VANISHING HOMOLOGY

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Abstract. In this paper we introduce a new homology theory devoted to the study of families such as semialgebraic or subanalytic families and in general to any family definable in an o-minimal structure (such as Denjoy-Carleman definable or $ln - exp$ definable sets). The idea is to study the cycles which are vanishing when we approach a special fiber. This also enables us to derive local metric invariants for germs of definable sets. We prove that the homology groups are finitely generated.

0. Introduction

The description of the topology of a set nearby a singularity is a primary focus of attention of algebraic geometers. We can regard a semialgebraic singular subset of $\mathbb{R}^n$ as a metric subspace. Then the behavior of the metric structure of a collapsing family reflects implicit information on the geometry of the singularity of the underlying set which is much more accurate than the one provided by the study of the topology.

In [V1], the author proved a bi-Lipschitz version of Hardt's theorem [H]. This theorem pointed out that semialgebraic bi-Lipschitz equivalence is a good notion of equisingularity to classify semialgebraic subsets from the metric point of view. For this purpose, it is also very helpful to find invariants such as homological invariants.

In this paper we introduce a homology theory for families of subsets which provides information about the behavior of the metric structure of the fibers when we approach a given fiber. This enables us to construct local metric invariants for singularities. We prove that these homology groups are finitely generated when the family is definable in an o-minimal structure. This allows, for instance, to define an Euler characteristic which is a metric invariant for germs of algebraic or analytic sets. These results were announced in [V4].

In [GM], M. Goresky and R. MacPherson introduced intersection homology and showed that their theory satisfies Poincaré duality for pseudo-manifolds which cover a quite large class of singular sets and turned out to be of great interest. They also managed to compute the intersection homology groups from a triangulation which yields that they are finitely generated. In [BB1] L. Birbrair and

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J.-P. Brasselet define their admissible chains to construct the metric homology groups. Both theories select some chains by putting conditions on the support of the chains. Our approach is similar in the sense that our homology groups will depend on a *velocity* which estimates the rate of vanishing of the support of the chains.

Our method relies on the result of [V1], where the author showed existence of a triangulation enclosing the metric type of a definable singular set. To compute the vanishing homology groups we will not use the triangulation constructed in [V1] but Proposition 3.2.6 of the latter paper (which was actually the main step of the construction). It makes it possible for the results proved below to go over non necessarily polynomially bounded o-minimal structures. It seems that the method of the present paper could be generalized to prove that the metric homology groups introduced in [BB1] are finitely generated as well.

It is well known that, given a definable family, we may always study the evolution of the fibers by studying what is called by algebraic geometers "the generic fiber" (see example 1.3.2 for a precise definition).

Therefore if we carry out a homology theory for definable subsets in an o-minimal structure expanding a given arbitrary real closed field, we will have a homology theory for families. This is the point of view of the present paper. Hence, even for families of subsets of $\mathbb{R}^n$, the case of an arbitrary real closed field will be required. Our approach will be patterned on the one of the classical homology groups as much as possible. Some statements (Theorem 3.2.2) are close to those given by Goresky and MacPherson for intersection homology but of course the techniques are radically different since the setting is not the same.

The admissible chains depend on a velocity which is a convex subgroup v of our real closed field R. For instance, if R is the field of real algebraic Puiseux series endowed with the order making the indeterminate T smaller than any positive real number, v may be the subgroup

$$\{x : \exists N \in \mathbb{N}, |x| \leq NT^2\}. \tag{0.1}$$

The *v-admissible chains* are the chains having a "v-thin" support. Roughly speaking, if v is as above, *v-thin* subsets of R^n are the generic fibers of families of sets whose fibers collapse onto a lower dimensional subset with at least the velocity Nt^2 (if t is the parameter of the family, $N \in \mathbb{N}$). For instance, let us consider the cycle given by Birbrair and Goldshtein's example. Namely, the subset of $X \subset R^4$ defined by:

$$\begin{aligned} x_1^2 + x_2^2 &= T^{2p}, \\ x_3^2 + x_4^2 &= T^{2q}. \end{aligned} \tag{0.2}$$

This set is the generic fiber of a family of tori, such that the support of the generators of $H_1(X)$ collapse onto a point at rate T^p and T^q respectively. Therefore, if for instance $p = 0$ and $q = 2$ then the 0-fiber is a circle and this family of torus is v-thin (with v like in (0.1)).

Taking all the v-admissible chains of a definable set X, we get a chain complex which immediately gives rise to the *v-vanishing homology groups* $H_j^v(X)$. We will show that these groups are finitely generated (Corollary 3.2.3).

