{F}}R \quad \textit{or}\ \lambda = 0\} \subset\ D_\infty,$$

which is independent of the choice of the right inverse R, provided that R is a Volterra operator;

• the space of *D-analytic* elements in a complete linear metric space X ($\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$)

$$A_R(D) = \{x \in D_\infty : x = \sum_{n=0}^{\infty} R^n F D^n x\} = \{x \in D_\infty : \lim_{n\to\infty} R^n D^n x = 0\},$$

where F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$.

Clearly, by definitions, we have $\mathbf{S},\ E(R) \subset D_\infty$. If X is a complete linear metric space then $\mathbf{S} \subset A_R(D) \subset D_\infty$.

2. Shifts in linear spaces.

Let X be a linear space over an algebraically closed field $\mathbb{F}$ of scalars. Recall that an operator $T \in L_0(X)$ is said to be *algebraic* on X if there is a polynomial $P(t) \in \mathbb{F}[t]$ such that $P(T)x = 0$ for all $x \in X$, i.e. $P(T) = 0$ on X. The operator T is algebraic of the *order* N if $\deg P(t) = N$ and there is no polynomial $P'(t) \in \mathbb{F}[t]$ of $\deg M < N$ such that $P'(T) = 0$ on X (we assume here and in the sequel that any polynomial under consideration is *normalized*, i.e. its coefficient of the term of the highest degree is equal 1). If it is the case, then $P(t)$ is said to be a *characteristic polynomial* of T and its roots are called *characteristic roots* of T. An operator T is algebraic on X of the order N with the characteristic polynomial

$$P(t) = \prod_{j=1}^{n} (t - t_j)^{r_j}, \quad t_j \neq t_k \ if\ j \neq k,\ r_1 + ... + r_n = N$$

if and only if X is the direct sum of the *principal spaces* of the operator T corresponding to the eigenvalues $t_1, ..., t_N$:

$$X = X_1 \oplus ... \oplus X_N, \quad where\ (T - t_j)^{r_j} x_j = 0\ for\ x_j \in X_j,$$

$$X_j = P_j X, \quad P_j = P_j(T)$$

and $P_1, ..., P_N$ are disjoint projectors giving the *partition of unit*:

$$P_j P_k = \delta_{jk} P_k \quad (j,k = 1, ..., N), \quad \sum_{j=1}^{N} P_j = I$$

which are polynomials in T uniquely determined for a given S (cf. PR[2]). If $r_1 = ... = r_N$, i.e. if the characteristic roots are *single*, then these projectors are of the form

$$P_j = P_j(T), \quad where \quad P_j(t) = \prod_{k=1,k\neq j}^{N} \frac{t - t_k}{t_j - t_k} \quad (j = 1, ..., N)$$

and $X_j = P_j X$ are eigenspaces of T (cf. PR[2], also PR[4], PR[15]).

Definition 2.1. (cf. PR[3], PR[4]). Suppose that X is a linear space over $\mathbb{C}$, $D \in R(X)$ and $S \in L_0(X)$ commute with D: $SD = DS$ on dom D. Let $N \in \mathbb{N}$ be arbitrarily fixed. If

$$X_{S^N} = \{x \in X : S^N x = x\} \neq \emptyset \tag{2.1}$$

then any element $x \in X_{S^N}$ is said to be S^N*-periodic*. □

Clearly, S is an *involution of order* N on the space X_{S^N}: $S^N = I$ on X_{S^N}. Thus there are N disjoint projectors P_j giving partition of unit, i.e.

$$P_k P_j = \delta_{jk} P_j \quad (j = 1, ..., N); \quad \sum_{j=0}^{N} P_j = I, \tag{2.2}$$

such that

$$SP_j = P_j S = \varepsilon^j P_j \quad (j = 1, ..., N), \quad S = \frac{1}{N} \sum_{j=0}^{N-1} \varepsilon^j P_j, \quad \textit{where } \varepsilon = \mathrm{e}^{2\pi i/N}. \tag{2.3}$$

Formulae (2.2) and (2.3) together imply that

$$X_{S^N} = X_{(1)} \oplus ... \oplus X_{(N)}, \quad \textit{where } X_{(j)} = P_j X_{S^N} \quad (j = 1, ..., N) \tag{2.4}$$

$$Sx_{(j)} = \varepsilon^j x_{(j)}, \quad \textit{where } x_{(j)} = P_j x,\ x \in X_{S^N} \quad (j = 1, ..., N). \tag{2.5}$$

Projectors P_j are of the form

$$P_j = \frac{1}{N} \sum_{k=0}^{N-1} \varepsilon^{-jk} S^k \quad (j = 1, ..., N). \tag{2.6}$$

□

Definition 2.2. Suppose that X is a linear space over $\mathbb{F}$, $D \in R(X)$, $\ker D \neq \{0\}$ and F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$.Then a family $\{S_h\}_{h\in\mathbb{R}} \subset L_0(X)$ of linear operators is family of R*-shifts* if

$$S_0 = I, \quad S_h R^k F = \sum_{j=0}^{k} \frac{h^{k-j}}{(k-j)!} R^j F \quad \textit{for } k \in \mathbb{N}_0. \tag{2.7}$$

□

Theorem 2.1. *Suppose that X is a linear space over $\mathbb{F}$, $D \in R(X)$, $\ker D \neq \{0\}$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$ and $\{S_h\}_{h\in\mathbb{R}}$ is a family of R-shifts* *defined by (2.7). Then R-shifts are D*-invariant, *i.e.* $S_hD = DS_h$ *on* dom D *whenever* $h \in \mathbb{R}$.

Theorem 2.2. *Suppose that all assumptions of Theorem 2.1 are satisfied. Write*

$$F_h = FS_{-h}, \quad D_h = D - \frac{1}{h}F_h, \tag{2.8}$$

$$R_h^0 = (I + F_h + \frac{1}{h}RF_h)R \quad for\ h \in \mathbb{R} \setminus \{0\}$$

$$\mathsf{E}_h = \{x \in X : S_hx = x\} \neq \emptyset, \quad \mathsf{E}_h^{(1)} = \mathsf{E}_h \cap \operatorname{dom} D \quad (h \in \mathbb{R}). \tag{2.9}$$

Then $D_h \in R(X)$, F_h is an initial operator for D_h corresponding to the right inverse $R_h = R - F_hR$. Moreover, the operator R_h^0 maps the space E_h onto the space $\mathsf{E}_h^{(1)}$ and

$$D_hR_h^0 = I \quad on\ \mathsf{E}_h, \quad R_h^0D_h = I \quad on\ \mathsf{E}_h^{(1)},\ i.e.\ D_h^{-1} = R_h^0. \tag{2.10}$$

Definition 2.3. Suppose that all conditions of Definition 2.1 are satisfied. Let $m \in \mathbb{N}$. An operator $A \in L_0(X)$ is said to be S^m*-periodic* if $S^mA = AS^m$. □

Clearly, if $D \in \mathsf{A}(X)$, $Ax = ax$ for an $a \in X$ and S is multiplicative, then $S^m(Ax) = S^m(ax) = (S^ma)S^mx$. Thus A is S^m-periodic if and only if $a \in X_{S^m}$ $(m \in \mathbb{N})$.

3. True shifts in linear metric spaces.

We begin with

Definition 3.1. (cf. PR[5], also PR[10]). Suppose that X is a complete linear metric locally convex space ($\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$), $D \in R(X)$ is closed, $\ker D \neq \{0\}$ and F is a continuous initial operator for D corresponding to a right inverse R almost quasinilpotent on $\ker D$. Let $A(\mathbb{R}) = \mathbb{R}_+$ or $\mathbb{R}$. If $\{S_h\}_{h \in A(\mathbb{R})} \subset L_0(X)$ is a family of continuous linear operators such that $S_0 = I$ and for $h \in A(\mathbb{R})$ either

$$S_hR^kF = \sum_{j=0}^{k} \frac{h^{k-j}}{(k-j)!}R^jF \quad for\ k \in \mathbb{N}_0$$

or

$$S_h(I - \lambda R)^{-1}F = e^{\lambda h}(I - \lambda R)^{-1}F \quad for\ \lambda \in v_{\mathbb{F}}R,$$

then S_h are said to be *true shifts*. The family $\{S_h\}_{h \in A(\mathbb{R})}$ is a semigroup (or group) with respect to the superposition of operators as a structure operation. □

Observe that, by definitions, there are such true shifts which are R-shifts and R-shifts which are true shifts.

Theorem 3.1. (cf. PR[5], also PR[10]). *Suppose that all conditions of Definition 3.1 are satisfied, $\{S_h\}_{h \in A(\mathbb{R})}$ is a strongly continuous semigroup (group) of true shifts and* *either* $\overline{P(R)} = X$ *or* $\overline{E(R)} = X$. *Then* D *is an infinitesimal generator for* $\{S_h\}_{h\in A(\mathbb{R})}$, *hence* $\overline{\operatorname{dom} D} = X$ *and* $S_h D = DS_h$ *on* $\operatorname{dom} D$. *Moreover, the canonical mapping* κ *defined as*

$$\kappa x = \{x^\wedge(t)\}_{t\in A(\mathbb{R})}, \quad \text{where} \quad x^\wedge(t) = FS_t x \quad (x \in X) \tag{3.1}$$

is an isomorphism (hence separate points) and

$$\kappa D = \frac{\mathrm{d}}{\mathrm{dt}}\kappa, \quad \kappa R = \int_0^t \kappa, \quad \kappa F x = \kappa x|_{t=0},$$

$$\text{and} \quad (\kappa S_h x)(t) = x^\wedge(t+h) \quad \text{for } x \in X,\ t, h \in A(\mathbb{R}).$$

Theorem 3.2. *Suppose that all conditions of Definition 3.1 are satisfied,* $\overline{E(R)} = X$ *and* $\{S_h\}_{h\in A(\mathbb{R})}$ *is a family of true shifts. Then the canonical mapping defined by (3.1) is a topological isomorphism.*

Theorem 3.3. (cf. PR[5], also PR[10]). *Suppose that all conditions of Definition 3.1 are satisfied and* $\{S_h\}_{h\in A(\mathbb{R})}$ *is a family of true shifts. Then for all* $h \in A(\mathbb{R})$ *and* $x \in A_R(D)$ *the series*

$$\mathrm{e}^{hD}x = \sum_{n=0}^{\infty} \frac{h^n}{n!} D^n x \qquad (\text{where we write } \mathrm{e}^h = \sum_{n=0}^{\infty} \frac{h^n}{n!})$$

is convergent,

$$S_h x = \mathrm{e}^{hD}x \quad \text{for } x \in A_R(D) \tag{3.2}$$

and e^{hD} *maps* $A_R(D)$ *into itself* (cf. also Formula(9.1)).

