# $C^0$ AND BI-LIPSCHITZ $\mathcal{K}$-EQUIVALENCE OF MAPPINGS

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Abstract. In this paper we investigate the classification of mappings up to $\mathcal{K}$-equivalence. We give several results of this type. We study semialgebraic deformations up to semialgebraic $C^0$ $\mathcal{K}$-equivalence and bi-Lipschitz $\mathcal{K}$-equivalence. We give an algebraic criterion for bi-Lipschitz $\mathcal{K}$-triviality in terms of semi-integral closure (Theorem 3.5). We also give a new proof of a result of Nishimura: we show that two germs of smooth mappings $f, g : \mathbb{R}^n \to \mathbb{R}^n$, finitely determined with respect to $\mathcal{K}$-equivalence are $C^0$-$\mathcal{K}$-equivalent if and only if they have the same degree in absolute value.

## 0. Introduction

The contact equivalence (or $\mathcal{K}$-equivalence) is a very important notion in the study of smooth mappings. For instance, the notion of $C^\infty$-$\mathcal{K}$-equivalence plays a crucial role in the construction of the stratification of the jet space [G, dPW].

Given two germs of a mappings having a singular point at the origin, contact equivalence compares the singularities resulting from the intersection of the graph of these mappings with the source axis. Let us start by precisely stating the definition.

**Definition 0.1.** Two map germs $f, g : (\mathbb{R}^n; 0) \to (\mathbb{R}^m; 0)$ will be said **$C^0$-$\mathcal{K}$-equivalent** whenever there exist germs of homeomorphisms $h : (\mathbb{R}^n; 0) \to (\mathbb{R}^n; 0)$ and $H : (\mathbb{R}^{n+m}, 0) \to (\mathbb{R}^{n+m}, 0)$ such that

$$\pi_n \circ H = h \circ \pi_n, \quad \pi_m \circ H(x, 0) = 0 \quad \text{and} \quad H \circ (i_n, f) = (i_n, g) \circ h \tag{0.1}$$

where $i_n$ is the identity map in $\mathbb{R}^n$ and $\pi_j : \mathbb{R}^n \times \mathbb{R}^j \to \mathbb{R}^j$ is the orthogonal projection onto $\mathbb{R}^j$, $j = n, m$.

We say that $f$ and $g$ are **bi-Lipschitz $\mathcal{K}$-equivalent** if the pair $(h, H)$ may be chosen bi-Lipschitz.

Most of the authors have focused their attention on $C^\infty$-$\mathcal{K}$-equivalence. In [N], Nishimura investigated the classification of smooth mappings up to $C^0$-$\mathcal{K}$-equivalence. See also [C] and [BCF] for other recent results. It seems that $C^0$-$\mathcal{K}$-equivalence has not been investigated in the very detail, so far. In this paper we consider $C^0$ and bi-Lipschitz $\mathcal{K}$-equivalence of mappings. Among other things, we prove a criterion for bi-Lipschitz $\mathcal{K}$-triviality of polynomial deformations in terms of semi-integral closures.

We start by studying $C^0$-$\mathcal{K}$-equivalence of deformations. We prove that local triviality of the zero locus of a continuous deformation ensures the topological triviality of the deformation with respect to $\mathcal{K}$-equivalence. This fact is very surprising since the theorem below clearly shows that the only data of the topology of the zero locus does not determine the $\mathcal{K}$-equivalence class of a mapping.

In [N], the following theorem is proved:

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**Theorem 0.2.** *(Nishimura) Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be two $C^\infty$ functions which are finitely $C^0$-determined with respect to $\mathcal{K}$-equivalence. Then, the following conditions are equivalent:*

1. *$f$ and $g$ are $C^0$-$\mathcal{K}$-equivalent*
2. *$|deg\, f| = |deg\, g|$ near zero.*

In particular, we will get an alternative proof of this theorem. Nishimura proved the latter theorem, using a result of Fukuda which relies on the Poincaré conjecture. Thus, the above theorem was actually stated for $n \neq 4$. However, Nishimura's proof actually yields the result for any $n$ since it is now agreed that the Poincaré conjecture holds true. Nevertheless, we give a proof avoiding this very deep theorem.

We carry out our proof of Nishimura's result using Thom-Mather isotopy theorem. We also show a theorem of $C^0$-$\mathcal{K}$-equivalence of semialgebraic deformations. In this case we provide a homeomorphism $H$ which is semialgebraic. This prevents from using integration of vector fields. In fact, in this case we use a technique which is similar to the one used in [N].

In section 3, we turn to study semialgebraic bi-Lipschitz $\mathcal{K}$-triviality of semialgebraic deformations. We give an algebraic triviality criterion in terms of semi-integral closure which holds for any polynomial deformation (Theorem 3.5). We prove that if the integral closure of the ideal generated by the components of the deformation coincides with the semi-integral closure of the ideal generated by the components of a fiber of the deformation at any generic point specializing at the parameter space, then the deformation is bi-Lipschitz $\mathcal{K}$-trivial.

We prove a theorem which is deduced from this result (Theorem 3.7). Roughly speaking this theorem yields that semialgebraic bi-Lipschitz $\mathcal{K}$-triviality holds generically for any semialgebraic Lipschitz deformation. It is well known, since the work of J-P. Henry and A. Parunsiński [HP] that bi-Lipschitz $\mathcal{A}$-equivalence of polynomial mappings does admit continuous moduli. This thus shows that bi-Lipschitz $\mathcal{K}$-equivalence is natural to investigate the metric properties of singular mappings.

Finally, we give an example showing that the algebraic criterion that we provided for bi-Lipschitz $\mathcal{K}$-triviality of deformations cannot be used to establish bi-Lipschitz $\mathcal{K}$-equivalence of two given mappings (not necessarily connected by a deformation). We then show that if the semi-integral closures of the ideals generated by *each component* coincide then the two given mappings are bi-Lipschitz $\mathcal{K}$-equivalent.

## 1. Basic definitions and notations.

**Definition 1.1.** A **deformation** is a map

$$F : U \times [0; 1] \to \mathbb{R}^m,$$

$(x; t) \mapsto F_t(x)$ with $U$ open neighborhood of the origin in $\mathbb{R}^n$.

We also say that $F$ is a **deformation between $F_0$ and $F_1$**. We will denote by $O_t$ the $t$-axis.

We say that the deformation $F$ is **$C^0$-$\mathcal{K}$-trivial** if there exist mappings $H : \mathbb{R}^{n+m} \times [0; 1] \to \mathbb{R}^{n+m}$ and $h : \mathbb{R}^n \times [0; 1] \to \mathbb{R}^n$ such that for any $t$ , the pair $(h_t, H_t)$ is a $C^0$-$\mathcal{K}$-equivalence between $F_0$ and $F_t$.

We say that it is **bi-Lipschitz** (resp. **semialgebraically**) $\mathcal{K}$-trivial if the homeomorphisms may be chosen bi-Lipschitz (resp. semialgebraic).