If X is the set defined by (0.2) with v like in (0.1), the v-vanishing homology groups depend on of p and q. For instance, we will prove (see Example 4.3.2) that if $p = 0$ and $q = 2$:

$$H_1^v(X) = \mathbb{Q}$$

(if $\mathbb{Q}$ is our coefficient group), and $H_2^v(X) = \mathbb{Q}$.

We may summarize it by saying that we get all the T^2-thin cycles of X. The group $H_j^v(X)$ is not always a subgroup of $H_j(X)$. In general we may also have cycles that do not appear in the classical homology groups, i. e. which are in the kernel of the natural map $H_j^v(X) \to H_j(X)$. The following picture illustrates an example for which such a situation occurs:

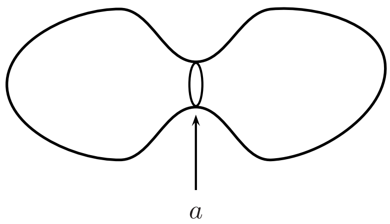

Figure 1.

The cycle a is collapsing onto a point faster than the set itself is collapsing. We see that we have an admissible one dimensional chain a which bounds a two dimensional chain which may fail to be admissible (depending on the velocity v). Therefore $H_1^v(X) \neq 0$ (while $H_1(X) = 0$).

This homology theory is not a homotopy invariant. It is preserved by Lipschitz homotopies but these are very hard to construct. For instance, given a function $f : \mathbb{R}^n \to \mathbb{R}$ it is well known that there exists a topological deformation retract of $f^{-1}(0; \varepsilon)$ onto $f^{-1}(0)$. It is easy to see that it is *not* possible to find such a retract which would be Lipschitz if $f(x; y) = y^2 - x^3$. The method used in this paper provides homotopies that are not Lipschitz but which preserve admissible chains. It seems that one could define various homology theories for which this method could be adapted. The theory developed below seemed to the author the simplest one and the most natural to start.

We compute the vanishing homology groups in terms of some basic sets obtained by constructing some nice cells decompositions (Theorem 3.2.2). For this we construct a homotopy which carries a given singular chain to a chain of these basic sets (Proposition 3.2.1). The homotopy has to preserve thin subsets. We are not able to construct such a homotopy for any admissible chain. Chains for which we can construct such a homotopy are called strongly admissible and are chains for which the distances in the support are known in a very explicit way. Therefore, the first step is to show that any class in $H_j^v(X)$ has a strongly admissible representant (Lemma 3.1.3). This is achieved by constructing some rectilinearizations of v-thin sets (Proposition 2.2.4). These are maps which transform our set into a union of hyperplanes crossing normally while controlling the distances in the transformation.

A non trivial convex subgroup v may be regarded as an interval in R which has no endpoint. This fact will somewhat complicate our task. To overcome this difficulty, we introduce an extra point u "at the end of v" which will fill the gap. This point living in an extension k_v of R, we will carry out most of the constructions rather in k_v than in R. The precise definition of k_v and the basic related notions are provided in the first section below. An advantage of using model theory is that we are able to carry out the theory for all the possible velocities (see example 1.1.2) in the same time.

Local invariants for singularities. An important application is that these homology groups may be used to derive local metric invariants for semialgebraic singularities.

We may associate to every germ of semialgebraic set the vanishing homology groups of the link introduced in [V2] which may be proved to be metric invariant of the singularity. Let us explain why the results of the latter paper make it possible.

We start by recalling the notion of link introduced in [V2]. Let us recall its definition. Let $k(0_+)$ be the field of real algebraic Puiseux series endowed with the order that makes the indeterminate T positive and smaller than any real number (see [BCR] example 1.1.2). Let $X \subset \mathbb{R}^n$ be a semialgebraic set and denote by d the Euclidian distance. Assume $0 \in X$ and set

$$X_t = S^{n-1}(0;t) \cap X.$$

It is well known that the homology of the germ of X at the origin is completely characterized by the topological type of X_t if $t > 0$ is small enough.

Observe that the family X_t is collapsing onto $X_0 = \{0\}$. It is known that the homology of the latter set is a topological invariant of A. The cycles of X_r are collapsing to a single point with a certain "rate". This rate is related to the metric type of the singularity.

Let L_X be the generic fiber of this family, that is to say, let:

$$L_X := \{x \in X_{k(0_+)} : d(x;0) = T\}$$

where $X_{k(0_+)}$ denotes the subset of $k(0_+)^n$ defined by the same equations as X.

In [V2] we proved that the set L_X is a metric invariant which characterizes the metric type of the singularity in the sense that:

Theorem 0.0.1. [V2] *L_X is semialgebraically bi-Lipschitz homeomorphic to L_Y iff Y is semialgebraically bi-Lipschitz homeomorphic to X.*

As the vanishing homology groups are metric invariants, this theorem admits the following immediate corollary.