This implies the *Lagrange-Poisson formula* for a right invertible operator D:

$$\Delta_h = \mathrm{e}^{hD} - I \quad \text{on } A_R(D), \quad \text{where} \quad \Delta_h = S_h - I \qquad (h \in A(\mathbb{R})) \tag{3.3}$$

(cf. PR[10]). Note that (under assumptions of Theorem 3.1) $v_{\mathbb{F}}(R_F S_h R) = v_{\mathbb{F}} R$ whenever F is an initial operator for D corresponding to R and S_h are true shifts. This means that the family $\{R_h\}_{h\in A(\mathbb{R})} = \{R - FS_h R\}_{h\in A(\mathbb{R})}$ of right inverses induced by shifts have the same regular values as R (cf. BPR[1]).

Definition 3.2. Let X be a linear metric space. Let $T \in L(X)$ and $x \in X$. The set $\mathcal{O}(T : x) = \{T^n x : n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ is said to be the *orbit* of x with respect to T (cf. Rolewicz R[1]). A continuous linear operator T acting in X is said to be *hypercyclic* if there is an element $x \in X$ (called later *hypercyclic vector*), such that its orbit $\mathcal{O}(T : x)$ is dense in X (cf. Shapiro S[1]). □

Recall the classical Birkhoff Theorem (cf. B[2]):

Theorem 3.4. *There exists an entire function Φ with the property: for every entire function f, every compact subset $\Omega \subset \mathbb{C}$ and every $\varepsilon > 0$ there exists a $\mu \in \mathbb{C}$ such that*

$$\max_{\lambda\in\Omega} |f(\lambda) - \Phi(\lambda+\mu)| < \varepsilon.$$

Looking over the proof, it is easy to observe that Theorem 3.4 can be formulated in a little stronger way. Namely, we have

Theorem 3.4′. (generalized Birkhoff theorem). *Let either $h \in \mathbb{R}$ or $h \in \mathbb{R}_+$. There exists an entire function Φ with the property: for every entire function f, every compact subset $\Omega \subset \mathbb{C}$ and every $\varepsilon > 0$ there exists a positive integer n such that*

$$\max_{\lambda\in\Omega} |f(\lambda) - \Phi(\lambda+nh)| < \varepsilon.$$

In other words, the generalized Birkhoff theorem says that in the space of all entire functions equipped with the topology induced by uniform convergence on compact sets the usual shift operator $(S_h f)(t) = f(t+h)$ $(t, h \in \mathbb{C})$ is hypercyclic and there is an entire function Φ which is a hypercyclic vector for S_h.

It will be shown that this property is much more general. Namely, true shifts generated by a right invertible operator D are hypercyclic and the corresponding hypercyclic vectors are D-analytic elements. In particular, the operator e^{hD} is hypercyclic, whenever $D \in L(X)$ is right invertible. In order to prove it, we need the following

Theorem 3.5. (cf. Rolewicz R[1]). *Let Y be a complete linear metric space with the F-norm $\|\cdot\|$. Let $\mathcal{Y} = (Y)_s$ be the space of all sequences $y = \{y_n\}$, $y_n \in Y$ $(n \in \mathbb{N})$ with the standard norm*

$$\|y\|_s = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\| y_n \|}{1 + \| y_n \|}.$$

Define the forward shift S acting in $\mathcal{Y}$ as follows: $S\{y_n\} = \{y_{n+1}\}$. Then for every $a > 0$ there is a y_a such that $\overline{\mathcal{O}(aS : y_a)} = \mathcal{Y}$, i.e. the forward shift S is hypercyclic in $\mathcal{Y}$ and the corresponding hypercyclic vector is y_a.

Theorem 3.6. (cf. PR[8]). *Suppose that all conditions of Definition 3.2 are satisfied and $\{S_h\}_{h\in A(\mathbb{R})}$ is a family of true shifts. Let $h \in A(\mathbb{R})$ be arbitrarily fixed. Then S_h is a hypercyclic operator and there is a $\chi \in A_R(D)$ which is a hypercyclic vector for S_h.*

Corollary 3.1. (cf. PR[8]). *Suppose that all conditions of Definition 3.2 are satisfied and $\{S_h\}_{h\in A(\mathbb{R})}$ is a family of true shifts. Let $h \in A(\mathbb{R})$ be arbitrarily fixed. Then the operator e^{hD} is hypercyclic and there is a $\chi \in A_R(D)$ which is a hypercyclic vector for e^{hD}.*

4. Multiplicative true shifts.

We shall consider now true shifts in the case when X is not only a linear metric space, but also a commutative algebra. Then we have

Theorem 4.1. (cf. PR[10]). *Let all conditions of Definition 3.1 be satisfied and let $D \in \mathsf{A}(X)$. Let $\{S_h\}_{h\in A(\mathbb{R})}$ be a family of true shifts. Let $\mathcal{A}_R(D) = \{x, y \in A_R(D) : xy \in A_R(D)\}$. Then for all $x, y \in \mathcal{A}_R(D)$, $h \in A(\mathbb{R})$*

$$S_h(xy) - (S_h x)(S_h y) = \sum_{n=0}^{\infty} \frac{h^n}{n!}[D^n(xy) - \sum_{k=0}^{n} \binom{n}{k}(D^k x)(D^{n-k}y)].$$

Theorem 4.2. (cf. PR[10]). *Let all conditions of Definition 3.1 be satisfied and let $D \in \mathsf{A}(X)$. Let $\{S_h\}_{h\in A(\mathbb{R})}$ be a family of true shifts. Then S_h are multiplicative on $\mathcal{A}_R(D)$ for all $h \in A(\mathbb{R})$: $S_h(xy) = (S_h x)(S_h y)$ for all $x, y \in \mathcal{A}_R(D)$, if and only if $D|_{\mathcal{A}_R(D)}$ satisfies the Leibniz condition, i.e. $D(xy) = xDy + yDx$.*

Note that in Leibniz algebras $xy \in A_R(D)$ whenever $x, y \in A_R(D)$. Thus in this case $\mathcal{A}_R(D) = A_R(D)$ and we have

Corollary 4.1. (cf. PR[6]). *Let all conditions of Definition 3.1 be satisfied and let $D \in \mathsf{A}(X)$. Let $\{S_h\}_{h\in A(\mathbb{R})}$ be a family of true shifts. If the restriction $D|_{A_R(D)}$ satisfies the Leibniz condition, then S_h are multiplicative on $A_R(D)$ for all $h \in A(\mathbb{R})$.*

Theorem 4.3. *Suppose that X is a complete linear metric locally convex space ($\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$) and a Leibniz D_i- algebra ($i = 1, 2$), $D_i \in R(X)$ are closed, $\ker D_i \neq \{0\}$ and F_i are continuous initial operators for D_i corresponding to a right inverse R_i almost quasinilpotent on $\ker D_i$, respectively. Let $A(\mathbb{R}) = \mathbb{R}_+$ or $\mathbb{R}$. Suppose that $\{S_{i,h}\}_{h\in A(\mathbb{R})}$ are strongly continuous semigroups (groups) of true shifts for D_i ($i = 1, 2$) respectively, and either $\overline{P(R_i)} = X$ or $\overline{E(R_i)} = X$ for $i = 1, 2$. Let κ_1, κ_2 be the canonical mappings for D_1, D_2, respectively. Then κ_i are algebra isomorphisms on $A_{R_i}(D_i)$ ($i = 1, 2$) and*

$$\kappa_1 D_1 \kappa_1^{-1} = \frac{\mathrm{d}}{\mathrm{d}t} = \kappa_2 D_2 \kappa_2^{-1} \quad \textit{on } X. \tag{4.1}$$

Corollary 4.1. *Suppose that all assumptions of Theorem 4.3 are satisfied. Then the operators satisfying the Leibniz condition are uniquely determined as $\frac{\mathrm{d}}{\mathrm{d}t}$ up to isomorphisms determined by the canonical mappings.*

It means that true shifts are, indeed, true.

5. True shifts in commutative algebras with logarithms.

Suppose that $D \in \mathsf{A}(X)$. Let a multifunction $\Omega : \mathrm{dom}\ D \longrightarrow 2^{\mathrm{dom}\ D}$ be defined as follows:

$$\Omega u = \{x \in \mathrm{dom}\ D : Du = uDx\} \quad \textit{for } u \in \mathrm{dom}\ D. \tag{5.1}$$

The equation

$$Du = uDx \quad \textit{for } (u, x) \in \mathrm{graph}\ \Omega. \tag{5.2}$$

is said to be the *basic equation*. Clearly,

$$\Omega^{-1}x = \{u \in \text{dom } D : Du = uDx\} \quad \textit{for } x \in \text{ dom D}.$$

The multifunction Ω is well-defined and dom $\Omega \supset \ker D \setminus \{0\}$.

Suppose that $(u, x) \in$ graph Ω, L is a selector of Ω and E is a selector of Ω^{-1}. By definitions, $Lu \in$ dom Ω^{-1}, $Ex \in$ dom Ω and the following equations are satisfied: $Du = uDLu$, $DEx = (Ex)Dx$.

Any invertible selector L of Ω is said to be a *logarithmic mapping* and its inverse $E = L^{-1}$ is said to be a *antilogarithmic mapping*. By $G[\Omega]$ we denote the set of all pairs (L, E), where L is an invertible selector of Ω and $E = L^{-1}$. For any $(u, x) \in$ dom Ω and $(L, E) \in G[\Omega]$ elements Lu, Ex are said to be *logarithm* of u and *antilogarithm* of x, respectively. The multifunction Ω is examined in PR[7].

Clearly, by definition, for all $(L, E) \in G[\Omega]$, $(u, x) \in$ graph Ω we have

$$ELu = u, \quad LEx = x; \qquad DEx = (Ex)Dx, \quad Du = uDLu. \tag{5.3}$$

A logarithm of zero is not defined. If $(L, E) \in G[\Omega]$ then $L(\ker D \setminus \{0\}) \subset \ker D$, $E(\ker D) \subset \ker D$. In particular, $E(0) \in \ker D$.

If $D \in R(X)$ then logarithms and antilogarithms are uniquely determined up to a constant.

Let $D \in \mathsf{A}(X)$ and let $(L, E) \in G[\Omega]$. A logarithmic mapping L is said to be of the *exponential type* if $L(uv) = Lu + Lv$ for $u, v \in$ dom Ω. If L is of the exponential type then $E(x + y) = (Ex)(Ey)$ for $x, y \in$ dom Ω^{-1}. We have proved that a logarithmic mapping L is of the exponential type if and only if X is a *Leibniz commutative algebra* (cf. PR[7]). Moreover, $Le = 0$, i.e. $E(0) = e$. In Leibniz commutative algebras with $D \in R(X)$ a necessary and sufficient conditions for u to belong to dom Ω is that $u \in I(X)$ (cf. PR[9]).