Given two functions $f$ and $g$, we will write that $f \sim g$ if there exist two positive constants $C$ and $C'$ such that

$$C'f \leq g \leq Cf.$$

The letter $C$ will sometimes stand for various positive constants, when no confusion may arise. We shall denote by $S^k(r)$ the $k$ sphere in $\mathbb{R}^{k+1}$ of radius $r > 0$ centered at the origin and, as usual, by $S^k$ the unit sphere.

We shall need the following lemma which is easily derived from Lemma 7 of [N].

**Lemma 1.2.** *(Nishimura) Let $f, g : \mathbb{R}^n \to \mathbb{R}^m$ be two germs of continuous mappings. Assume:*

(1) $f^{-1}(0) = g^{-1}(0)$
(2) $\frac{g(x)}{|g(x)|} \neq \frac{-f(x)}{|f(x)|}$ *for any $x$ close to the origin at which $|f(x)| \neq 0$.*

*Then $f$ is $C^0$-$\mathcal{K}$-equivalent to $g$.*

## 2. $C^0$-$\mathcal{K}$-equivalence of mappings

2.1. **$C^0$-$\mathcal{K}$-equivalence of deformations.** In this section we study $C^0$-$\mathcal{K}$-equivalence of deformations. The main result is Theorem 2.4 which asserts that for any deformation, topological triviality of the zero locus implies $C^0$-$\mathcal{K}$ triviality.

It is easy to find mappings $f, g : \mathbb{R}^n \to \mathbb{R}^n$ vanishing only at 0, with different degrees, which by Theorem 0.2 are not $C^0$-$\mathcal{K}$-equivalent. This means that in Theorem 2.4, continuity of the deformation with respect to $t$ is crucial.

Let us start by proving the following lemma. The point (2) deals with Lipschitz geometry and will only be used in the last part of this paper but it is more convenient to present its proof here.

**Lemma 2.1.** *Let $F : U \times [0;1] \to \mathbb{R}^m$ be a $C^0$ deformation such that $F_t^{-1}(0)$ is constant with respect to $t$. Let $G_t(x) := \frac{|F_0(x)|}{|F_t(x)|} \cdot F_t(x)$ (with $G_t(x) = 0$ if $F_t(x) = 0$). Then:*

(1) *$G_t$ is $C^0$-$\mathcal{K}$-equivalent to $F_t$; if $F$ is semialgebraic, the equivalence is semialgebraic.*
(2) *If $F$ is semialgebraic and Lipschitz and if $|F_t(x)| \sim |F_0(x)|$ (on $U \times [0;1]$) then $G_t$ is semialgebraically bi-Lipschitz $\mathcal{K}$-equivalent to $F_t$.*

*Proof.* The proof of part (1) follows the same steps as in [N] (Lemma 7): we first define a function $\alpha$ on the complement of $F^{-1}(0)$ by distinguishing several cases.

Assume first $0 < |F_0(x)| \leq |F_t(x)|$.

Let:

$$\alpha(x;t;r) := \begin{cases} \frac{|F_0(x)|}{|F_t(x)|}, & \text{if } 0 < r \leq |F_t(x)|; \\ \frac{2(|F_0(x)| - |F_t(x)|)}{r} + \frac{2|F_t(x)| - |F_0(x)|}{|F_t(x)|}, & \text{if } |F_t(x)| \leq r \leq 2|F_t(x)|; \\ 1, & \text{if } 2|F_t(x)| \leq r. \end{cases}$$

Assume now $0 < |F_t(x)| < |F_0(x)|$. In that case we define $\alpha$ as follows.

Let:

$$\alpha(x;t;r) := \begin{cases} \frac{|F_0(x)|}{|F_t(x)|}, & \text{if } 0 < r \leq |F_t(x)|; \\ \frac{2|F_0(x)|(|F_0(x)|-|F_t(x)|)}{r(2|F_0(x)|-|F_t(x)|)} + \frac{|F_0(x)|}{2|F_0(x)|-|F_t(x)|}, & \text{if } |F_t(x)| \leq r \leq 2|F_0(x)|; \\ 1, & \text{if } r \geq 2|F_0(x)|. \end{cases}$$

Observe that the different functions involved in the definition of $\alpha$ glue together into a continuous function on its domain. The function $\alpha$ is bounded which means that the mapping

$$H(x;t;y) = (x;t;\alpha(x;t;|y|) \cdot y)$$

extends continuously when $|y|$ tends to zero. Thus, to show that $H$ is a homeomorphism, it is enough to check that the map $r \mapsto r\alpha(x;t;r)$ is a homeomorphism for any fixed $(x;t)$, which is clear in view of the definition of $\alpha$.

Furthermore, we see that $H_t$ preserves the source axis and maps the graph of $F_t$ onto the graph of $G_t$. The map $H$ is a $C^0$-$\mathcal{K}$-equivalence between $F$ and $G$.

To show (2), it is enough to see that all the derivatives of $r\alpha$ are bounded and that $\frac{\partial(r\alpha)}{\partial r}$ is bounded below away from zero (derivatives exist generically if $F$ is semialgebraic). Observe that it is easy to derive from the definition of $\alpha$ that, in the setting of (2), the derivatives of $r\alpha$ are bounded in each case. To complete the proof (2), notice that a straightforward computation shows that the derivative of $r\alpha$ with respect to $r$ is bounded away from zero. □

**Theorem 2.2.** *Let $F : U \times [0;1] \to \mathbb{R}^m$ be a $C^0$ deformation. Assume that $F_t^{-1}(0)$ is locally topologically trivial at the origin. Then $F$ is $C^0$-$\mathcal{K}$-trivial.*

*Proof.* As $F^{-1}(0)$ is topologically trivial, we may assume that the zero locus is constant with respect to $t$. Moreover, we can find an approximation $G$ of $F$, which is smooth outside the zero locus and which satisfies:

$$|G_t(x) - F_t(x)| \leq \frac{|F_t(x)|}{2}.$$

This implies that the sine of the angle between $G_t(x)$ and $F_t(x)$ is less than $\frac{1}{2}$, which, thanks to Lemma 1.2 implies that $G$ is $C^0$-$\mathcal{K}$-equivalent to $F$. Hence, possibly replacing $F$ by $G$, we may assume that $F$ is smooth outside $F^{-1}(0)$ (and continuous everywhere). Observe that, thanks to Lemma 2.1, we may also assume that the norm of $F_t$ is constant with respect to $t$.

Thus, we have reduced the proof to the case where $F$ is smooth in the complement of $F^{-1}(0)$ and $F_t$ has constant norm with respect to $t$. We will assume these facts without changing the notations.