Corollary 0.0.2. *For any convex subgroup $v \subset k(0_+)$, the groups $H_j^v(L_X)$ are semialgebraic bi-Lipschitz invariants of X.*

Corollary 3.2.3 will imply that these groups are finitely generated so that the vanishing homology Euler characteristic $\chi_v(L_X)$ (see (3.16)) is a semialgebraic bi-Lipschitz invariant of the germ X. Observe that the results of the present paper show that these invariant are finitely generated and may be computed explicitly from suitable decompositions. Some concrete examples are discussed in the last section of the paper.

One may observe that the latter corollary is devoted to the only case of semialgebraic sets. Although the results may go over any polynomially bounded o-minimal structure, they are no longer true in the non polynomially bounded case as it is shown by an explicit example at the end of the paper.

Content of the paper. In section 1, we give all the basic definitions and introduce our vanishing homology. We prove in the next section some cell decomposition theorems and rectilinearization theorems necessary to compute the vanishing homology groups. In section 3, we compute the v-vanishing homology groups in terms of this cell decomposition. The main result is Theorem 3.2.2 which yields that the homology groups are finitely generated. The last section computes the vanishing homology groups on some examples.

The reader is referred to [C] or [vD] for basic facts about o-minimal structures.

Notations and conventions. Throughout this paper we work with a fixed o-minimal structure expanding a real closed field R. Let $\mathcal{L}_R$ be the first order language of ordered fields together with an n-ary function symbol for each function of the structure. The word definable means $\mathcal{L}_R$-definable. The language $\mathcal{L}_R(u)$ is the language $\mathcal{L}_R$ extended by an extra symbol u.

The letter G will stand for an abelian group (our coefficient group). *Singular simplices* will be definable continuous maps $c : T_j \to X$, T_j being the j-simplex spanned by $0, e_1, \dots, e_j$ where $e_1, \dots, e_j$ is the canonical basis of R^j. Sometimes, we will work in an extension k_v of R and simplices will actually be maps $c : T_j(k_v) \to k_v^n$ where $T_j(k_v)$ is the extension of T_j to k_v. Given a definable set $X \subset R^n$ we denote by $C(X)$ the chain complex of definable chains with coefficients in a given group G. We will write $|c|$ for the support of a chain c.

By **Lipschitz function** we will mean a function f satisfying

$$|f(x) - f(x')| \le N|x - x'|$$

for some integer N. It is important to notice that we require the constant to be an integer for R is not assumed to be archimedean. A map $h : A \to R^n$ is Lipschitz if all its components are, and a homeomorphism h is bi-Lipschitz if h and h^{-1} are Lipschitz.

We denote by $\pi_n : R^n \to R^{n-1}$ the canonical projection and by $cl(X)$ the closure of a definable set X.

Let $k(0_+)$ be the field of real algebraic Puiseux series endowed with the order that makes the indeterminate T positive and smaller than any real number (see [BCR] example 1.1.2).

1. Definition of the vanishing homology.

1.1. **The velocity v.** We shall use some very basic facts of model theory. We refer the reader to [M] for basic definitions.

The vanishing homology depends on a **velocity** v which estimates the rate of vanishing of the cycles. This is a convex subgroup v of $(R;+)$ (convex in the sense that it is a convex subset of R).

We then define a 1-type by saying that a sentence $\psi(u) \in \mathcal{L}_R(u)$ is in this type iff the set

$$\{x \in R : \psi(x)\}$$

contains an interval $[a;b]$ with $a \in v$ and $b \notin v$. This type is complete due to the o-minimality of the theory.

We will denote by k_v an $\mathcal{L}_R$-elementary extension of R realizing this type.

Roughly speaking we can say that the velocity is characterized by a cut in R, at which the gap is "bigger" than the distance to the origin. This is to ensure that the sum of two admissible chains will be admissible (see section 1.3).

Notations. Throughout this paper, a velocity v is fixed and u is the point realizing the corresponding type in k_v.

We define a convex subgroup w of $(k_v;+)$ extending the group v in a natural way:

$$w := \{x \in k_v : \exists\, y \in v, |x| \leq y\}.$$

Remark 1.1.1. Given $z \in R$ we may define a velocity $\mathbb{N}z$ by setting:

$$\mathbb{N}z := \{x \in R : \exists\, N \in \mathbb{N}, |x| \leq Nz\}.$$

Example 1.1.2. As in the above remark, the element T^k gives rise to a subgroup $\mathbb{N}T^k$ of $(k(0_+);+)$ which is constituted by all the series $z \in k(0_+)$ having a valuation greater or equal to k. One could also consider the velocity v defined by the set of x satisfying $|x| \leq NT^k$ for any N in $\mathbb{Q}$. In the field of $ln-exp$ definable germs of one variable functions (in a right-hand side neighborhood) one may consider the set of all the L^p integrable germs of series.