By $\mathbf{Lg}(D)$ we denote the class of these commutative algebras with $D \in R(X)$ and with unit $e \in$ dom Ω for which there exist invertible selectors of Ω, i.e. there exist $(L, E) \in G[\Omega]$. By $\mathsf{L}(D)$ we denote the class of these commutative Leibniz algebras with unit $e \in$ dom Ω for which there exist invertible selectors of Ω. By these definitions, $X \in \mathbf{Lg}(D)$ is a Leibniz algebra if and only if $X \in \mathsf{L}(D)$ and $D \in R(X)$. This class we shall denote by $\mathit{L}(D)$. It means that $\mathit{L}(D)$ is the class of these commutative Leibniz algebras with $D \in R(X)$ and with unit $e \in$ dom Ω for which there exist invertible selectors of Ω, i.e. there exist $(L, E) \in G[\Omega]$.

If $\ker D = \{0\}$ then either X is not a Leibniz algebra or X has no unit. Thus, by our definition, if $X \in \mathit{L}(D)$ then $\ker D \neq \{0\}$, i.e. the operator D is right invertible but not invertible.

Theorem 5.1. *Suppose that $X \in \mathit{L}(D)$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$, $(L, E) \in G[\Omega]$ and A is an algebra isomorphism of X. Let $D' = A^{-1}DA$ and let Ω' : dom $D' \longrightarrow 2^{\text{dom } D'}$ be defined as follows:*

$$\Omega' u = \{x \in \text{dom } D' : D'u = uD'x\} \quad \textit{for } u \in \text{dom } D'. \tag{5.4}$$

Then there are $(L', E') \in G[\Omega']$ *and* $L' = A^{-1}LA$, $E' = A^{-1}EA$.

Theorem 5.2. *Suppose that* X *is a complete linear metric locally convex space* ($\mathbb{F} = \mathbb{C}$ *or* $\mathbb{F} = \mathbb{R}$), $D \in R(X)$ *is closed,* $\ker D \neq \{0\}$ *and* F *is a continuous initial operator for* D *corresponding to a right inverse* R *almost quasinilpotent on* $\ker D$. *Let* $A(\mathbb{R}) = \mathbb{R}_+$ *or* $\mathbb{R}$, $\{S_h\}_{h\in A(\mathbb{R})}$ *is a strongly continuous semigroup (group) of true shifts and either* $\overline{P(R)} = X$ *or* $\overline{E(R)} = X$. *Suppose, moreover, that* $X \in \mathsf{L}(D)$, $(L, E) \in G[\Omega]$.*Write:* $D' = \kappa' \frac{\mathrm{d}}{\mathrm{d}t} \kappa'^{-1}$. *Let* Ω' *be defined by (5.4), where* κ *is the canonical mapping defined by (3.5). Then there are* $(L', E') \in G[\Omega']$ *such that* $L' = \ln$, $E'(\cdot) = \exp(\cdot)$.

Note that for $X \in \mathsf{L}(D)$, $(L, E) \in G[\Omega]$ we have

$$FE = EF, \quad FL = LF \quad \text{whenever} \quad F \in \mathcal{F}_D \tag{5.5}$$

(cf. PR[15]).

6. Periodic problems.

By Theorem 2.1, true shifts are D-invariant, i.e. $S_hD = DS_h$ on $\operatorname{dom} D$ for all $h \in A(\mathbb{R})$.

Theorem 6.1. *Suppose that* $X \in \mathbf{Lg}(D)$ *has the unit* e *and is a complete linear metric space over* $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$), $(L, E) \in G[\Omega]$, $D \in R(X)$ *is closed,* $\{S_h\}_{h\in A(\mathbb{R})}$ *is a family of true shifts induced by an* $R \in \mathcal{R}_D \cap AQN(\ker D)$, $g = Re$, $\lambda g \in \operatorname{dom} \Omega^{-1}$ *for* $\lambda \in v_{\mathbb{F}}R$. *Then* $S_hE(\lambda g) = \mathrm{e}^{\lambda h}E(\lambda g)$ *whenever* $\lambda \in v_{\mathbb{F}}R$, $h \in A(\mathbb{R})$.

Theorem 6.2. *Suppose that* $X \in \mathbf{Lg}(D)$ *has the unit* e *and is a complete linear metric space over* $\mathbb{C}$, $(L, E) \in G[\Omega]$, $D \in R(X)$ *is closed,* $\{S_h\}_{h\in\mathbb{R}}$ *is a family of true shifts induced by an* $R \in \mathcal{R}_D \cap AQN(\ker D)$, S_{-r} *is multiplicative for an* $r \in \mathbb{R}$, $\omega = Nr$ ($N \in \mathbb{N}$), $g = Re$ *and* $\frac{2\pi i}{\omega} g \in \operatorname{dom} \Omega^{-1}$ *whenever* $\frac{2\pi i}{\omega} \in v_{\mathbb{C}}R$. *Let*

$$\mathsf{E}_{nr} = X_{S^n_{-r}} = \{x \in X : S^n_{-r}x = x\} = \{x \in X : S_{-nr}x = x\} \neq \emptyset \quad (n \in \mathbb{N}) \tag{6.1}$$

and let $\mathsf{E}_{(j)} = P_j\mathsf{E}_\omega = P_j\mathsf{E}_{Nr}$, *where* P_j *are determined by Formulae (2.6). If* $u = E(-\frac{2\pi i}{\omega} jg)v$, *where* $v \in \mathsf{E}_r$, *i.e.* $S_{-r}v = v$ ($j = 1, ..., N$), *then* $u \in \mathsf{E}_{(j)}$, *i.e. if* $S_{-r}u = \varepsilon^j u$.

In order to prove the necessity of the condition given in Theorem 6.1 we need the assumption that X is a Leibniz algebra. Namely, we have

Theorem 6.3. *Suppose that* $X \in \mathsf{L}(D)$ *has the unit* e *and is a complete linear metric space over* $\mathbb{C}$, $(L, E) \in G[\Omega]$, $D \in R(X)$ *is closed,* $\{S_h\}_{h\in\mathbb{R}}$ *is a family of true shifts induced by an* $R \in \mathcal{R}_D \cap AQN(\ker D)$, S_-r *is multiplicative for an* $r \in \mathbb{R}$, $\omega = Nr$ ($N \in \mathbb{N}$), $g = Re$ *and* $\frac{2\pi i}{\omega} g \in \operatorname{dom} \Omega^{-1}$ *whenever* $\frac{2\pi i}{\omega} \in v_{\mathbb{C}}R$. *Let* $\mathsf{E}_{(j)} = P_j\mathsf{E}_\omega = P_j\mathsf{E}_{Nr}$, *where* P_j *and* E_{Nr} *are determined by Formulae (2.4) and (6.1), respectively. Then* $u = E(-\frac{2\pi i}{\omega} jg)v$, *where* $v \in \mathsf{E}_r$, *i.e.* $S_{-r}v = v$ ($j = 1, ..., N$), *only if* $u \in \mathsf{E}_{(j)}$, *i.e.* $S_{-r}u = \varepsilon^j u$.

An immediate consequence of Theorems 6.2, 6.3 and the decomposition (2.4) onto the direct sum is

Corollary 6.1. *Suppose that all assumptions of Theorem 6.3 are satisfied. Then $x \in \mathsf{E}_\omega$ if and only if*

$$x = \sum_{j=1}^{N} E(-\frac{2\pi i j}{\omega} g) v_j, \quad \textit{where } S_{-r} v_j = v_j \quad (j = 1, ..., N)$$

(cf. PR[2], also PR[7]).

Note 6.1. In the classical case of the space $X = C(\mathbb{R})$ over $\mathbb{C}$, when $D = \frac{\mathrm{d}}{\mathrm{d}t}$, $(S_h x)(t) = x(t+h)$ for $x \in X$, $t, h \in \mathbb{R}$, we find that E_ω, $\omega = Nr$, is the space of of *ω-periodic functions*. The operator S_{-r} is an involution of order N on E_ω for $(S_{-r}^N x)(t) = x(t - Nr) = x(t - \omega) = x(t)$. Thus any ω-periodic function $x \in X$ is of the form

$$x = \sum_{j=1}^{N} \mathrm{e}^{2\pi i j t/\omega} v_j, \quad \textit{where } v_1, ..., v_N \textit{ are } r\textit{-periodic functions.}$$

Functions of the form $\mathrm{e}^{\lambda t} v$, where $\lambda \in \mathbb{C}$, v is a periodic function, and their linear combinations are said to be *exponential-periodic functions*. Elements of the form $E(\lambda g)v$, where $\lambda \in \mathbb{C}$, v is a periodic element, and their linear combinations are said to be *exponential-periodic elements*. Recall that in Corollary 6.2 X is a Leibniz algebra, hence antilogarithms are exponentials. For an arbitrary $M \in \mathbb{N}$ and a true shift S_h we denote the *space of exponential-periodic elements* by

$$X_{EP}(h; \lambda_1, ..., \lambda_M) =$$

$$= \operatorname{lin} \left\{ E(\lambda_j g) v_j : v_j \in X_{S_h}; \lambda_j \in \mathbb{C}; \lambda_m \neq \lambda_j + \frac{2\pi i k}{h}, m \neq j; k \in \mathbb{Z} \quad (m, j = 1, ..., M) \right\}.$$

By Corollary 6.1, $x \in \mathsf{E}_\omega$ if and only if $x \in X_{EP}(h; -\frac{2\pi i}{\omega}, ..., -\frac{2\pi i N}{\omega})$ $(S_h = S_{-r})$. More details about spaces of exponential-periodic functions and their applications can be found in PR[2], PR[7]. □

In order to generalize the Floquet theorem, we use Theorem 6.1 and Corollary 6.1. The classical Floquet theorem says that *every linear ordinary differential equation with periodic coefficients has at least one non-zero exponential-periodic solution*, i.e. a solution of the form $e^{\lambda t} u(t)$, where $u \neq 0$ is a periodic function and $\lambda \in \mathbb{C}$ is properly chosen (cf. Arscott Ar[1], Ince In[1]).