For $(x;y;t) \in U \times \mathbb{R}^m \times [0;1]$, let us define the vector field $v(x;y;t) := (0; \frac{\partial F}{\partial t}(x;t); 1)$. The vector $v$ is tangent to the graph of $F_t$ at any point of the graph. As the norm of $F_t$ is constant with respect to $t$, the vector $v(x;F_t(x);t)$ is tangent to the manifold $U \times S^{m-1}(|F_t(x)|) \times [0;1]$, for any $(x;t) \in U \times [0;1]$.

Given a point $q = (x;y;t) \in U \times \mathbb{R}^m \times [0;1]$, let now $P_q$ denote the projection onto the tangent space at $q$ of $U \times S^{m-1}(|y|) \times [0;1]$. Then we can set:

$$w(q) := P_q(v(q)).$$

We get a vector field which is tangent to the set:

$$\{(x;y;t) : |y| = |F_t(x)|\},$$

and which coincides with $v$ on the graph of $F_t$.

Define now:

$$\eta(x;y;t) := \begin{cases} \partial_t, & \text{if } F_t(x) = 0 \text{ or } y = 0\ ; \\ \alpha(\frac{|y|}{|F_t(x)|})w(x;\frac{|F_t(x)|}{|y|}y;t) + (1-\alpha(\frac{|y|}{|F_t(x)|}))\partial_t, & \text{if } y \neq 0 \text{ and } F_t(x) \neq 0 \end{cases}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is a smooth function which is 1 at 1 and which is identically zero on the complement of $[\frac{1}{2};2]$ and $\partial_t = (0;0;1)$.

We claim that this vector field satisfies:

(1) $\pi(\eta(q)) = \eta(\pi(q)) = (0;0;1)$ (if $\pi$ is the orthogonal projection onto $\mathbb{R}^m$)
(2) $\eta(x;y;t)$ is tangent to the manifold $U \times S^{m-1}(|y|) \times [0;1]$.
(3) $\eta$ is tangent to the graph of $F$

The point (3) is clear from the construction since $\alpha$ is one on the graph of $F$. Remark that, as (1) and (2) hold for $\partial_t$ it is enough to check them for $w$. The point (2) is clear for $w$ since we projected on the tangent space to the cylinder. To show the first statement, we may fix a point $q = (x;y;t)$ with $y$ nonzero. We have by definition $\pi(v(q)) = \partial_t$ and $w_{|\mathbb{R}^n \times \{0\} \times [0;1]} \equiv \partial_t$. The mapping $(Id - P_q)$ is the orthogonal projection onto $\mathbb{R}^n \times \mathbb{R}y \times \{0\}$. Observe that, as $v$ is normal to $\mathbb{R}^n$, so is $(v - P_q(v))$. This means that $\pi$ maps this vector to zero, which shows that $\pi(v) = \pi(P_q(v))$. This proves that $w$ satisfies (1).

As, due to the above reductions, $F$ is smooth outside $F^{-1}(0)$ we get a vector field which is smooth outside $F^{-1}(0) \times \mathbb{R}^m$. Existence of the integral curves is clear in view of conditions (1) and (2). Although this vector field is not continuous everywhere, we claim that it is integrable and gives rise to a continuous flow $H$.

To see this, observe that the flow exists locally at any point for which $F_t(x)$ is nonzero since the vector field is smooth. On $F^{-1}(0)$ the vector $\eta$ is identically equal to $\partial_t$ and generates integral curves as well.

Uniqueness of the integral curves is clear in view of (2) since it implies that the flow preserves the distance to $\mathbb{R}^n \times 0$, which means that the integral curves may not fall into $F^{-1}(0)$. Continuity of the flow only needs to be checked along $F^{-1}(0)$. Property (1) implies that the first component is continuous and condition (2) implies that the normal component tends to zero when we approach $F^{-1}(0)$.

This one parameter group provides a family of homeomorphisms which constitutes the desired trivialization. By (3) and (1), it preserves the graph of $F$ and $\mathbb{R}^n \times 0$, and if we set $h = Id_{\mathbb{R}^n}$ then (1) obviously implies that (0.1) holds. ☐

2.2. **About Nishimura's theorem.** Nishimura's proof of Theorem 0.2 makes use of the Poincaré conjecture. We indicate how to spare this very involved result. The idea is to replace the result of Fukuda's used in Nishimura's proof by the second Thom-Mather's isotopy theorem. Using Theorem 2.2, we get an alternative proof of Nishimura's theorem.

**Proposition 2.3.** *Let $f, g : (\mathbb{R}^n;0) \to (\mathbb{R}^n;0)$ be two finite-to-one polynomial maps with $|deg\, f| = |\deg\, g|$. Then there exists a continuous deformation $F : U \times [0;1] \to \mathbb{R}^n$ between $f$ and either $g$ or $-g$, satisfying $F^{-1}(0) = O_t$.*

*Proof.* Let $\rho(x) := \sum_{i=1}^n f_i^2(x)$, and $U_\varepsilon := \{0 < \rho \leq \varepsilon\}$ where $f = (f_1; \dots, f_n)$. Then $\rho$ is a semialgebraic function and the origin is an isolated point in its zero locus. Thanks to the uniqueness of the link and the local conic structure (see [CK]), we can find a semialgebraic homeomorphism $\mu : S^{n-1} \times (0;\varepsilon] \to U_\varepsilon$ (with $\varepsilon > 0$) such that:

$$\rho(\mu(x;r)) = r.$$

The map $f$ is finite-to-one by assumption. Therefore, the mapping $\theta^f : S^{n-1} \times (0;\varepsilon) \to S^{n-1}$ defined by

$$(x;r) \mapsto \theta^f_r(x) = \nu \circ f \circ \mu(x;r),$$

with $\nu(x) := \frac{x}{|x|}$ is finite to one and thus "sans éclatement" (see for instance [G]). Similarly, we may define a mapping $\theta_g$.

As the maps $\theta^f$ are semialgebraic we may stratify them, that is, find two Whitney stratifications, respectively of the source and the target, in such a way that $\theta^f$ is a Thom map, sending submersively strata onto strata (again see [G]). By the second Thom's isotopy Lemma we get two families of homeomorphisms of $S^{n-1}$, $h_r$ and $H_r$, $r \in (0;r_0]$ such that:

$$\theta^f_r \circ H_r^{-1} = h_r \circ \theta^f_{r_0},$$

for some $r_0 > 0$ and small enough. Furthermore $H_r$ is continuous with respect to $r$ and $h_{r_0} = H_{r_0} = Id_{S^{n-1}}$.

Alike, we may apply the second Thom isotopy Lemma to $\theta^g$ to get two families of homeomorphisms $h'_r$ and $H'_r$, $r \in (0;r_0]$ (we may assume that $r_0$ is the same) satisfying the same property.

By assumption, the maps $\theta^f_{r_0}$ and $\theta^g_{r_0}$ have the same degree in absolute value.

This implies that there is a continuous homotopy:

$$\psi : S^{n-1} \times [0;1] \to S^{n-1}$$

such that $\psi(x;0) = \theta^f_{r_0}$ and $\psi(x;1) = \theta^g_{r_0}$ or $\theta^{-g}_{r_0}$. Let us assume for simplicity that $\psi(x;1) = \theta^g_{r_0}$.