Extension of functions. On the other hand, as k_v is an elementary extension of R, it is well known that we may define X_v, the extension of X to k_v, by regarding the formula defining X in k_v^n. Every mapping $\sigma : X \to Y$ may also be extended to a mapping $\sigma_v : X_v \to Y_v$.

1.2. **v-thin sets.** We give the definition of the v-thin sets which is required to introduce the vanishing homology.

The "v-thin" sets will be the sets collapsing faster than the velocity v. Before stating the precise definition, we give an explicit example.

Example 1.2.1. Consider an algebraic set $X \subset \mathbb{R}^n$. As in the end of the introduction, assume $0 \in X$ and set $X_t = S(0; t) \cap X$.

Observe that the family X_t is collapsing onto $X_0 = \{0\}$. Let us also recall the notion of link introduced in [V2]:

$$L_X := \{x \in X_{k(0_+)} : d(x; 0) = T\},$$

$X_{k(0_+)}$ being the extension of X to $k(0_+)$.

Define a velocity by $v := \{x \in k(0_+) : \exists N \in \mathbb{N}, |x| \leq NT^2\}$. Let us discuss what it will mean for L_X to be v-thin on some particular cases. We denote by $S^{n-1}(0; T)$ the sphere in $k(0_+)^n$ of radius T centered at the origin.

Roughly speaking, a L_X will be v-thin if the distance to a lower codimensional subset is less than NT^2 for some integer N. This means for instance that if L_X is a sphere in $S^{n-1}(0; T)$ such as

$$S(x; r) \cap S^{n-1}(0; T),$$

for $x \in S^{n-1}(0; T)$ and $r \in k(0_+)$, then L_X is v-thin iff $r \leq NT^2$ and in this case the family X_t is a family of spheres of radius decreasing faster than Nt^2. Then, in particular no orthogonal projection of L_X on a $(n-2)$ dimensional vector space contains a ball of radius $(N+1)T^2$. The set is said $(n-2; v)$-thin.

A more sophisticated example is provided by the equation (0.2). Assume these are the equations of L_X. The set L_X is supposed to sit in $S^{n-1}(0; T)$. Nevertheless if we identify the complement of a ball in $S^4(0; T)$ with $k(0_+)^4$ we may consider the link as a subset of $k(0_+)^4$.

Then, L_X is v-thin if p or q is greater or equal to 2. The tori X_t are collapsing onto the origin and the v-vanishing homology will account for the rate of vanishing of this cycle.

Let us emphasize the difference with the approach adopted by Birbrair and Brasselet in [BB1],[BB2]. In the theory of the latter paper the authors consider the rate of the volume of the subsets, so that the sum $p + q$ matters for the torus to be admissible. Here, for being v-thin, the integer $\max(p; q)$ is determinant, since it characterizes the distance to either $S^1 \times \{0\}$ or $\{0\} \times S^1$, which are lower dimensional subsets.

We now turn to the precise definition. We denote by $\mathbb{G}_n^j$ the Grassmaniann of j dimensional vector spaces of R^n. Given $P \in \mathbb{G}_n^j$ we denote by π_P the orthogonal projection onto P.

Definitions 1.2.2. Let $j \leq n$ be integers. A j-dimensional definable subset X of R^n is called **v-thin** if there exists $z \in v$ such that, for any $P \in \mathbb{G}_n^j$, no ball (in P) of radius z entirely lies in $\pi_P(X)$.

For simplicity we say that X is $(j;v)$**-thin** if either X is v-thin or $\dim X < j$. A set which is not v-thin will be called v**-thick**.

Note that in the above definition it is actually enough to require that the property holds for a sufficiently generic linear projection $\pi : R^n \to R^j$. As we said in the introduction, roughly speaking, $\mathbb{N}T^2$-thin sets of $k(0_+)^n$ are the generic fibers of one parameter families whose fibers "collapse onto a lower dimensional subset at rate at least t^2" (if t is the parameter of the family). Also, by convention $R^0 = \{0\}$ so that a 0-dimensional subset is never v-thin. This is natural in the sense that a family of points never collapses onto a lower dimensional subset.

Basic properties of $(j;v)$-thin sets. (1) If a definable subset $A \subset X$ is $(j;v)$-thin and if $h : X \to Y$ is a definable Lipschitz map then $h(A)$ is $(j;v)$-thin.
(2) Given j, $\cup_{i=1}^p X_i$ is $(j;v)$-thin iff X_i is $(j;v)$-thin for any $i = 1, \dots, p$.