Proposition 6.1. *Suppose that all assumptions of Theorem 6.3 are satisfied. Write*

$$Q(D) = \sum_{k=0}^{K} Q_k D^k, \quad Q_k \in L_0(X),\ S_{-r} Q_k = Q_k S_{-r}\ (k = 0, 1, .., K), \tag{6.3}$$

i.e. Q_k are S_{-r}-periodic. Suppose, moreover, that $v \in \operatorname{dom} D^K$ is an S_{-r}-periodic element and $u = E(\lambda g)v \in \ker Q(D)$. Then

$$\lambda = \frac{2\pi i}{\omega} j \quad \textit{where } j \in \mathbb{Z} \textit{ is arbitrary.}$$

Proposition 6.2. *Suppose that all assumptions of Proposition 6.1 are satisfied. Let $\lambda = \frac{2\pi i}{\omega} j$, where $j \in \mathbb{Z}$ is arbitrary. Then $Q(D+\lambda I)v = 0$. Conversely, if $Q(D+\lambda I)v = 0$, then $u = E(\lambda g)v \in \ker Q(D)$.*

Theorem 6.4. (Generalized Floquet Theorem). *Suppose that all assumptions of Theorem 6.3 are satisfied. Let D_r, F_r, R_r^0 and $Q(D)$ be defined by Formulae (2.8). Write*

$$\widetilde{Q}_k = \sum_{k=m}^{K} \binom{m}{k} \lambda^{m-k} Q_m \quad (k = 0, 1, ..., K), \tag{6.4}$$

$$\widetilde{Q}(t,s) = \sum_{k=0}^{K} \widetilde{Q}_k t^k s^{K-k}, \quad \widetilde{Q}(t) = \widetilde{Q}(t,1). \tag{6.5}$$

If the operator $\widetilde{Q}(I, R_{-r}^0)$ is invertible in the space E_r then the equation

$$Q(D)x = 0 \tag{6.6}$$

has exponential-periodic solutions which are of the form

$$x = E(\lambda g)v, \quad \textit{where } S_{-r}v = v, \ \lambda = \frac{2\pi i}{\omega} j \quad (j \in \mathbb{Z}), \tag{6.7)}$$

$$v = (R_{-r}^0)^K [\widetilde{Q}(I, R_{-r}^0)]^{-1} \sum_{k=0}^{K} \widetilde{Q}_k \sum_{j=0}^{k-1} r^{-k+j} z_j, \tag{6.8}$$

$$z_0, ..., z_{K-1} \in \ker D \ \textit{are arbitrary.}$$

In order to reduce an equation with an involution of order N, in particular, an equation with a true shift in the space of periodic elements, we need

Proposition 6.3. (cf. PR[7]). *Suppose that X is a linear space over $\mathbb{C}$, $D \in R(X)$, $S \in L_0(X)$ commutes with D: $SD = DS$ on dom D, the operators $Q_{km} \in L_0(X)$ are S-periodic, i.e. $SQ_{km} = Q_{km}S$ ($k = 0, 1, ..., N-1$; $N \geq 2$; $m = 0, 1, ..., M$). Write*

$$Q_m(S) = \sum_{k=0}^{N-1} Q_{km} S^k, \quad Q(D,S) = \sum_{m=0}^{M} D^{m+M_1} \quad (M_1 \in \mathbb{N}_0)). \tag{6.9}$$

If $X_{S^N} \neq \emptyset$ then

$$Q(D,S) = \sum_{j=1}^{N} Q(D, \varepsilon^j) P_j \quad \textit{on } X_{S^N}, \quad \textit{where } \varepsilon = \mathrm{e}^{\frac{2\pi i}{N}}, \tag{6.10}$$

and projectors $P_1, ..., P_N$ defined by (2.4) commute each with another.

Proposition 6.4. (cf. PR[7]). *Suppose that all assumptions of Proposition 6.3 are satisfied. Then the equation*

$$Q(D,S)x = y, \quad y \in X_{S^N}, \tag{6.11}$$

is equivalent in the space X_{S^N} to N independent equations

$$Q(D,\varepsilon^j)x_j = y_j, \quad \textit{where } x_j = P_j x, y_j = P_j y \in X_{(j)},\ (j = 1, ..., N), \tag{6.12}$$

and $X_{(j)}$ are defined by the decomposition (2.5), x_j, y_j are defined by Formulae (2.6).

Theorem 6.4. (cf. PR[7]) *Suppose that all assumptions of Proposition 6.3 are satisfied. If each of Equations (6.12) has a solution $x_j \in X_j$ $(j = 1, ..., N)$ then Equation (6.11) has a solution $x = x_1 + ... + x_N \in X_{S^N}$. Conversely, if Equation (6.11) has an S^N-periodic solution x then the j-th Equation (6.12) has a solution $x_j = P_j x \in X_j$.*

Theorem 6.4 has several applications. We may use this theorem when $S = S_h$ is a true shift (under appropriate additional assumptions). Some of these applications are quite far from classical differential-difference equations (cf. PR[2], PR[7]). For instance, this method can be used for a reduction of a stochastic differential-difference equation to a stochastic differential equation in order to find its periodic solutions (cf. Wilkowski Wi[1]). In a similar manner one can consider the operator

$$Q(D,S) = D^{M_1} \sum_{m=0}^{M} Q_m(S) D^m.$$

Another possibility is given by

Theorem 6.5. *Suppose that all assumptions of Theorem 6.3 are satisfied. Consider a nonlinear equation*

$$Dx = G(x, S_{-\omega_1}x, ..., S_{-\omega_m}x), \tag{6.13}$$

where the mapping

$$G \ : \ \underbrace{X \times ... \times X}_{m-times} \longrightarrow X$$

is continuous in each variable and all numbers $\omega_1, ..., \omega_m$ are commensurable, i.e. there exist an $r \in \mathbb{R} \setminus \{0\}$ and $\eta_1, ..., \eta_m \in \mathbb{Z}$ such that $\omega_j = \eta_j r$ $(j = 1, ..., m)$. We admit $\omega_0 = \eta_0 = 0$. Let N be a common multiple of positive integers $|\eta_1|, ..., |\eta_m|$ and let $\omega = Nr$. Then Equation (6.13) has a solution $x \in \mathsf{E}_\omega$ if and only if the following system of N equations without shifts

$$D\tilde{x}_j = \frac{2\pi i j}{\omega}\tilde{x}_j + \widetilde{G}_j(\tilde{x}_1, ..., \tilde{x}_N) \quad (j = 1, ..., N), \tag{6.14}$$

has an r-periodic solution, where

$$(6.15)\qquad \widetilde{G}(\tilde{x}_1,...\tilde{x}_N)=G\big(\sum_{k=1}^{N}\ \mathrm{e}^{2\pi k\eta_0}E(\frac{2\pi ik}{\omega}g)\tilde{x}_k,...,\sum_{k=0}^{N}\ \mathrm{e}^{2\pi k\eta_m}E(\frac{2\pi ik}{\omega}g)\tilde{x}_k\big),$$

$$\widetilde{G}_j(\tilde{x}_1,...,\tilde{x}_N)=E(\frac{2\pi ij}{\omega}g)P_j\widetilde{G}(\tilde{x}_1,...,\tilde{x}_N)\quad (j=1,...,N).$$

If it is the case, then

$$(6.16)\qquad x=\sum_{j=1}^{N}\ E(\frac{2\pi ij}{\omega}g)\tilde{x}_j,$$

where $(\tilde{x}_1,...,\tilde{x}_N)$ is an r-periodic solution of the system (6.14).

Theorem 6.5 can be used also in order to study linear equations with shifts, in particular, differential-difference equations with periodic coefficients. In a similar manner one can consider other periodic problems (cf. PR[2], also PR[7], Section 15). Note that in Theorem 6.5 we obtain a system of equations which, in general, are not independent.

Consider the space of exponential-periodic elements (defined in Note 6.1 in the classical case) for an arbitrary $M\in\mathbb{N}$ and a true shift S_h

$$(6.17)\qquad X_{EP}(h;\lambda_1,...,\lambda_M)=$$

$$=\mathrm{lin}\big\{E(\lambda_j g)v_j : v_j\in X_{S_h};\lambda_j\in\mathbb{C};\ \lambda_m\neq\lambda_j+\frac{2\pi ik}{h},\ m\neq j;\ k\in\mathbb{Z}\ \ (m,j=1,...,M)\big\}$$

We have

Theorem 6.6. *Suppose that all assumptions of Theorem 6.3 are satisfied. Let $\lambda_j=\frac{2\pi i}{\omega}j$ $(j=1,...,N)$, and let $h=-r$. Then S_h is an algebraic operator on the space $X_{EP}(h;\lambda_1,...,\lambda_N)$ with the characteristic polynomial*

$$P(t)=\prod_{j=1}^{N}\ (t-t_j),\quad where\ t_j=e^{\lambda_j h}\ (j=1,...,N)$$

(with single characteristic roots).

An immediate consequence of Theorem 6.6 is

Corollary 6.2. *Suppose that all assumptions of Theorem 6.6 are satisfied. Then $X=X_{EP}(h;\lambda_1,...,\lambda_N)$ is a direct sum of the eigenspaces of the operator S_h corresponding to the eigenvalues $t_1,...,t_N$:*

$$(6.18)\qquad X_{EP}(h;\lambda_1,...,\lambda_N)=X_1\oplus...\oplus X_N,\quad where\ S_hx_j=t_jx_j\ for\ x_j\in X_j,$$

$$X_j = P_j X,\ P_j = P_j(S_h), \quad P_j(t) = \prod_{k=1, k\neq j}^{N} \frac{t - t_k}{t_j - t_k}$$

Consider now combinations of exponential-periodic elements and D-polynomials. However, in that case S_h is an algebraic operator with *multiple* characteristic roots. Note that polynomial-periodic solutions of differential-difference equations firstly have been studied by WŁODARSKA-DYMITRUK (cf. WD[1]). A generalization for polynomial-exponential-periodic elements and shifts induced by right invertible operators was given in PR[7]).

