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A new necessary condition for coexistence of species in equilibrium states of the Wanner-Gujer-Kissel biofilm model

We consider the classical Wanner-Gujer-Kissel 1D-model [1,2] in the case of two bacterial species competing for space and a single limiting substrate in a biofilm of a given fixed thickness. We focus on the model's ability to describe equilibrium states in which the two species coexist. If we let $f(z,t) = (f_1(z,t), f_2(z,t))$, $0 \leq z \leq L$, $t \geq 0$, denote the volume fractions of the two species and $S(z,t)$ the concentration of the limiting substrate, then the model consistes of the following system of non-linear PDEs:

$$f_t + (vf)_z = A(S)f, \quad f_1(z,t) + f_2(z,t) = 1, \quad v(0,t) = 0, \tag{1}$$

and

$$S_t - DS_{zz} + \lambda^T A(S)f = 0, \quad S_z(0) = 0, \quad S(L) = S^0, \tag{2}$$

along with appropriate initial data. Here $v = v(z,t)$ is a (scalar) velocity field, $A(S) = \mathrm{diag}(a_1(S), a_2(S))$ the growth matrix, and S^0 the bulk concentration of the substrate at the biofilm-water interface $z = L$. Moreover, D denotes diffusivity and λ is a vector containing reciprocal yield coefficients. More about mathematical biofilm modelling can be found in a recent overview by Klapper and Dockery [3]

In this work we derive a new necessary condition, in the form of an inequality, for the existence of coexistence equilibrium states to the model (1) and (2). This condition is used in numerical experiments to locate model parameters which exibit coexistence states, something which would be difficult otherwise. The equilibrium is computed using a robust numerical method developed by the author and presented at the ECMTB 2008 in Edinburgh. It is hoped that our necessary condition could be a stepping stone in the search for a mathematically rigorous proof of the existence of coexcistence equilibrium states for biofilm models of this class.

A motivation for this work is a recent article by Klapper and Szomolay [4], where an exclusion principle for ruling out occurence of certain coexistence equilibrium states is presented. While this principle is correct, it is exemplified with a biofilm system, of the kind studied here, for which the authors seem to imply that a co-existence equilibrium may occur only for one special value of the applied substrate bulk concentration S^0. Our investigations indicate that the situation is far more favorable, and that coexistence equilibria actually exists for a whole range of values of S^0, and that for each such value, the system is actually attracted to a coexistence equilibrium state.

References

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[4] Klapper, I. and B. Szomolay, *An Exclusion Principle and the Importance of Motility for a Class of Biofilm Models*. Bull. Math. Biol. Published online: 15 January 2011, DOI: 10.1007/s11538-010-9621-5.