1.3. **Definition of the vanishing homology.** Given a definable set X let $C_j^v(X)$ be the G-submodule of $C_j(X)$ generated by all the singular chains c such that $|c|$ is $(j;v)$-thin and $|\partial c|$ is $(j;v)$-thin as well. We endow this complex with the usual boundary operator and denote by $Z_j^v(X)$ the cycles of $C_j^v(X)$.

A chain $\sigma \in C_j^v(X)$ is said v**-admissible**. We denote by $H_j^v(X)$ the resulting homology groups which we call the v**-vanishing homology groups**.

If v is $\mathbb{N}z$, for some $z \in R$ (see Remark 1.1.1), then we will simply write $C_j^z(X)$ and $H_j^z(X)$ (rather than $C_j^{\mathbb{N}z}$ and $H_j^{\mathbb{N}z}$).

Remark 1.3.1. If X is v-thin and if $j = \dim X$ then every j-chain is v-admissible. Moreover every $(j+1)$-dimensional chain is admissible by definition. Hence the map $H_j^v(X) \to H_j(X)$ induced by the inclusion of the chain complexes is an isomorphism. Note also that the map $H_{j-1}^v(X) \to H_{j-1}(X)$ is a monomorphism.

Every Lipschitz map sends a $(j;v)$-thin set onto a $(j;v)$-thin set. Thus, every Lipschitz map $f : X \to Y$, where X and Y are two definable subsets, induces a sequence of mappings $f_{j,v} : H_j^v(X) \to H_j^v(Y)$. In consequence, the vanishing homology groups are preserved by definable bi-Lipschitz homeomorphisms.

As we said in the introduction this homology gives rise to a metric invariant for families (preserved by families of bi-Lipschitz homeomorphisms) by considering the generic fiber as described in the following example.

Example 1.3.2. Given an algebraic family $X \subset \mathbb{R}^n \times \mathbb{R}$ defined by $f_1 = \dots = f_p = 0$, we set

$$X_{0_+} := \{x \in k(0_+)^n : f_1(x;T) = \dots = f_p(x;T) = 0\}.$$

Hence, $H_j^v(X_{0_+})$ is a metric invariant of the family.

1.4. **The complex $C_j^v(X;\mathcal{F})$.** Given a finite family $\mathcal{F}$, of closed subsets of X, we write $C_j(X;\mathcal{F})$ for the j-chains of $\underset{F\in\mathcal{F}}{\oplus} C_j(F)$. Similarly we set:

$$C_j^v(X;\mathcal{F}) := \underset{F\in\mathcal{F}}{\oplus} C_j^v(F)$$

and denote by $H_j^v(X;\mathcal{F})$ the corresponding homology groups. By Remark 1.3.1, if τ is a chain of $Z_j^v(|\sigma|)$ whose class is σ in $H_j(|\sigma|)$ then $\tau = \sigma$ in $H_j^v(|\sigma|)$ as well. Therefore, as $H_j(|\sigma|;\mathcal{F}) = H_j(|\sigma|)$ we get:

$$H_j^v(X;\mathcal{F}) \simeq H_j^v(X). \tag{1.3}$$

1.5. **Strongly admissible chains.** It is difficult to construct homotopies between v-admissible chains. To overcome this difficulty we introduce strongly v-admissible chains.

Definition 1.5.1. We denote by T_j^q the set of all $(x;\lambda) \in T_j \times R$ such that $x+\lambda e_q$ belongs to T_j. A simplex $\sigma : T_j \to R^n$ is **strongly v-admissible** if there exists q such that for any $(x;\lambda) \in T_j^q$:

$$(\sigma(x) - \sigma(x+\lambda e_q)) \in v. \tag{1.4}$$

A chain is strongly admissible if it is a combination of strongly admissible simplices. We denote by $\widehat{C}_j^v(X)$ the chain complex generated by the strongly admissible chains σ for which $\partial\sigma$ is strongly admissible, and by $\widehat{Z}_j(X)$ the strongly admissible cycles. The resulting homology is denoted by $\widehat{H}_j^v(X)$. If $\mathcal{F}$ is a family of closed subsets of X, we also define $\widehat{C}_j^v(X;\mathcal{F})$, $\widehat{Z}_j^v(X;\mathcal{F})$, and $\widehat{H}_j^v(X;\mathcal{F})$ in an analogous way (see section 1.4).