Proposition 6.4. *Let $D \in R(X)$, $\ker D \neq \{0\}$ and let F be an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Let $\{S_h\}_{h\in\mathbb{R}} \subset L_0(X)$ be a family of R-shifts, i.e. such operators that $S_0 = I$ and for $h \in \mathbb{R}$*

$$S_h R^k F = \sum_{j=0}^{k} \frac{h^{k-j}}{(k-j)!} R^j F \quad for\ k \in \mathbb{N}_0 \tag{6.19}$$

(cf. Proposition 6.1). *Then for all $h \in \mathbb{R}$, $z \in \ker D$, $k, n \in \mathbb{N}$, we have*

$$S_h^n R^k z = \sum_{m=0}^{k} h_{k,m}^{(n)} R^m z, \quad where \tag{6.20}$$

$$h_{k,j}^{(1)} = \frac{1}{(k-j)!} h^{k-j} \quad (j = 0, 1, ..., k),\ n \geq 1, \tag{6.21}$$

$$h_{k,j}^{(n+1)} = \sum_{m=1}^{k} h_{k,m}^{(n)} h_{m,j}^{(1)}, \quad h_{k,k}^{(n+1)} = 1. \tag{6.22}$$

By induction, we get

Proposition 6.5. *Suppose that all assumptions of Proposition 6.4 are satisfied. Write*

$$P^{(n+1)}(t) = \sum_{m=0}^{M} p_{M,m} t^m, \quad P^{(0)}(t) = P(t) = \sum_{m=0}^{M} p_m t^m, \tag{6.23}$$

$$p_{M,m}^{(0)} = p_m, \quad p_{M,j}^{(n+1)} = \sum_{m=j}^{M} p_{M,m}^{(n)} h_{m,j}^{(n)}, \quad p_{M,M}^{(n+1)} = 1, \tag{6.24}$$

$$p_m \in \mathbb{C},\ p_M = 1 \quad (j, m = 0, 1, ..., M;\ n \in \mathbb{N}_0),$$

where $h_{m,j}^{(n)}$ *are defined by Formulae (6.21), (6.22). Then* $p_{M,M}^{(n)} = 1$ *for* $M \in \mathbb{N}$, $n \in \mathbb{N}_0$ *and*

$$(6.25) \qquad S_h^n P^{(n)}(R) = P^{(n+1)}(R) \quad \textit{for all } h \in \mathbb{R},\ z \in \ker D \quad (n \in \mathbb{N}_0).$$

Proposition 6.6. *Suppose that all assumptions of Proposition 6.1 are satisfied. Then for every* $h \in \mathbb{R}$ *and* $k \in \mathbb{N}_0$ *the operator* S_h *is algebraic on the space* $\ker D^{k+1}$ *with the characteristic polynomial* $P(t) = (t-1)^{k+1}$, *i.e.*

$$(S_h - I)^{k+1} R^k z = 0 \quad \textit{for every } h \in \mathbb{R},\ z \in \ker D,\ k \in \mathbb{N}_0.$$

Writing

$$\widetilde{h}_{k,\nu}^{j+1} = \sum_{\mu=\nu}^{k-j} h_{k,\mu+j}^{(1)} h_{\mu+j,\nu}^{(1)} \quad \textit{for } j = 1, ..., k;\ \nu = 0, 1, ..., k-j,$$

we get

$$(6.26) \qquad (S_h - I)^2 R^k z = \sum_{\nu=0}^{k-2} \widetilde{h}_{k,\nu}^{(1)} R^\nu z,$$

Corollary 6.3. *Let* $D \in R(X)$, $\ker D \neq \{0\}$ *and let* F *be an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$. *Let* $D \in \mathsf{A}(X)$. *Let* $\{S_h\}_{h\in\mathbb{R}} \subset L_0(X)$ *be a family of multiplicative R-shifts* (cf. Proposition 6.5). *Then*

$$(S_h^N - I)^{k+1}(vR^k z) = 0 \quad \textit{for all } z \in \ker D,\ v \in X_{S_h^N},\ k, N \in \mathbb{N},$$

i.e. S_h *is an algebraic operator on the space*

$$\mathrm{lin}\ \{vR^k z : z \in \ker D,\ S_h^N v = v\ (k \in \mathbb{N}_0)\}$$

with the characteristic polynomial $P(t) = (t^N - 1)^{k+1}$ *and with the characteristic roots* $\varepsilon^j = \mathrm{e}^{2\pi i j/N}$ $(j = 0, 1, ..., N)$, *each of multiplicity* $k+1$.

Corollary 6.4. *Suppose that all assumptions of Corollary 6.3 are satisfied. Write for* $M, n_0, ..., n_M$, $n_j \neq n_k$ *if* $j \neq k$, $h \in \mathbb{R}$

$$(6.27) \qquad X_{PP}(h; n_0, ..., n_M) =$$

$$= \{u = \sum_{m=0}^{M} v_m R^m z_m : z_m \in \ker D,\ S_h^{n_m} v_m = v_m\ (m = 0, 1, ..., M)\}.$$

Then

$$(6.28) \qquad P(S_h) = 0 \quad \textit{on } X_{PP}(h; n_0, ..., n_M), \quad \textit{where } P(t) = \prod_{m=0}^{M} (t^{n_m} - I)^{M+1},$$

i.e. S_h *is an algebraic operator on the space* $X_{PP}(h; n_0, ..., n_M)$ *with the characteristic polynomial* $P(t)$. *The characteristic roots are* $\varepsilon_m^j = \mathrm{e}^{\frac{2\pi ij}{n_m}}$ $(m = 0, 1, ..., M;\ j = 0, 1, ..., n_m - 1)$, *each of the multiplicity* $M + 1$. *Thus* $X_{PP}(h; n_0, ..., n_M)$ *is a direct sum of principal spaces* X_{jm} *such that* $(S_h - \varepsilon_m^j I)^{M+1} = 0$ *on* X_{jm} $(j = 0, 1, ..., n_m - 1;\ m = 0, 1, ..., M)$.

Elements of spaces $X_{PP}(h; n_0, ..., n_M)$ are said to be *D-polynomial-periodic elements.*

Theorem 6.7. *Suppose that* $X \in \mathsf{L}(D)$ *has the unit* e *and is a complete linear metric space over* $\mathbb{C}$, $(L, E) \in G[\Omega]$, $D \in R(X)$ *is closed,* $\{S_h\}_{h\in\mathbb{R}}$ *is a family of multiplicative true shifts induced by an* $R \in \mathcal{R}_D \cap AQN(\ker D)$, $g = Re$, $M, K_j, N_j \in \mathbb{N}$ *and* $\lambda_j g \in \mathrm{dom}\ \Omega^{-1}$ *whenever* $\lambda_j \in v_{\mathbb{C}}R$ $(\lambda_j \neq \lambda_k + 2\pi il/h$ *if* $j \neq k$; $l \in \mathbb{Z}$; $j, k = 0, 1, ..., M)$. *Then* S_h *is an algebraic operator on the space of D-polynomial-exponential-periodic elements:*

$$X_{PEP}(h; \lambda_j; K_j; N_j; M) \tag{6.29}$$

$$= \{x = \sum_{j=0}^{M} \big(\sum_{k=0}^{K_j} v_{jk} R^k z_{jk}\big) E(\lambda_j g) : z_{jk} \in \ker D,\ v_{jk} \in X_{S_h^{N_j}},\ k = 0, 1, ..., K_j\}$$

with the characteristic polynomial

$$P(t) = \prod_{j=0}^{M} (t^{N_j} - t_j^{N_j})^{K_j+1}, \quad \textit{where } t_j = \mathrm{e}^{\lambda_j h}\ (j = 0, 1, ..., M). \tag{6.30}$$

The characteristic roots of the polynomial $P(t)$ *are*

$$t_{jm} = t_j \varepsilon_j^m, \quad \textit{where } \varepsilon_j = \mathrm{e}^{\frac{2\pi i}{N_j}} \quad (m + 0, 1, ..., N_j - 1;\ , j = 0, 1, ..., M) \tag{6.31}$$

of the multiplicity $K_j + 1$, *respectively.*

Corollary 6.5. *Suppose that all assumptions of Theorem 6.7 are satisfied. Then a principal space corresponding to the root 1 of the multiplicity* $k + 1$ *is* $\mathrm{lin}\ \{R^j z : z \in \ker D,\ j = 0, 1, ..., k\}$.

We can solve now equations with shifts in the spaces

$$X_{PP}(h; n_0, ..., n_M) \quad \textit{and} \quad X_{PEP}(h; \lambda_j, Kj; N_j; M)$$

in the same manner as we did it in the space of periodic elements and in $X_{EP}(h; \lambda_j, ..., \lambda_M)$ (cf. for instance, Proposition 6.2, Theorem 6.4).

Note 6.2. Theorems 6,6, 6.7 and Corollary 6.2 are proved for Leibniz algebras. A modified proof could be used for quasi Leibniz algebras, i.e. commutative algebras with the product rule $D(xy) = xDy + yDx + d(Dx)(Dy)$ for $x, y \in \mathrm{dom}\ D$, where $d \neq 0$ is a scalar.We have only remember that in this case logarithms (provided that they exist) are not of of the exponential type, but they satisfy the following equation

$$DL(uv) = D(Lu + Lv) + d(DLu)(DLv) \quad for\ u, v \in \mathrm{dom}\ \Omega,\ (L, E) \in G[\Omega].$$

Hence for antilogarithms we have the equation

$$(Ex)(Ey) = E\{x + y + dR[(Dx)(Dy)] + z\} \quad \textit{where } z \in \ker D,\ x = Lu,\ y = Lv.$$

We cannot use similar arguments for simple Duhamel algebras, i.e. algebras with the product rule $D(xy) = xDy$, what can be written $D(xy) = \frac{1}{2}(xDy+yDx)$ for $x, y \in \mathrm{dom}\, D$. Logarithmic (hence also antilogarithmic) mappings in that case do not exist. $\square$

It should be mentioned that R-shifts S_h defined by Formula (2.7) correspond to R-shifts S_{-h} studied in PR[5], PR[7]. This change of sign is not essential, however, it is convenient in order to have a unified approach to different questions considered here. Some results of this chapter are true also in the case when $h \in A(\mathbb{R}) = \mathbb{R}_+$.

7. Harmonic logarithms.

We shall use the so-called *Roman factorial* defined as

$$[n]! = \begin{cases} n! & \text{if } n \geq 0,\ (0! = 0); \\ \frac{(-1)^{n+1}}{(-n-1)!} & \text{if } n < 0 \end{cases} \qquad (n \in \mathbb{N}) \tag{7.1}$$

and *Roman coefficients*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} \qquad (n, k \in \mathbb{Z}) \tag{7.2}$$

(cf. Roman and Rota RR[10]). In particular, we have $\begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ -k \end{bmatrix} = \frac{(-1)^{k+1}}{k!}$ for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 7.1. (cf. PR[9]). Suppose that $X \in \mathbf{Lg}(D)$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$), F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$ and there is $(L, E) \in G_{R,1}[\Omega]$ *. We admit the following convention: $R^{-n}L = D^n L$ ($n \in \mathbb{N}$) for $FL = 0$. *Harmonic logarithms of order* $p \in \mathbb{N}_0$ are elements

$$\lambda_n^{(p)}(u) = [n]!R^n(Lu)^p \quad \textit{for } u \in I(X) \cap \mathrm{dom}\, \Omega,\ n \in \mathbb{Z},\ p \in \mathbb{N}_0. \tag{7.3}$$

$\square$

For instance, if $g = Re \in I(X) \cap \mathrm{dom}\, \Omega$, then

$$\lambda_0^{(p)}(g) = (Lg)^p \quad (p \in \mathbb{N}_0) \quad ; \quad \lambda_n^{(1)}(g) = \begin{cases} g^n[Lg - (1 + \frac{1}{2} + ... + \frac{1}{n})e] & \text{if } n \in \mathbb{N}_0; \\ g^{-n} & \text{if } -n \in \mathbb{N}. \end{cases}$$

Note that harmonic logarithms are not logarithms defined in Section 5, although they are constructed with the use of these logarithms.

* Let $(L, E) \in G[\Omega]$. If $FD^jL = 0$ for $j = 0, ..., m-1$ then (L, E) is said to be m-*normalized* by R and we write $(L, E) \in G_{R,m}[\Omega]$.