We are now ready to construct the desired deformation. For simplicity, we first set:

$$h''_r := h'_{(1-t)r_0+tr} \circ h_{tr_0+(1-t)r},$$

and

$$H''_{t,r} := H'_{tr+(1-t)r_0} \circ H_{tr_0+(1-t)r}.$$

We define a family of homotopies, for $0 < r \leq r_0$:

$$\psi'_r : S^{n-1} \times [0;1] \to S^{n-1},$$

between $\theta^f_r$ and $\theta^g_r$ by

$$\psi'_r(x;t) := h''_{t,r} \circ \psi(H''_{t,r}(x);t).$$

We have $\psi'_r(x;0) = \theta^f_r$ and $\psi'_r(x;1) = \theta^g_r$. By construction the mapping $\psi$ is continuous with respect to $r$ and $t$. It is clear from the definition of $\theta^f_r$ and $\theta^g_r$ that this homotopy gives rise to a homotopy between $f$ and $g$. $\square$

As a matter of fact, we get an alternative proof of Nishimura's Theorem:

*Proof of Theorem 0.2.* If $f$ and $g$ do not have the same degree in absolute value, they are not be $\mathcal{K}$-equivalent. Conversely, as $f$ and $g$ are finitely determined with respect to $\mathcal{K}$-equivalence, we may assume that they are finite-to-one polynomial mappings. By the preceding Proposition, there exists a continuous deformation $F$ between $f$ and $g$ or $-g$. But, by Theorem 2.2, the deformation $F$ is $C^0$-$\mathcal{K}$-trivial. As $-g$ is $C^0$-$\mathcal{K}$-equivalent to $g$, we are done. $\square$

2.3. **$C^0$-$\mathcal{K}$-equivalence of semialgebraic mappings.** We are going to establish a semialgebraic version of Theorem 2.2. It means that we are going to show that the trivialization may be required to be semialgebraic if the deformation $F$ is assumed to be so. This prevents from constructing the isotopy by integrating a vector field, as we did in the proof of Theorem 2.2. We will actually use a technique which is close to the one used by Nishimura for proving Lemma 1.2.

The problem is one more time that $F_t$ and $F_{t'}$ might have opposite directions even if $t$ is close to $t'$. The idea is to work only on areas on which this does not happen by considering a well chosen cell decomposition of $\mathbb{R}^n \times [0;1]$.

Given two functions $\zeta$ and $\xi$ on a set $C \subset \mathbb{R}^n$ with $\xi \le \zeta$ we define the **closed inteval** as the set:

$$[\xi;\zeta] := \{(x;y) \in C \times \mathbb{R} : \xi(x) \le y \le \zeta(x)\}.$$

We also set

$$C_\varepsilon := \{x \in C : d(x;\partial C) \ge \varepsilon\}.$$

We denote by $\xi_A$ the restriction of $\xi$ to a subset $A$.

**Theorem 2.4.** *Let $F : \mathbb{R}^n \times [0;1] \to \mathbb{R}^m$ be a semialgebraic deformation such that the family $F_t^{-1}(0)$ is semialgebraically topologically trivial. Then $F$ is semialgebraically $C^0$-$\mathcal{K}$-trivial.*

*Proof.* By Lemma 2.1, we may assume that the norm of $F_t$ is constant (possibly replacing the deformation $F$ by the one provided by this Lemma). Let:

$$D := \{(u;v) \in \mathbb{R}^m \times \mathbb{R}^m : |u| = |v| \neq 0, \text{ and } u \neq -v\}.$$

We start by constructing a mapping

$$\Lambda : D \times \mathbb{R}^m \to D \times \mathbb{R}^m \tag{2.2}$$

such that:

1. For any $(u;v)$ in $D$ we have $\Lambda(u;v;w) = (u;v;\Lambda_{u,v}(w))$ where $\Lambda_{u,v}$ is a homeomorphism with $\Lambda_{u,v} = \Lambda_{v,u}^{-1}$ and $\Lambda_{u,v}(u) = v$.
2. For any $(u;v)$ in $D$, the map $\Lambda_{u,v}$ is norm preserving
3. $\Lambda_{u,v}(w) = w$ if $|w| \ge 2|u|$, and $\Lambda_{u,u}$ is the identity for any $u \in \mathbb{R}^m \setminus 0$.

Define the set:

$$B := \{(u;v) \in D : u \neq v\}.$$

Let $V_{u,v}$ be the vector space generated by $u$ and $v$. We endow this plane with the orientation making $(u;v)$ direct. By definition of $D$, this is a vector space of dimension 2 if $(u;v) \in B$. Let $\theta_{u,v}$ be the angle between $u$ and $v$.

Then, for $(u;v)$ in $D \setminus B$:

$$\alpha_{u,v}(s) := \begin{cases} \theta_{u,v}, & \text{if } 0 \le s \le |u|; \\ \frac{-s\theta_{u,v}}{|u|} + 2\theta_{u,v}, & \text{if } |u| \le s \le 2|u|; \\ 0, & \text{if } 2|u| \le s. \end{cases}$$

We then get a function $\alpha$ on $D \times \mathbb{R}^m$ by setting $\alpha(u;v;w) = \alpha_{u,v}(|w|)$ if $(u;v) \in D \setminus B$ and $\alpha(u;u;w) = (u;u;w)$ for any $u \in \mathbb{R}^m$.

Let $\widetilde{V}_{u,v}$ be the orthogonal supplement of $V_{u,v}$ in $\mathbb{R}^m$ and let $\phi_\alpha$ be the rotation by angle $\alpha$ in the plane $V_{u,v}$. We now define for $w \in \mathbb{R}^m$ the map:

$$\Lambda(u;v;w) := (\phi_{\alpha(u;v;w)}(w_1); w_2)$$

where $w = (w_1; w_2)$ in $V_{u,v} \oplus \widetilde{V}_{u,v}$. This map is well defined outside $B \times \mathbb{R}^m$ but extends continuously on this set since $\phi_{\theta_{u,v}}$ tends to the identity.

For each couple $(u; v) \in D$, this provides a map $\Lambda_{u,v}$ from $\mathbb{R}^m$ to itself.

Observe that, as by assumption $|u| = |v|$ on $D$, we have:

$$\Lambda(u; v; u) = v.$$

Thus, by construction, (1),(2) and (3) hold.