Example 1.5.2. Let $\sigma : T_2(k(0_+)) \to k(0_+)^2$ be defined by

$$\sigma(x_1;x_2) = (\frac{1}{T} + x_1; x_2T^2).$$

Fix $x \in T_2(k(0_+))$. Then, observe that, for any $\lambda \in k(0_+)$ such that $(x+\lambda e_2)$ belongs to $T_2(k(0_+))$ we have $(\sigma(x) - \sigma(x+\lambda e_2)) = \lambda T^2$. Therefore σ is strongly v-admissible if v is the subgroup of $k(0_+)$ defined as in (0.1). We see that the support is v-thin and that $T_2(k(0_+))$ is contracted along e_2 by T^2. This is the difference between admissible and strongly admissible simplices: the condition is not only on the image but also on the way the mapping operates. It must contract the standard simplex along a vector of the canonical basis.

Remark 1.5.3. Let $\sigma : T_j \to R^n$ be a strongly admissible simplex with $j \leq n$. Then by definition, there exists $z \in v$ such that for any $x \in T_j$:

$$d(\sigma(x); \sigma(\partial T_j)) \leq z.$$

As $\sigma(\partial T_j)$ is of dimension strictly inferior to j we see that the image of this set under a projection onto R^j contains no open ball in R^j. In other words, if σ and $\partial\sigma$ are strongly admissible chains then σ is admissible. In consequence, a strongly admissible cycle is admissible.

2. Rectilinearizations of v-thin sets.

2.1. **Regular directions.** We recall a result proved in [V1] which will be very useful to compute our vanishing homology. We start by the definition of a regular direction. We denote by X_{reg} the set of points $x \in X$ at which X is a C^1 manifold.

Definition 2.1.1. Let X be a definable set of R^n. An element λ of S^{n-1} is said **regular for** X if there exists a positive $\alpha \in \mathbb{Q}$:

$$d(\lambda; T_x X_{reg}) \geq \alpha,$$

for any $x \in X_{reg}$.

Not every definable set has a regular line. However, we have:

Proposition 2.1.2. [V1] *Let A be a definable subset of R^n of empty interior. Then there exists a definable bi-Lipschitz homeomorphism $h : R^n \to R^n$ such that e_n is regular for $h(A)$.*

Remark 2.1.3. When e_n is regular for a set X, we may find finitely many Lipschitz definable functions, say $\xi_i : R^{n-1} \to R$, $i = 1, \dots, s$, satisfying

$$\xi_1 \leq \cdots \leq \xi_s, \tag{2.5}$$

and such that the set X is included in the union of their respective graphs.

2.2. **Cell decompositions.** In order to fix notations we recall the definition of the cells, which, as usual, are introduced inductively. All the definitions of this section deal with subsets of R^n, but since R stands for an arbitrary real closed field, we will use them for subsets of k_v^n as well.

Definitions 2.2.1. For $n = 0$ a cell of R^n is $\{0\}$. A **cell** E of R^n is either the graph of a definable function $\xi : E' \to R$, where E' is a cell of R^{n-1} or a band of type:

$$\{x = (x'; x_n) \in E' \times R : \xi_1(x') < x_n < \xi_2(x')\}, \tag{2.6}$$

where $\xi_1, \xi_2 : E' \to R$ are two definable functions satisfying $\xi_1 < \xi_2$ or $\pm\infty$. The cell E is **Lipschitz** if E' is Lipschitz and if ξ_1 and ξ_2 (or ξ) are Lipschitz functions (and $\{0\}$ is Lipschitz). A **closed cell** is the closure of a cell (which is obtained by replacing $<$ by $\leq$ in the definition).

Given $z \in R$, the Lipschitz cell E is **z-admissible** if

(1) E' is z-admissible
(2) If E is a band defined by two functions ξ_1 and ξ_2, then either $(\xi_2 - \xi_1)(x) \leq z$ for any $x \in E'$, or $(\xi_2 - \xi_1)(x) \geq z$ for any $x \in E'$.

Set also that the cell $\{0\}$ is z-admissible.

A cell E of dimension j is canonically homeomorphic to $(0; 1)^j$. The **barycentric subdivision of** E is the partition defined by the image by this homeomorphism of the barycentric subdivision of $(0; 1)^j$.

We shall need the following very easy lemma.

Lemma 2.2.2. *Let E' be a w-thick Lipschitz cell of k_v^{n-1} and let $\xi_s : E' \to k_v$, $s = 1, 2$, be two Lipschitz functions such that $\xi_1 < \xi_2$ and $(\xi_1 - \xi_2)(x) \notin w$, for any $x \in E'$. Then the band:*

$$E := \{(x; y) \in E' \times k_v : \xi_1(x) < y < \xi_2(x)\}$$

is w-thick.

Proof. We may assume that E' is open in k_v^{n-1} since we may find a bi-Lipschitz homeomorphism which carries E' onto an open cell. Then E is also an open cell and, for any positive $z \in w$ the cell E' contains a ball of radius z, say $B(x_0; z)$. In this case, let

$$t_0 := \frac{\xi_2(x_0) - \xi_1(x_0)}{2},$$

and $y_0 := (x_0; t_0)$. We shall show that that $B(y_0; z) \subset E$.