Theorem 7.1. (cf. PR[4]) *Suppose that X is a complete linear metric locally convex space ($\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$), $D \in R(X)$ is closed, $\ker D \neq \{0\}$ and F is a continuous initial operator for Dcoresponding to a right inverse R almost quasinilpotent on $\ker D$. Suppose, moreover, that $X \in \mathbf{Lg}(D)$, there are $(L, E) \in G_{R,1}[\Omega]$, $g = Re \in I(X) \cap \mathrm{dom}\, D$, $g^{-1} \in A_R(D)$ and $\{S_h\}_{h \in A(\mathbb{R})}$ is a family of multiplicative true shifts. Then*

$$\lambda_n^{(p)}(g + he) = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} h^k \lambda_{n-k}^{(p)}(g) \qquad for\ n \in \mathbb{Z},\ p \in \mathbb{N}_0. \tag{7.4}$$

Theorem 7.1 is a generalization of the well-known binomial theorem with harmonic logarithms appearing in Umbral Calculus (cf. Roman and Rota RR[1], Loeb and Rota LR[1]) for harmonic logarithms induced by a right invertible operator $D \in L(X)$ and $(L, E) \in G[\Omega]$.

Let $X \in \mathbf{Lg}(D)$, $(L, E) \in G_{R,1}[\Omega]$ $(R \in \mathcal{R}_D)$ and let $g = Re \in I(X) \cap \mathrm{dom}\ \Omega$. Consider the algebra

$$\mathcal{X}(L; g) = \mathrm{lin}\ \{g^n (Lg)^p : n \in \mathbb{Z},\ p \in \mathbb{N}_0\}.$$

Clearly, $\mathcal{X}(L; g)$ is a Leibniz algebra whenever X is a Leibniz algebra.

Theorem 7.2. (cf. PR[9]) *Suppose that $X \in \mathbf{Lg}(D)$ is a Leibniz algebra, $(L, E) \in G_{R,1}[\Omega]$ for an $R \in \mathcal{R}_D$ and $g = Re \in I(X) \cap \mathrm{dom}\ \Omega$. Then*

$$\mathcal{X}(L; g) = \mathrm{lin}\ \{\lambda_n^{(p)}(g) : n \in \mathbb{Z},\ p \in \mathbb{N}_0\}. \tag{7.5}$$

Now one can extend results obtained for algebras considered in Umbral Calculus to algebras $\mathcal{X}(L; g)$ induced by a right invertible operator $D \in L(X)$ and $(L, E) \in G[\Omega]$.

Concerning linear equations with scalar coefficients and with the right-hand side belonging to $\mathcal{X}(L; g)$, we have the following

Theorem 7.3. (cf. PR[9]). *Suppose that X is a complete linear metric space ($\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) and a commutative Leibniz algebra with unit e, $D \in R(X)$, $\ker D \neq \{0\}$, F is a multiplicative initial operator for D corresponding to an $R \in \mathcal{R}_D \cap AQN(\ker D)$, $X \in \mathbf{Lg}(D)$, $(L, E) \in G[\Omega]$ and $g \in I(X) \cap \mathrm{dom}\ \Omega$. Then every equation*

$$P(D)x = y, \qquad y \in \mathcal{X}(L; g) \quad \big(P(t) \in \mathbb{F}[t]\big) \tag{7.6}$$

has all solutions belonging again to $\mathcal{X}(L; g)$. If, in addition, $g^{-1} \in A_R(D)$ then $\mathcal{X}(L; g) \subset A_R(D)$.

Note that in the proof of Theorem 7.3 we have applied in an essential way properties of the so-called *D-R hulls* (cf. von Trotha T[1], also PR[2]).

An analogue of Theorem 7.1 for $u \neq g = Re$ is

Theorem 7.4. (cf. PR[9] *Suppose that X is a complete linear metric locally convex space ($\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$), $D \in R(X)$ is closed, $\ker D \neq \{0\}$ and F is a continuous initial* *operator for* D *corresponding to a right inverse* R *almost quasinilpotent on* $\ker D$*. Suppose, moreover, that* $X \in \mathbf{Lg}(D)$*, there are* $(L, E) \in G_{R,m}[\Omega]$ $(m \in \mathbb{N})$*,* $u \in I(X) \cap \operatorname{dom} D^m$*,* $u^{-1} \in A_R(D)$ *and* $\{S_h\}_{h \in A(\mathbb{R})}$ *is a family of multiplicative true shifts. Then*

$$\lambda_n^{(p)}(S_h u) = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} h^k \lambda_{n-k}^{(p)}(u) \qquad for\ n \in \mathbb{Z},\ p \in \mathbb{N}_0. \tag{7.7}$$

Corollary 7.1. (cf. PR[9]). *Suppose that all assumptions of Theorem 7.4 are satisfied. Then*

$$\lambda_n^{(p)}(u) = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} h^k \lambda_{n-k}^{(p)}(S_{-h} u) \qquad for\ n \in \mathbb{Z},\ p \in \mathbb{N}_0, \tag{7.8}$$

$$(L S_h u)^p = \frac{1}{n!} D^n \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} h^k \lambda_{n-k}^{(p)}(u) \qquad for\ n \in \mathbb{Z},\ p \in \mathbb{N}_0. \tag{7.9}$$

Denote by $I_n(Y)$ the set of all elements from $Y \subset X$ having *n-th roots*:

$$I_n(Y) = \{x \in Y : \underset{y \in I(Y)}{\exists} \ y^n = x\} \quad (n \in \mathbb{N}). \tag{7.10}$$

If $x \in I_n(Y)$ and $y^n = x$ then we write $y = x^{1/n}$, $(n \in \mathbb{N})$.

Theorem 7.5. (cf. PR[9]. *Suppose that* X *is a complete linear metric locally convex space (*$\mathbb{F} = \mathbb{C}$ *or* $\mathbb{F} = \mathbb{R}$*),* $D \in R(X)$ *is closed,* $\ker D \neq \{0\}$ *and* F *is a continuous initial operator for* D *corresponding to a right inverse* R *almost quasinilpotent on* $\ker D$*. Suppose, moreover, that* $X \in \mathbf{Lg}(D)$*, there are* $(L, E) \in G_{R,m}[\Omega]$ $(m \in \mathbb{N})$*,* $u \in I_p(X) \cap \operatorname{dom} D^m$*,* $u^{-1} \in A_R(D)$ *and* $\{S_h\}_{h \in A(\mathbb{R})}$ *is a family of multiplicative true shifts. Then*

$$S_h u = \Big(\sum_{k=0}^{\infty} [k]! h^k D^k u^p\Big)^{1/p} \qquad for\ n \in \mathbb{Z},\ p \in \mathbb{N}_0. \tag{7.11}$$

Theorem 7.6. (cf. PR[9]). *Suppose that all assumptions of Theorem 7.4 are satisfied. Then*

$$\lambda_n^{(p)}(S_h u) = S_h \lambda_n^{(p)}(u) \qquad for\ n \in \mathbb{Z},\ p \in \mathbb{N}_0. \tag{7.12}$$

8. Characteristic quasipolynomials.

Suppose that $X \in \mathbf{Lg}(D))$ is a complete linear metric space over $\mathbb{C}$, $(L, E) \in G[\Omega]$, $D \in R(X)$, there is an $R \in \mathcal{R}_D \cap AQN(\ker D)$, F is an initial operator for D corresponding to R, $g = Re$, $\lambda g \in \mathrm{dom}\ \Omega$ for $\lambda \in v_{\mathbb{C}}R$ and $\{S_h\}_{h\in\mathbb{R}}$ is a family of true shifts (induced by R). Write

$$\text{(8.1)} \qquad W(tD) = \sum_{k=0}^{n} \sum_{j=0}^{m} a_{kj} t^k S_{-h_j}, \quad W^\wedge(t) = \sum_{k=0}^{n} \sum_{j=0}^{m} a_{kj} t^k \mathrm{e}^{-h_j t}$$

$$0 = h_0 < h_1 < ... < h_m,\ a_{kj} \in \mathbb{C}\ (k = 0, ..., n;\ j = 0, ..., m),$$

Then $E(\lambda g) \in \ker W(D)$ if and only if $W^\wedge(\lambda) = 0$. It means that in order to determine solutions of the equation $W(D)x = 0$, which are of the form $E(\lambda g)$, it is enough to find *characteristic roots*, i.e. zeros of the *characteristic quasipolynomial* $W^\wedge(t)$.

In particular, the characteristic quasipolynomials for homogeneous linear differential-difference equations with scalar coefficients and their roots are often studied in order to determine the corresponding solutions.

9. Oscillations.

We begin with

Definition 9.1. Let $D \in R(X)$ and let $d(D) = \{1, 2, ..., \dim\ \ker D \leq +\infty\}$, $(0 < d(D) \leq +\infty)$. Then $\ker D = \mathrm{lin}\ \{z_n\}_{n\in d(D)}$, where $z_1, ..., z_{d(D)} \in \ker D$ are linearly independent. By Theorem 3.1, to every $x \in X$ there corresponds a function $x^\wedge : A(\mathbb{R}) \longrightarrow \ker D$ defined as $x^\wedge(t) = F_t x$, where $F_t = FS_t$. Thus there exist scalar functions $\mathbb{S}_{x;1}, ..., \mathbb{S}_{x;d(D)} : A(\mathbb{R}) \longrightarrow \mathbb{F}$ such that

$$\text{(9.1)} \qquad x^\wedge(t) = \{\mathbb{S}_{x;n} z_n\}_{n\in d(D)} \quad \textit{for all } t \in A(\mathbb{R}) \quad (x \in X).$$

The sequence $\mathbb{S}_x = \{\mathbb{S}_{x;n}\}_{n\in d(D)}$ is said to be the *symbol* of the element x. Its nth component is said to be *nth symbol function* *. □

From Definition 9.1 it follows that the symbol is *linear* in its index, i.e.

$$\text{(9.2)} \qquad \mathbb{S}_{cx} = c\mathbb{S}_x, \quad \mathbb{S}_{x+y} = \mathbb{S}_x + \mathbb{S}_y \quad \textit{for all } x, y \in X,\ c \in \mathbb{F}.$$

Theorem 9.1. *Suppose that all assumptions of Theorem 3.1 are satisfied and $x \in X$, $t, h \in A(\mathbb{R})$. Then*

(i) $\mathbb{S}_{D^k x} = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbb{S}_x$ for $x \in \mathrm{dom}\ D^k$ $(k \in \mathbb{N})$;

(ii) all nth symbol functions are infinitely differentiable (with respect to t for $x \in D_\infty$ $(n \in d(D))$;

---

* The symbol functions for D-polynomials and exponentials has been introduced in PR[2], p. 357. The case dim ker $D = n$ has been examined in PR[11].