Let $q_1, \dots, q_c$ in $S^{m-1}$ be such that $\cup B(q_i; \frac{1}{4}) = S^{m-1}$. Consider a cell decomposition $\mathcal{C}$ of $\mathbb{R}^n \times [0; 1]$ compatible with $F^{-1}(0)$, the hyperplanes $t = 0$ and $t = 1$, as well as the sets:

$$\{(x; t) \in \mathbb{R}^n \times [0; 1] : F_t(x) \neq 0, \frac{F_t(x)}{|F_t(x)|} \in B(q_i; \frac{1}{4})\}. \tag{2.3}$$

Fix $\varepsilon > 0$ and a cell $E \subset \mathbb{R}^n \times \mathbb{R}$ which is the graph of a function $\eta : C \to \mathbb{R}$ (where $C \subset \mathbb{R}^n$ is another cell). The function $\eta_{|C_\varepsilon}$ may be extended to a continuous function on $\mathbb{R}^n$. Doing this for all the cells $E$ we get finitely many functions $\xi_1, \dots, \xi_\nu$. Using the min function, we may change these functions and assume that

$$\xi_1 \leq \dots \leq \xi_\nu$$

(without changing the union of the graphs of this family).

We claim that, if the above $\varepsilon$ is chosen small enough, for any $i \leq \nu$ and every cell $C \subset \mathbb{R}^n$, we have for any $(x; t)$ in $[\xi_{i,C}; \xi_{i+1,C}]$,

$$d(F_t(x); F_{\xi_{i+1}(x)}(x)) < \frac{1}{2}|F_t(x)|. \tag{2.4}$$

Let us check it by induction on the dimension of $C$. This is clear for the empty cell (of dimension $-1$). For a given $s$ dimensional cell $C$, (2.4) holds in a neighborhood of the boundary (in $\mathbb{R}^n \setminus \{0\}$) thanks to the induction hypothesis. It holds above $C_\varepsilon$ by construction since $[\xi_{i,C_\varepsilon}; \xi_{i+1,C_\varepsilon}]$ is included in a cell of $\mathcal{C}$ on which the property is true thanks to (2.3). This yields the claim.

Thanks to (2.4), $F(x; \xi_i(x))$ and $F(x; t)$ may not be antipodal for $(x; t) \in [\xi_i; \xi_{i+1}]$ . Let

$$G_i(x; t) = \Lambda_{F(x;\xi_i(x)),F(x;t)}.$$

This defines a family of mappings $G_i : [\xi_i; \xi_{i+1}] \times \mathbb{R}^m \to \mathbb{R}^m$ which is continuous with respect to $(x; t)$ in $[\xi_i; \xi_{i+1}]$. If $F_t(x)$ is zero, then define $G_i(x; t)$ as the identity mapping.

Define now inductively some isotopies $H_i$, $i = 1, \dots, \nu - 1$, on $[\xi_i; \xi_{i+1}] \times \mathbb{R}^m$ as follows. Let $H_{0,t} = Id$ and

$$H_{i,t}(x; y) = G_i(x; t)(H_{(i-1),\xi_i(x)}(x; y)).$$

To define $H$, take $H(x; y; t) := (x; H_{i,t}(x; y); t)$ where $i$ is the greatest integer such that $(x; t) \in [\xi_i; \xi_{i+1}]$. We claim that this map is continuous.

Observe that $G_i$ tends to the identity when we approach $F^{-1}(0)$. Hence $H$ is continuous at any point of $F^{-1}(0)$. Above each $[\xi_i; \xi_{i+1}]$ this is clear for the maps $G_i$ and $H_i$ are continuous. We thus have to check that the mappings induced by $H_i$ above the graph of $\xi_i$ glue together into a continuous map. As $\Lambda_{y,y} = Id$ for any $y$, $G_i$ is the identity on the graph of $\xi_i$. We are done.

As $\Lambda$ is a homeomorphism, the map $H$ is a family of one-to-one mappings. As the map $H$ preserves the distance to $\mathbb{R}^n \times [0; 1]$ it is proper. This implies that it is a family of homeomorphism.

Moreover, it is easy to prove by induction on $i$ that $H_{i,t}(x;\cdot)$ maps $F_0(x)$ to $F_t(x)$ for $(x;t) \in [\xi_i;\xi_{i+1}] \times [0;1]$. As $\pi \circ H = Id$, $H$ is the desired trivialization. $\square$

## 3. Bi-Lipschitz $\mathcal{K}$-equivalence

We now turn to the study of bi-Lipschitz $\mathcal{K}$-equivalence of deformations $F : \mathbb{R}^n \times [0;1] \to \mathbb{R}^m$. The case $m = 1$ appears in [BCFR]. The problem is more complicated than the study of $C^0$-equivalence. First observe that Theorem 2.2 no longer holds in the Lipschitz setting. If $F$ is bi-Lipschitz $\mathcal{K}$-trivial then we must have a family of bi-Lipschitz germs $h_t : (\mathbb{R}^n;0) \to (\mathbb{R}^n;0)$ such that:

$$|F_t \circ h_t| \sim |F_0|,$$

in a neighborhood of zero. We are going to see that this condition is actually sufficient for semialgebraic Lipschitz deformations.

This will imply for instance that, when the mapping is a polynomial, an algebraic criterion for semialgebraically bi-Lipschitz $\mathcal{K}$-triviality in terms of semi-integral closure.

3.1. **Bi-Lipschitz $\mathcal{K}$-triviality of polynomial mappings.** We first carry out the technical part of this section by proving the theorem below in which we construct the desired trivialization. The proof is at some places similar to the proof of Theorem 2.2 but is actually more delicate since we now work in the Lipschitz setting.

**Theorem 3.1.** *Let $F : \mathbb{R}^n \times [0;1] \to \mathbb{R}^m$ be a semialgebraic Lipschitz deformation such that $\frac{|F_t(x)|}{|F_0(x)|}$ is uniformly bounded above and below on $(x;t)$ in $U \times [0;1]$. Then $F_t$ is semialgebraically bi-Lipschitz $\mathcal{K}$-trivial.*

*Proof.* Thanks to Lemma 2.1, we may assume that the norm of $F_t$ is constant with respect to $t$.

Write $F := (F^1;\ldots;F^n)$, and for each $x$, denote by $s_x^i(t)$ the length of the arc $F^i(x;\tau)$ (in $\mathbb{R}$), $0 \le \tau \le t$, and set

$$s_x(t) := \frac{1}{|F_0(x)|}\sum_{i=1}^{n} s_x^i(t) + t.$$

The function $s$ is semialgebraic and strictly increasing with respect to $t$.

Let now $a_x := s_x(1)$ and:

$$F'(x;u) := F(x;s_x^{-1}(u)).$$

This mapping is well defined at $(x;u)$ if $0 \le u \le a_x$.

As $F$ is semialgebraic, for any $x$, the sign of $\dfrac{\partial F}{\partial t}$ may only change finitely times (partial derivatives exist for almost every $t$). Furthermore the number of changes is bounded uniformly in $x$ by a constant $N$. Therefore, the length $s_x^i(t)$ of the arc is bounded by $N \sup_{t\in[0;1]} |F_t^i(x) - F_0^i(x)|$. Hence, $s_x^i$ is bounded by $N \sup_{t\in[0;1]} |F_t^i(x) - F_0^i(x)|$ for any $x$. Thus, the function $a_x$ is bounded above by a constant $C$ since the quotient $\frac{|F_t^i(x)|}{|F_0(x)|}$ is bounded.