For any $y = (x; t) \in B(y_0; z)$, we may write

$$t - \xi_1(x) = (t - t_0) + (t_0 - \xi_1(x_0)) + (\xi_1(x_0) - \xi_1(x)), \tag{2.7}$$

and observe that as $|t - t_0| \le t \in w$, we certainly have $(t - t_0) \in w$. On the other hand, since ξ_1 is Lipschitz $(\xi_1(x_0) - \xi_1(x))$ belongs to w as well. We have by assumption $2(t_0 - \xi_1(x_0)) \notin w$ so that $(t - \xi_1(x_0)) \notin w$. Therefore,

$$(t_0 - \xi_1(x_0)) > |t - t_0| + |\xi_1(x_0) - \xi_1(x)|,$$

which, thanks to (2.7), implies that $(t_0 - \xi_1(x_0))$ and $(t - \xi_1(x))$ have the same sign. This means that $(t - \xi_1(x_0))$ is positive. Arguing in the same way, we could show that $(t - \xi_2(x))$ is negative. This shows that $y \in E$. □

Definition 2.2.3. The subset $\{0\}$ is an **L-cell decomposition of** R^0. For $n > 0$, an **L-cell decomposition** of R^n is a cell decomposition of R^n satisfying:

(i) The cells of R^{n-1} constitute an L-cell decomposition of R^{n-1}
(ii) There exist finitely many Lipschitz functions $\xi_1, \dots, \xi_s : R^{n-1} \to R$ satisfying (2.5) such that the union of all the cells which are graphs of a function on a subset of R^{n-1}, is the union of the graphs of the ξ_i's.

An L-cell decomposition is said **compatible with** finitely many given definable subsets $X_1, \dots, X_m$ if these subsets are union of cells. It is said **z-admissible** if every cell is z-admissible. Taking the barycentric subdivision of every cell, we get a **barycentric subdivision** of an L-cell decomposition.

We are going to show that, we may find a u-admissible L-cell decomposition which is compatible with some given $\mathcal{L}_R(u)$-definable subsets of k_v^n. This will be helpful to prove that the homology groups are finitely generated, since we will show that only the $\mathbb{N}u$-thin cells are relevant to compute the homology groups. The following proposition deals with subsets of k_v since we will apply it to k_v but of course the proof goes over an arbitrary model of the theory.

Proposition 2.2.4. *Let $X_1, \dots, X_m$ be $\mathcal{L}_R(u)$-definable subsets of k_v^n. There exists a $\mathcal{L}_R(u)$ definable bi-Lipschitz homeomorphism $h : k_v^n \to k_v^n$ such that we can find a u-admissible L-cell decomposition of k_v^n compatible with $h(X_1), \dots, h(X_m)$.*

Proof. For $n = 0$ there is nothing to prove. Assume $n > 1$ and apply Proposition 2.1.2 to $\cup_{j=1}^m \partial X_j$ (where ∂ denotes the topological boundary). Then (see Remark 2.1.3) there exist finitely many definable Lipschitz functions ξ_i, $i = 1, \dots, s$ satisfying (2.5). Consider a cell decomposition of k_v^n compatible with $X_1, \dots, X_m$, all the graphs of the ξ_i's, as well as all the sets

$$\{x \in k_v^{n-1} : \xi_{i+1}(x) - \xi_i(x) = u\}.$$

Now apply the induction hypothesis to all the cells of this decomposition which lie in k_v^{n-1} to get a cell decomposition $\mathcal{E}$ of k_v^{n-1}. Then set $\xi_0 := -\infty$, $\xi_{s+1} := \infty$, and consider the cell decomposition of k_v^n constituted by the graphs of the restrictions of the functions ξ_i's to an element of $\mathcal{E}$ on the one hand, and all the subsets of type:

$$\{(x; x_n) \in E \times k_v : \xi_i(x) < x_n < \xi_{i+1}(x)\},$$

where $E \in \mathcal{E}$, on the other hand. The required properties hold. $\square$

2.3. **Rectilinearization of v-thin sets.** We introduce the notion of rectilinearization. This is a mapping which transforms a set into a union of coordinate hyperplanes and which induces an isomorphism in homology (the usual one). Admissible rectilinearizations will be very helpful to construct strongly admissible chains (see section 1.5). It will be convenient to "rectilinearize" several sets simultaneously in order to get simplicial chains compatible with a given family (see section 1.4).

We are going to show that we can always find a v-admissible rectilinearization compatible with a given family of v-thin sets.