(iii) $x \in A_R(D)$ if and only if all nth symbol functions $\mathbb{S}_{x;n}$ $(n \in d(D))$ are analytic at $t = 0$ and

$$\mathbb{S}_{x;n}(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbb{S}_{x;n}^{(k)}(0) \quad for\ t \in A(\mathbb{R}) \quad \big(x \in A_R(D),\ n \in d(D)\big).$$

Corollary 9.1. *Suppose that all assumptions of Theorem 3.1 are satisfied. Let $P(t) \in \mathbb{F}[t]$. Then the equation*

$$P(D)x = y, \qquad y \in X \tag{9.3}$$

has a solution x if and only if each nth symbol function $\mathbb{S}_{x;n}$ $(n \in d(D))$ satisfies an ordinary differential equation:

$$P\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)\mathbb{S}_{x;n} = \mathbb{S}_{y;n} \qquad \big(n \in d(D)\big). \tag{9.4}$$

Definition 9.2. Suppose that all assumptions of Theorem 3.1 are satisfied. Then true shifts have the *the intermediate value property* (shortly: IVP) if for every $x \in \operatorname{dom} D$ if for every $t, h \in A(R)$ there exists a $\theta = \{\theta_n\}$, $0 < \theta_n < 1$ $(n \in d(D))$ such that

$$\mathbb{S}_{x;n}(t+h) - \mathbb{S}_x(t) = h\mathbb{S}_{Dx;n}(t + \theta_n h) \quad \big(n \in d(D)\big). \tag{9.5}$$

$\square$

Since the family $\{S_h\}_{h\in A(\mathbb{R})}$ of true shifts is at least a semigroup, in order to show that they have IVP it is enough to prove that for every $x \in \operatorname{dom} D$, $h \in A(\mathbb{R})$ there is a $\theta = \{\theta_n\}_{n\in d(D)}$, $\theta_n \in (0, 1)$, such that

$$\mathbb{S}_{x;n}(h) - \mathbb{S}_x(0) = h\mathbb{S}_{Dx;n}(\theta_n h) \quad \big(n \in d(D)\big). \tag{9.6}$$

Theorem 9.2. *Suppose that all assumptions of Theorem 3.1 are satisfied. Then true shifts S_h have* IVP *on* dom D.

Corollary 9.2. *Suppose that all assumptions of Theorem 3.1 are satisfied. Then the initial operators $F_h = FS_h$ $(h \in A(\mathbb{R}))$ have* IVP.

This Corollary has deep consequences. Namely, we have

Theorem 9.3. *Suppose that all assumptions of Theorem 3.1 are satisfied. Then the following theorems on intermediate value hold:*

(i) If $a \neq b$, $x \in$ dom D *and $F_a x = 0$, $F_b x = 0$ then there exists a $\theta = \{\theta_n\}_{n\in d(D)}$ such that*

$$\mathbb{S}_{F_b x - F_a x;n} = (b-a)\mathbb{S}_{F_{a+\theta(b-a)}Dx} \quad \big(n \in d(D)\big)$$

(ii) If $a \neq b$, $x \in$ dom D *and $F_b x = F_a x$, then there exists a $\theta = \{\theta_n\}_{n\in d(D)}$ such that*

$$\mathbb{S}_{F_{a+\theta_n(b-a)}Dx;n} = 0 \quad \big(n \in d(D)\big);$$

(iii) If $a \neq b$ and $x \in X$ then there exists a $\theta = \{\theta_n\}_{n \in d(D)}$ such that

$$\frac{1}{b-a}\mathbb{S}_{I_b^a x;n} = \mathbb{S}_{F_{a+\theta(b-a)}x;n}, \quad \textit{where } I_a^b = (F_b - F_a)R, \quad \big(n \in d(D)\big);$$

(iv) If $a \neq b$ and $x \in \operatorname{dom} D$ then

$$\mathbb{S}_{F_b x - F_a x;n} = (b-a)\mathbb{S}_{[\int_0^1 F_{a+\theta_n(b-a)}d\theta_n]Dx;n} \quad \big(n \in d(D)\big).$$

(v) If $\dim \ker D = 1$ *(i.e. $d(D) = 1$), then to (i)-(iv) there correspond the classical* Lagrange *and* Rolle *theorems, theorem on intermediate value of a definite integral and* Hadamard *Lemma (where $0 < \theta < 1$):*

$$F_b x - F_a x = (b-a)F_{a+\theta(b-a)}Dx \quad \textit{whenever } F_a = F_b = 0;$$

$$F_{a+\theta(b-a)}Dx = 0 \quad \textit{whenever } F_b x = F_a x;$$

$$\frac{1}{b-a}I_a^b x = F_{a+\theta(b-a))}x, \quad \textit{where } I_a^b = (F_b - F_a)R;$$

$$F_b x - F_a x = (b-a)\Big[\int_0^1 F_{a+\theta(b-a)}d\theta\Big]Dx.$$

In order to examine solutions of linear equations in a right invertible operator we need

Definition 9.3. Let X be a linear space over the field $\mathbb{F}$ and let $D \in R(X)$. Suppose that $\{F_a\}_{a \in A(\mathbb{R})} \subset \mathcal{F}_D$ is a family of initial operators for D. A point $a \in A(\mathbb{R})$ is said to be a *zero* of an element $x \in X$ if $F_a x = 0$. An element $x \in X$ is said to be *oscillatory* if there is a sequence $\{a_n\} \subset \mathbb{R}$ such that $F_{a_n} x = 0$ for $n \in \mathbb{N}$, i.e. if x has infinitely many zeros. □

Proposition 9.1. *Let F be an initial operator for $D \in R(X)$ corresponding to a right inverse R and let be given a semigroup $\{S_h\}_{h \in \mathbb{R}} \subset L_0(X)$. If $x \in X$ is S_h-periodic and $Fx = 0$ then x has infinitely many zeros jh for $j \in \mathbb{Z}$, i.e. x is oscillatory.*

Suppose that $D \in R(X)$ and $R \in \mathcal{R}_D$. An operator $A \in L_0(X)$ is said to be *stationary* if $DA = AD$ and $RA = AR$. Clearly, scalar multiples of the identity are stationary. In general, a converse statement is not true.

Theorem 9.4. (Sturm Separation Theorem) *Suppose that all assumptions of Theorem 3.1. are satisfied. Let u and Rv be two linearly independent solutions of the equation*

$$(9.7) \qquad Q(D)x = 0, \quad \textit{where} \quad Q(D) = \sum_{k=0}^{N} Q_k D^k, \quad Q_N = I, Q_0, ..., Q_{N-1} \in L_0(X),$$

$Q_0, ..., Q_{N-1}$ are stationary, the operator

$$(9.8) \qquad Q(I,R) = \sum_{k=0}^{N} Q_k R^{N-k}$$

is invertible and

$$F_a v = 0, \quad F_b v = 0 \quad (b \neq a). \tag{9.9}$$

Then there exists a $\theta = \{\theta_n\}_{n \in d(D)}$, $0 < \theta_n < 1$ *for* $n \in d(D)$, *such that*

$$\mathbb{S}_{F_{a+\theta(b-a)}u;n} = 0 \quad \textit{for all } n \in d(D). \tag{9.10}$$

In particular, if $\dim\ \ker D = 1$, *then there is a* $\theta \in (0,1)$ *such that*

$$F_{a+\theta(b-a)}u = 0. \tag{9.11}$$

Corollary 9.3. *Suppose that all assumptions of Theorem 3.1 are satisfied and* $A(\mathbb{R}) = \mathbb{R}$. *If* v *is* S_h*-periodic then there exists a* $\theta = \{\theta_n\}_{n \in d(D)}$ $(\theta_n \in (0,1))$ *such that* $h'_{jn} = (j + \theta_n)h$ *are zeros of* u *for* $j \in \mathbb{Z}$.

Theorem 9.5. *Suppose that all assumptions of Theorem 3.1 are satisfied and* $A(\mathbb{R}) = \mathbb{R}$. *If* v *is oscillatory then* u *is oscillatory and for every* $n \in \mathbb{N}$ *there exists a* $\theta_n \in (0,1)$ *such that*

$$\mathbb{S}_{F'_{h_n}u;n} = 0, \quad \textit{where } h'_n = h_n + \theta_n(h_{n+1} - h_n). \tag{9.12}$$

Moreover,

(i) *if* $|h_{n+1} - h_n| \to 0$ *then* $|h'_{n+1} - h'_n| \to 0$ *as* $n \to \infty$;

(ii) *if* $|h_{n+1} - h_n| \to \infty$ *then* $|h'_{n+1} - h'_n| \to \infty$ *as* $n \to \infty$;

i.e. two linearly independent solutions u *and* $v = Ru$ *of Equation (9.7) have similar kind of oscillations.*

10. Periodicity in locally pseudoconvex algebras.

We start with

Definition 10.1. X is said to be a complete *m-pseudoconvex algebra* if it is an algebra and a complete locally pseudoconvex space with the topology induced by a sequence $\{\|\cdot\|_n\}$ of *submultiplicative* p_n-homogeneous F-norms, i.e. such pseudonorms that $\|xy\|_n \leq \|x\|_n \|y\|_n$ for all $x, y \in X$, $n \in \mathbb{N}$. (cf. Rolewicz R[1]). □

Theorem 10.1. (PR[7]). *Suppose that either* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$, $X \in \mathsf{L}(D)$ *with unit* $e \in \operatorname{dom} \Omega^{-1}$ *is a complete* m*-pseudoconvex algebra and* $(L, E) \in G[\Omega]$. *Let* D *be closed. Let* $g = Re$ *and let* $\lambda g \in \operatorname{dom} \Omega^{-1}$ *for an* $R \in \mathcal{R}_D$ *and a* $\lambda \in \mathbb{F}$. *Let the initial operator* F *corresponding to* R *be multiplicative. Write*

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{10.1}$$

whenever this series is convergent. Then $\lambda \in v_{\mathbb{F}} R$ *and*

$$\mathrm{e}^{\lambda g} = (I - \lambda R)^{-1} e = E(\lambda g), \quad L\mathrm{e}^{\lambda g} = \lambda g. \tag{10.2}$$

In the sequel we assume Condition $[\mathbf{C}]_\mathbf{1}$:

$$-x \in \mathrm{dom}\ \Omega^{-1} \quad \textit{whenever} \quad x \in \mathrm{dom}\ \Omega^{-1}. \tag{10.3}$$

Definition 10.2. (cf. PR[7]) Suppose that Condition $[\mathbf{C}]_\mathbf{1}$ holds and $(L, E) \in G[\Omega_1]$. For $ix \in \mathrm{dom}\ \Omega_1$ we write

$$Cx = \frac{1}{2}[E(ix) + E(-ix)], \quad Sx = \frac{1}{2i}[E(ix) - E(-ix)]. \tag{10.4}$$

The mappings C and S are said to be *cosine* and *sine mappings*, respectively, or *trigonometric mappings*. Elements Cx and Sx are said to be *cosine* and *sine elements* or *trigonometric elements*. □

These mappings and elements have all properties of the classical cosine and sine functions. Indeed, trigonometric mappings C and S are well-defined for all $ix \in \mathrm{dom}\ \Omega_1$ and they are even and odd functions of their argument respectively, i.e. $C(-x) = Cx$, $S(-x) = -Sx$ for $ix \in \mathrm{dom}\ \Omega_1$. Moreover, $C(0) = z \in \ker D \setminus \{0\}$, $S(0) = 0$. If $X \in L(D)$ then the *Trigonometric Identity* holds, i.e.

$$(Cx)^2 + (Sx)^2 = e \quad \textit{whenever } ix \in \mathrm{dom}\ \Omega_1. \tag{10.5}$$

Moreover, since $[E(ix)]^n = E(inx)$ in $X \in L(D)$, the *De Moivre formula* hold: $(Cx + iSx)^n = C(nx) + iS(nx)$ for $ix \in \mathrm{dom}\ \Omega_1^{-1}$.