Furthermore, by definition we have

$$|\frac{\partial s_x}{\partial t}| \ge \varepsilon \frac{|\frac{\partial F_t}{\partial t}(x;t)|}{|F_0(x)|}. \tag{3.5}$$

It means that the derivative of $s_x^{-1}(u)$ (with respect to $u$) is not greater than

$$|F_0(x)|/|\frac{\partial F_t}{\partial t}(x; s_x^{-1}(u))|,$$

which implies that (recall that $|F_t(x)|$ is constant with respect to $t$):

$$|\frac{\partial F'_u}{\partial u}(x)| \leq C|F_0(x)| \leq C|F'_u(x)|, \tag{3.6}$$

for some constant $C$.

We also claim that $F'$ is a Lipschitz map. To see this, observe first that

$$|\frac{\partial s_x}{\partial x_i}| \leq \frac{C}{|F_0(x)|}. \tag{3.7}$$

As

$$\frac{\partial s_x^{-1}(u)}{\partial x_i} = \frac{\partial s_x(s_x^{-1}(u))}{\partial x_i} / \frac{\partial s_x}{\partial t}(s_x^{-1}(u)),$$

we may conclude from (3.5) that for some constant $C$:

$$|\frac{\partial F'}{\partial x_i}| \leq |\frac{\partial F}{\partial x_i}| + |\frac{\partial F}{\partial t} \cdot \frac{\partial s_x^{-1}}{\partial x_i}| \leq C.$$

Observe that, by (3.6), the sine of the angle between $F'(x;t)$ and $F'(x;t+\frac{1}{M})$, for some positive integer $M$ large enough, is less than or equal to $\frac{1}{2}$ (in absolute value).

Let $\Lambda$ be the map constructed in the proof of Theorem 2.4, satisfying $(1)-(3)$. By definition, $\Lambda_{u,v}$ is well defined if $u$ and $v$ are not opposite and have the same norm.

Hence, we can set $H(x;y;t) := (x; G(x;t)(y))$, where $G(x;t) : \mathbb{R}^m \to \mathbb{R}^m$ is defined as the composite:

$$\Lambda_{F'(x;E(s_x(t))),F'(x;s_x(t))} \circ \Lambda_{F'(x;E(s_x(t))-\frac{1}{M}),F'(x;E(s_x(t)))} \circ \cdots \circ \Lambda_{F'(x;0),F'(x;\frac{1}{M})},$$

$E(.)$ denoting the integral part.

This mapping is clearly continuous since $\Lambda_{u,u}$ is the identity for any $u$. Furthermore, by definition, the map $H$ tends to the identity as we draw near $F^{-1}(0)$.

We need to check that $H$ is bi-Lipschitz. We start by proving the following:

**Claim.** $\Lambda$ is Lipschitz.

We use of the same notations as in the proof of Theorem 2.4 i. e. we denote by $\theta_{u,v}$ the angle between $u$ and $v$, and by $V_{u,v}$ (resp. $\widetilde{V}_{u,v}$) the vector space spanned by (resp. normal to) $u$ and $v$. Let also $\alpha$ be the function defined in the proof of the Theorem 2.4.

A straightforward computation shows that there is a constant $C$ such that

$$|d_{(u;v)}\theta| \leq \frac{C}{\min(|u|;|v|)} \tag{3.8}$$

(considering $\theta_{u,v}$ as a function of $u$ and $v$). Observe that by definition and (3.8):

$$|d_{(u;v;w)}\alpha| \leq \frac{C}{|u|} \tag{3.9}$$

for some constant $C$ and any $(u;v;w)$ in $D \times \mathbb{R}^m$.

If $|w| \geq 2u$ then $\Lambda(u;v;w) = (u;v;w)$ and therefore the claim is clear. Thus, we may assume $|w| \leq 2u$. By (3.8), the spaces $V_{u,v}$ and $\widetilde{V}_{u,v}$ have derivatives bounded by $\frac{C}{\min(|u|;|v|)}$, for some constant $C$ (as mappings taking their values in the Grassmanian). Therefore, for $|w| \leq 2|u|$, the vectors $w_1$ and $w_2$ are Lipschitz functions of $w$. Furthermore, we clearly have $|w_1| \leq 2|u|$ which means that (3.9) implies that the map $\Lambda$ is Lipschitz. This shows the claim.

By (3.6) and (3.7), the mapping $H$ is Lipschitz. As $\Lambda_{u,v}^{-1} = \Lambda_{v,u}$, the mapping $H$ is indeed bi-Lipschitz and thus realizes a bi-Lipschitz $\mathcal{K}$-trivialization of the deformation $F$. $\square$

**Remark 1.** In the proof of Theorem 3.1, we show that $\Lambda$ is a Lipschitz map. This may be used to get a Lipschitz version of Lemma 1.2. Namely we can show:

Let $f, g : \mathbb{R}^n \to \mathbb{R}^m$ be two germs of smooth mappings. Assume:

(1) $|f| \sim |g|$
(2) There exists $\eta > 0$ such that $d(\frac{g(x)}{|g(x)|}; \frac{-f(x)}{|f(x)|}) \geq \eta$ for any $x$ close to the origin at which $|f(x)| \neq 0$.

Then $f$ is bi-Lipschitz $\mathcal{K}$-equivalent to $g$ at the origin.

**Corollary 3.2.** *Let $F : \mathbb{R}^n \times [0;1] \to \mathbb{R}^m$, $(x;t) \mapsto F(x;t)$, be a polynomial deformation. Write $F = (F^1; \dots; F^m)$ and assume that near $\mathbb{R}^n \times \{t_0\}$ we have for any $i$:*

$$|(t - t_0)\frac{\partial F^i}{\partial t}| \ll |F^i|.$$

*Then for any $t$ the map $F_t$ is semialgebraically bi-Lipschitz $\mathcal{K}$-equivalent to $F_{t_0}$ for $t$ close to $t_0$.*

*Proof.* First observe that, thanks to Łojasiewicz inequality and our assumption on $F$, we can find a neighborhood of $\mathbb{R}^n \times \{t_0\}$ such that:

$$|\frac{\partial F^i}{\partial t}(x;t)| \leq C\frac{|F^i(x;t)|}{|t - t_0|^\alpha},$$

for some positive constant $C$ and $\alpha < 1$.

Integrating this inequality with respect to $t$ we get for any $x$ and $t_0$:

$$|F^i(x;t_0)|e^{-C|t-t_0|^{1-\alpha}} \leq |F^i(x;t)| \leq |F^i(x;t_0)|e^{C|t-t_0|^{1-\alpha}}$$

For $t$ close to $t_0$ this implies that

$$|F(x;t) - F(x;t_0)| \leq \frac{|F(x;t_0)|}{2},$$

which means that the hypotheses of Theorem 3.1 hold. Therefore $F_t$ is semialgebraically bi-Lipschitz $\mathcal{K}$-equivalent to $F_{t_0}$ if $t$ is chosen close to $t_0$. $\square$

**Corollary 3.3.** *Let $F : U \times [0;1] \to \mathbb{R}^m$ be a semi-algebraic Lipschitz deformation. Then $F$ is semialgebraically bi-Lipschitz $\mathcal{K}$-trivial if and only if there exists a semialgebraic bi-Lipschitz trivialization $h$ of $F^{-1}(0)$ such that:*

$$|F \circ h| \sim F_0.$$

*Proof.* The "only if" part is clear. For the if part apply Theorem 3.1 to $F \circ h$. $\square$

**Bi-Lipschitz $\mathcal{K}$-equivalence and semi-integral closure.** Theorem 3.1 will enable us to provide an algebraic criterion for bi-Lipschitz triviality of deformations. For this purpose, we recall the notion of semi-integral closure. We recall some basic facts about the semi-integral closure. The reader may find in [B] (Proposition 8) the following proposition which defines and characterizes the semi-integral closure of an ideal.

**Proposition 3.4.** *Let $(K;\leq)$ be an ordered field and $A$ a subring and $I$ an ideal of $A$. The following subsets of $K$ coincide:*

(1) $\displaystyle\bigcap_{\substack{A_v=convex\ val.\ ring \\ A_v\supset A}} IA_v$
(2) $\{x\in K : \exists a_1,\dots,a_{2n},\ a_i\in I^i,\ x^{2n}+a_1x^{2n-1}+\dots+a_{2n}\leq 0\}$
(3) $\{x\in K : |x|$ *is bounded by an element of* $I\}$

*We call this subset the* **semi-integral closure** *of $I$ (relatively to $\leq$).*

We refer the reader to [BCR] for all the basic facts about the real spectrum. All the necessary definitions may be found in this reference. Let $\alpha\in spec_{\mathbb{R}}(\mathbb{R}[X_1;\dots;X_n;T])$ be a cone of dimension $(n+1)$. The cone $\alpha$ induces an order on the field $K$ of rational fractions. Given a polynomial mapping $F=(F^1,\dots,F^m):\mathbb{R}^n\times[0;1]\to\mathbb{R}^m$ we denote by $\mathcal{I}_F$ the ideal generated by $F^1,\dots,F^m$ in $\mathbb{R}[X_1;\dots;X_n;T]$. We denote by $\overline{\mathcal{I}_F}^{\alpha}$ the semi-integral closure of the ideal $\mathcal{I}_F$ in $(K[X_1;\dots;X_n;T];\leq_\alpha)$.

Below we identify $F_0$ with a $(n+1)$ variable mapping, considering that it is constant with respect to $t$.

**Theorem 3.5.** *Let $F:\mathbb{R}^n\times[0;1]\to\mathbb{R}^m$ be a polynomial deformation. Assume that*
$$\overline{\mathcal{I}_F}^{\alpha}=\overline{\mathcal{I}_{F_0}}^{\alpha},$$
*for any $\alpha\in Spec(\mathbb{R}^n\times[0;1])$ of dimension $(n+1)$ specializing in an element of the real spectrum of $\{0_{\mathbb{R}^n}\}\times[0;1]$. Then $F$ is semialgebraically bi-Lipschitz $\mathcal{K}$-trivial.*

*Proof.* Let $\alpha\in spec_{\mathbb{R}}(\mathbb{R}[X_1;\dots;X_n;T])$ be a cone of dimension $n$ specializing at an element of the real spectrum of $\{0_{\mathbb{R}^n}\}\times[0;1]$. Thanks to the assumption on the integral closure and Proposition 3.4 (3) we know that there exists an element $U_\alpha$ of the ultrafilter corresponding to $\alpha$ and a positive constant $C_\alpha$ such that on $U_\alpha$:
$$\frac{|F_0|}{C_\alpha}\leq|F|\leq C_\alpha|F_0|. \tag{3.10}$$

On the other hand for any cone $\alpha$ of dimension less than $(n+1)$ we can choose an element $U_\alpha$ of the corresponding ultrafilter which is open, and for any $\alpha$ which does not specialize at an element of the real spectrum of $\{0_{\mathbb{R}^n}\}\times[0;1]$ we can choose a semialgebraic set $U_\alpha$ in the corresponding ultrafilter whose closure does not meet $\{0_{\mathbb{R}^n}\}\times[0;1]$. Now, all these subsets $U_\alpha$ induce a covering of the real spectrum, which by compactness gives rise to a finite covering.

On each open set $U_\alpha$ of this finite covering whose closure meets the $t$-axis (between 0 and 1), the inequality (3.10) holds for some constant $C$. As the partition is finite we may choose the same $C$ for all the elements of the partition. This means that (3.10) holds in a dense subset of a neighborhood of $\{0_{\mathbb{R}^n}\}\times[0;1]$. As $F$ is continuous this amounts to say that it holds in a neighborhood of $\{0_{\mathbb{R}^n}\}\times[0;1]$. But, by Theorem 3.1 this implies that the deformation is semialgebraically bi-Lipschitz $\mathcal{K}$-trivial. □

**Remark 2.** The latter theorem is devoted to polynomial deformations, although Theorem 3.1 holds for any semialgebraic Lipschitz deformation. Nevertheless, it is possible to provide an analogous criterion in terms of integral closure which applies to any semialgebraic Lipschitz deformation. However, when the map is no longer polynomial, we have to consider the ring of semialgebraic Lipschitz functions, which is a rather wide ring (observe that Proposition 3.4 holds for any ordered field).

3.2. **Semialgebraic bi-Lipschitz $\mathcal{K}$-triviality holds generically.** We are going to use the results of the previous section to prove that a semialgebraic family is semialgebraically bi-Lipschitz $\mathcal{K}$-trivial for generic parameters.

Using Lipschitz stratifications J. C. Ferreira Costa [C] has proved that such a deformation was bi-Lipschitz $\mathcal{K}$-trivial, but, as his proof involves integration of vector fields, it does not provide a semialgebraic trivialization.

We shall make use of the following lemma proved in [V1]:

**Lemma 3.6.** *Let $f$ be a semialgebraic function. Then there exist a finite number of semialgebraic subsets $A_1, \dots, A_p$, and a partition of $\mathbb{R}^n$ such that the restriction of $f$ to each element of this partition is equivalent to a product of powers of distances to the $A_j$'s.*

This Lemma is a consequence of the so-called preparation theorem and is useful for the purpose of the present section since we have proved (Theorem 3.1) that the $\sim$ class is responsible for bi-Lipschitz triviality of deformations. It should be noted that the exponents in the above lemma are rational numbers which may be negative. Also, the subsets $A_i$ are in general bigger than the zero locus of the function.