Note that the mappings C', S' defined as follows: $C'x = C(x + z)$, $S'x = S(x + z)$ for $ix \in \mathrm{dom}\ \Omega^{-1}$, $z \in \ker D$ are again trigonometric mappings.

Recall that a necessary and sufficient condition for the trigonometric identity to be satisfied is that X is a Leibniz algebra.

Here and in the sequel we shall assume that the trigonometric identity holds. Let $w = u + iv \in \mathrm{dom}\ \Omega$, $w^* = u - iv$ and let

$$\mathbb{C}(X) = \{w = u + iv \in \mathrm{dom}\ \Omega : ww^* \in I_2(\mathrm{dom}\ \Omega)\}. \tag{10.6}$$

By definition, $w \in I(X)$.

Suppose that, in addition, D is closed and $(L, E) \in G_{R,1}[\Omega]$. Then for all $w \in \mathbb{C}(X)$ and $x = \arg w \in \mathcal{E}_D(X)$ the mapping Ei is $2\pi e$-periodic, i.e. $E[i(x + 2\pi e)] = E(ix)$.

Suppose then that all the listed conditions are satisfied, a λ satisfies the equation $W^\wedge(\lambda) = 0$ and $\lambda g \in \mathcal{E}_D(X)$. Then $E(\lambda g) = E[i(-\lambda i g)]$ is $2\pi e$-periodic, i.e. $E(\lambda g) = E(\lambda g + 2\pi i k e)$, whenever $k \in \mathbb{Z}$.

11. Perturbations of shifts.

In Section 6 we have considered shifts as algebraic operators (in particular, involutions of order N) on properly chosen subspaces of elements. Namely, in the classical case of a differential-difference equation with commensurable deviations due the decomposition of the space under consideration onto direct sum of principal spaces for the given shift one can obtain a system of differential equations without deviations (i.e. without shifts). Here we will show how to use these results in the case of non-commensurable deviations. Tn other words, we shall try to answer for the following questions: 1^o when do linear differential-difference equations have periodic solutions and how they are determined? 2^o Is it possible to approximate ω-periodic solutions of such equations with deviations, "near" in a sense, to the previous ones but commensurable? The answers to both questions are positive under some additional restrictions regarding the form of equations under consideration.

Denote by C_ω^n the space of all n-times continuously differentiable complex valued ω-periodic functions x of the real argument t with the norm

$$\|x\|_n = \sum_{k=0}^{n} \sup_{0\leq t\leq\omega} |x^{(k)}(t)| \qquad (n = 0, 1, ...). \tag{11.1}$$

Clearly, the spaces $C_\omega = C_\omega^0 \supset ... \supset C_\omega^n ...$ $(n \in \mathbb{N}_0))$ are Banach spaces.

Let $h = (h_1, ..., h_m)$ be a system of real numbers. Consider a linear differential-difference operator

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \sum_{j=0}^{m} a_j(t)x(t - h_j) = y \qquad (h_0 = 0), \tag{11.2}$$

where the functions $a_1, ..., a_\omega, y$ are ω periodic complex valued functions defined for all $t \in \mathbb{R}$. We are looking for solutions x of differential-difference Equation (11.2) belonging to C_ω^1. Without loss of generality we can assume $\omega \neq 0$.

When all $h_1, ..., h_m$ are commensurable with ω then there is a positive integer N and a real $r \neq 0$ such that $h_j = jr$, $\omega = Nr$. Namely, ω is the greatest common multiple of the numbers $h_1, ..., h_m$. We therefore can apply here results of Section 6. On the other hand, one can prove the following

Theorem 11.1. (cf. Rolewicz R[2], also PR[2]). *If the homogeneous equation (11.2) (i.e. this equation with $y = 0$) has only the ω-periodic solution zero, then Equation (11.2) has a unique solution $x_h \in C_\omega^1$ for every $y \in C_\omega$.*

Theorem 11.2. (cf. Rolewicz R[2], also PR[2]). *If Equation (11.2) has a unique solution $x_h \in C_\omega^1$, then for arbitrary reals $h_1', ..., h_m'$ such that the values $|h_j' - h_j|$ $(j = 1, ..., m)$ are sufficiently small the equation*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \sum_{j=0}^{m} a_j(t)x(t - h_j') = y \qquad (h_0' = 0) \tag{11.2$'$}$$

has a unique solution $x_{h'} \in C^1_\omega$ *(where* $h' = (h'_1, ..., h'm)$*. Moreover, if* $h'_j \to h_j$ *for* $j = 1, ..., m$ *then* $x_{h'}$ *tends uniformly to* x_h.

Note 11.1. Theorems 11.1 and 11.2 remain valid if instead of scalar functions we consider vector-valued functions of an arbitrary dimension k. If it is the case, then coefficients a_j $(j = 1, ..., m)$ are taken to be $k \times k$ matrices. However, one can show that proofs of these theorems do not work in the case of functions taking values in an infinite dimensional space (cf. Rolewicz R[2]). □

Theorems 11.1 and 11.2 are also true for the equation

$$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} + \sum_{k=0}^{n-1} \sum_{j=0}^{m} a_{kj}(t) x^{(k)}(t - h_j) = y \qquad (h_0 = 0), \quad (n \in \mathbb{N},\ n \neq 1) \tag{11.3}$$

where we are looking for solutions belonging to C^n_ω.

Suppose now that the numbers $h_1, ..., h_m$ in Equation (11.3) are not commensurable with ω nor, possibly, commensurable with one another. Since we are looking for ω-periodic solutions, we can assume without loss of generality, that

$$0 \ < \ h_j \ \leq \ \omega \qquad \textit{for } j = 1, ..., m. \tag{11.4}$$

For a given number $\delta > 0$ we can find numbers ω_1,...,ω_m commensurable with ω and with one another, such that

$$0 \leq \omega_j \leq \omega \quad \textit{and} \quad |\omega_j - h_j| < \delta \quad (j = 1, ..., m). \tag{11.5}$$

Indeed, without loss of generality we can assume that $\delta < 1$. Let $N = [\frac{1}{\delta}]$ *. For a fixed j there is an integer $\eta_j \leq N$ such that

$$\frac{\eta_j}{N}\omega - h_j \leq \delta \leq \frac{1}{N}.$$

Write

$$r = \frac{\omega}{N}, \quad \eta_{m+j} = N, \quad \eta_0 = 0, \quad \omega_j = \eta \quad (j = 1, ..., m).$$

So that we have $m+1$ commensurable numbers ω_1,...,ω_{m+1} satisfying the required condition (11.5) and such that $0 \leq \omega_j \leq \omega_{m+1} \leq \omega$. Consider now the perturbed equation (11.3), where instead of deviations h_j we put the numbers ω_j:

$$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} + \sum_{k=0}^{n-1} \sum_{j=0}^{m} a_{kj}(t) x^{(k)}_\omega(t - \omega_j) = y \qquad (n \in \mathbb{N},\ n \neq 1) \tag{11.3$'$}$$

Using the method described in Section 2, we conclude that this equation is equivalent in the space C^n_ω to a system of N independent differential equations without deviation of

* The symbol $[x]$ read "integer part of x, denotes the greatest integer $M \leq x$.

argument. If each of the corresponding homogeneous equations has only zero as an ω-periodic solution, then all non-homogeneous equations has the unique ω-periodic solution X. We therefore conclude that Equation $(11.3')$ has a unique ω-periodic x_ω for every $y \in C^n_\omega$ tending uniformly to X as $\omega_j - h_j$ tends to zero for $j = 1, ..., m$.

Note 11.2. We should point out that the operator $R_\omega = R - F_\omega R$, where $F_\omega = FS_\omega$ (which appears in Section 2) is of the form

$$(R_\omega x)(t) = \int_0^t x(s)ds - \left(\frac{t}{\omega} + 1\right) \int_0^\omega x(s)ds \quad for\ x \in C_\omega.$$

It is clear that R_ω maps continuous functions into differentiable functions and ω-periodic functions into ω-periodic functions for $(R_\omega x)(t+\omega) - (R_\omega x)(t) = 0$ for arbitrary $x \in C_\omega$. Similar properties have the operators R^n_ω $(n > 1)$. □

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A B S T R A C T

MSC 2010: 47A05, 47D03, 47A16, 16R99, 30D20, 34C10, 34K15, 37A32

Keywords: algebraic analysis, right invertible operator, algebraic operator, involution of order n, shift, R-shift, true shift, Leibniz condition, Leibniz algebra, multiplicative operator, hypercyclic operator, almost quasinilpotent operator, logarithmic mapping, antilogarithmic mapping, periodic element, polynomial-periodic element, exponential-periodic element, symbol function, intermediate value, oscillatory element, harmonic logarithm, umbral calculus, binomial formula, perturbation, differential-difference equation

Shifts and periodicity for functional-differential equations and their generalizations have been studied by the author in in various aspects. Here we would like to give a comprehensive survey of some of these results (without proofs) in order to recall the most important properties of considered shifts In particular, there is shown that the so-called true shifts in complete linear metric spaces are hypercyclic and that a necessary and sufficient condition for true shifts in commutative algebras to be multiplicative is that the generating operator D satisfies the Leibniz condition. A consequence of this fact is that in commutative Leibniz algebras with logarithms the operator D is uniquely determined by an isomorphism acting on $\frac{\mathrm{d}}{\mathrm{d}t}$. There are also studied generalized periodic and exponential-periodic solutions of linear and some nonlinear equations with shifts and generalizations of the classical Birkhoff theorem and Floquet theorem. These results are obtained by means of tools given by Algebraic Analysis (cf. the author PR[4]). A generalization of binomial formula of Umbral Calculus is shown in Section 7. Section 11 contains a perturbation theorem for linear differential-difference equations with non-commensurable deviations and some its consequences.