**Theorem 3.7.** *Let $A \subset \mathbb{R}^n \times \mathbb{R}^m$ be a Lipschitz semialgebraic family of sets and let $f_1, \dots, f_k : \mathbb{R}^{n+m} \to \mathbb{R}$ be finitely many semialgebraic functions. There exists a semialgebraic partition $V_1, \dots, V_s$ of $\mathbb{R}^m$ such that for any $j = 1, \dots, s$ there exists a bi-Lipschitz trivialization*

$$h : \mathbb{R}^n \times V_j \to \mathbb{R}^n \times V_j,$$

*of type $(x;t) \mapsto (h_t(x);t)$ of the family $A$, such that $h_t$ is Lipschitz for any $t$ and such that for any $j$ we have:*

$$f_{j,t}(h_t(x)) \sim f_{j,t_j}(x) \tag{3.11}$$

*for some $t_j \in V_j$.*

*Proof.* By Lemma 3.6 we know that, for each function $f_j$, we have a partition $(C_i)_{i\in I}$ of $\mathbb{R}^n$ and also finitely may semialgebraic subsets $A_1, \dots, A_s$ such that over each $C_i$ the functions $f_j$'s are equivalent to product of powers of distances to the $A_k$'s. We may find a common refinement of all these partitions. Assume that this is done without changing the notations.

By Theorem 2.2 of [V1], we have a finite partition of such that along each element of this partition we may find semialgebraic bi-Lipschitz trivialization of the family $A$. Again thanks to [V1] (see Remark 2.3 (a)), we may assume that this trivialization preserves finitely many given subsets. So, in particular we may assume that it preserves the $C_i$'s, $i \in I$, above as well as the $A_k$'s, $k = 1, \dots, s$. We get a trivialization

$$h_t : \mathbb{R}^n \times V_j \to \mathbb{R}^n \times V_j$$

But, as $h_t$ is a family of bi-Lipschitz homeomorphisms, it certainly fulfills

$$d(h_t(x); A_k) \sim d(x; A_k)$$

which clearly establishes (3.11). □

**Corollary 3.8.** *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be a polynomial family of maps. Then there exists a semialgebraic stratification of $\mathbb{R}^m$, such that $f_t$ and $f_{t'}$ are bi-Lipschitz $\mathcal{K}-$equivalent if $t$ and $t'$ are chosen in the same stratum. Moreover, the equivalence $H$ may be chosen semialgebraic.*

*Proof.* Apply Theorem 3.7 to the functions $f_j$'s, $j = 1, \dots, p$ to get a stratification of $\mathbb{R}^m$ such that (3.11) holds along each stratum. Then the result follows from Theorem 3.1. □

3.3. **On bi-Lipschitz $\mathcal{K}$-equivalence of mappings.** We saw in section 3.1 that, if the semi-integral closure of the ideal generated by the components of a semialgebraic deformation is equal to the semi-integral closure of the zero fiber of the deformation, then the deformation is bi-Lipschitz $\mathcal{K}$-trivial.

We provide an example to show that this is no longer true with mappings. The semi-integral closure of the ideal generated by the components of the mapping does not determine the class up the semialgebraic bi-Lipschitz $\mathcal{K}$-equivalence. This means that it is crucial to have a deformation from $f$ to $g$.

**Example 1.** Let $f, g : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = z^3$ and $g(z) = |z|^2 \cdot z$. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ we get two mappings from $\mathbb{R}^2$ to itself. Observe that the quotient $\frac{|f|}{|g|}$ is bounded away from zero and infinity. Clearly $f$ is of degree three and $g$ of degree one. Therefore, by Theorem 0.2, the two mappings may not be $C^0$-$\mathcal{K}$-equivalent and thus cannot be bi-Lipschitz $\mathcal{K}$-equivalent either.

However, if the semi-integral closure of all the ideals generated by each of the respective components of the two mappings coincide then the two mappings are bi-Lipschitz $\mathcal{K}$-equivalent. This is what is established by the following Proposition.

**Proposition 3.9.** *Let $f, g : (\mathbb{R}^n; 0) \to (\mathbb{R}^m; 0)$ be semialgebraic Lipschitz map germs $f, g : (\mathbb{R}^n; 0) \to (\mathbb{R}^m; 0)$ satisfying $f_i \sim g_i$ for $i = 1, \dots, p$. Then $f$ and $g$ are semialgebraically bi-Lipschitz $\mathcal{K}-$equivalent.*

*Proof.* We construct by induction on $i$ a bi-Lipschitz mapping

$$H_i : \mathbb{R}^{n+i} \times \{0_{\mathbb{R}^{p-i}}\} \to \mathbb{R}^{n+i} \times \{0_{\mathbb{R}^{p-i}}\}$$

which sends the graph of $(f_1; \dots; f_i)$ onto the graph of $(g_1; \dots; g_i)$ and which is the identity on $\cup_{i=1}^n \{f_i = 0\}$ (observe that the $f_i$'s and the $g_i$'s have the same zero locus by assumption). For $i = 0$ just set $H_0 = Id$. Assume that $H_i$ is defined and extend $f_{i+1}$ and $g_{i+1}$ to $\mathbb{R}^{n+i}$ in a trivial way. Obviously, we still have $f_{i+1} \sim g_{i+1}$.

Now define for $(x; y) \in \mathbb{R}^{n+i} \times \mathbb{R}$ with $0 \leq y \leq f_{i+1}(x)$ or with $f_{i+1}(x) \leq y \leq 0$:

$$H_{i+1}(x; y) := (H_i(x); \frac{g_{i+1}(H_i(x))}{f_{i+1}(x)} y).$$

We may extend $H_{i+1}$ by setting

$$H_{i+1}(x; y) := (H_i(x); y + g_{i+1}(H_i(x)) - f_{i+1}(x)),$$

when $y \geq f_{i+1}(x) \geq 0$ or $y \leq f_{i+1}(x) \leq 0$. Clearly, this function is continuous. Moreover, thanks to the fact that $f_{i+1} \sim g_{i+1}$, a straightforward computation yields that $H_{i+1}$ is a Lipschitz mapping.

This map is indeed a bi-Lipschitz homeomorphism. Actually the inverse is given by:

$$H_{i+1}^{-1}(x; y) = (H_i^{-1}(x); \frac{f_{i+1}(H_i^{-1}(x))}{g_{i+1}(x)} y)$$

when $0 \leq y \leq g_{i+1}(x)$ or if $g_{i+1}(x) \leq y \leq 0$, and:

$$H_{i+1}^{-1}(x;y) = (H_i^{-1}(x); f_{i+1}(H_i^{-1}(x)) + y - g_{i+1}(x)).$$

if $y \geq g_{i+1}(x)$ or $y \leq g_{i+1}(x)$. For the same reasons we see that $H_{i+1}$ is a Lipschitz mapping.

Moreover it is clear in view of the definition of $H_{i+1}$, that it sends the graph of $(f_1; \dots; f_{i+1})$ onto the graph of $(g_1; \dots; g_{i+1})$. $\square$

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