Definitions 2.3.1. A **hyperplane complex** is a subset W of R^n, which is a union of finitely many coordinate hyperplanes of type $x_j = s$ where, for each hyperplane, s is an integer. There is a canonical cell decomposition of R^n compatible with W. We refer to the cells (resp. closure of the cells) as the **cells of W** (resp. **closed cells of W**).

Let $X_1, \dots, X_m$ be definable subsets. A **rectilinearization** of $X_1, \dots, X_m$ is a continuous mapping $h : R^n \to R^n$, such that the $h^{-1}(X_i)$'s are union of cells of W and such that for any $i = 1, \dots, m$ the mapping $h_i : h^{-1}(X_i) \to X_i$ induces an isomorphism in homology (the usual one).

If $X_1, \dots, X_m$ are v-thin, a rectilinearization of $X_1, \dots, X_m$ is **v-admissible** if for each cell σ of W included in $h^{-1}(X_i)$ there exists an integer q with e_q tangent to σ for which

$$(h(x) - h(x + \lambda e_q)) \in v \tag{2.8}$$

for any $x \in \sigma$ and $\lambda \in R$ such that $x + \lambda e_q \in \sigma$.

Remark 2.3.2. After a barycentric subdivision of $h^{-1}(X_i)$, we get a simplicial complex K_i and a map $h_i : K_i \to X_i$ which induces an isomorphism in homology. Note that, thanks to (2.8) each simplicial chain gives rise (identifying each j-simplex to T_j in a linear way) to a strongly admissible chain (see Definition 1.5.1). Moreover, as h induces an isomorphism in homology, this identification defines an isomorphism in homology $H_j(K_i) \to H_j(X_i)$.

Proposition 2.3.3. *Let $X_1, \dots, X_m$ be closed definable v-thin subsets of R^n. Then there exists a v-admissible rectilinearization of $X_1, \dots, X_m$.*

Proof. We start by proving the following statements $(\mathbf{H_n})$ by induction on n.

$(\mathbf{H_n})$. Let $\mathcal{E}$ be a u-admissible L-cell decomposition of k_v^n and let $Y_1, \dots, Y_r$ denote the w-thin closed cells. Then there exists a $\mathbb{N}u$-admissible rectilinearization $h : k_v^n \to k_v^n$ of $Y_1, \dots, Y_r$ such that, for every E in $\mathcal{E}$, $h^{-1}(cl(E))$ is a union of closed cells of W and there exists a strong deformation retract $r_E : h^{-1}(cl(E)) \times I \to C_E$, where C_E is a closed cell of W.

Note that it follows from the existence of this deformation retract that h induces an isomorphism in homology above any union of closed cells of $\mathcal{E}$. Actually, the existence of r_{Y_i} implies

$$H_j(h^{-1}(Y_i)) \simeq H_j(C_{Y_i}) \simeq H_j(Y_i),$$

and the map $h_{|h^{-1}(Y_i)} : h^{-1}(Y_i) \to Y_i$ induces an isomorphism in homology. Therefore, thanks to the Mayer-Vietoris property and to the 5-Lemma, we see that for any subset X constituted by the union of finitely many closed cells the map $h_{|h^{-1}(X)} : h^{-1}(X) \to X$ induces an isomorphism in homology.

Note that nothing is to be proved for $n = 0$ and assume $(\mathbf{H_{n-1}})$. Apply the induction hypothesis to the family constituted by the closure of the cells of $\mathcal{E}$ in k_v^{n-1} which are w-thin to get a rectilinearization $h : k_v^{n-1} \to k_v^{n-1}$ and a hyperplane complex W.

Note that by definition, the cells of $\mathcal{E}$ on which the restriction of π_n is one-to-one are included in the union of finitely many graphs of definable Lipschitz functions $\xi_1, \dots, \xi_s : k_v^{n-1} \to k_v$ satisfying (2.5).

We obtain a hyperplane complex $\widetilde{W}$ by taking the inverse image of W by π_n, and by adding the hyperplanes defined by $x_n = i$, $i = 1, \dots, s$.

Define now the desired mapping $\widetilde{h}$ as follows:

$$\widetilde{h}(x; i + t) = (h(x);\ (1 - t)\xi_i(h(x)) + t\xi_{i+1}(h(x)))$$

for $1 \le i < s$ integer, $x \in k_v^n$ and $t \in [0; 1)$. Define also:

$$\widetilde{h}(x;\ 1 - t) = (h(x); \xi_1(h(x)) - t)$$

and

$$\widetilde{h}(x;\ s + t) = (h(x); \xi_s(h(x)) + t)$$

for $t \in [0;\infty)$. This defines a mapping $\widetilde{h} : k_v^n \to k_v^n$. We are going to check that

$$|\widetilde{h}(x) - \widetilde{h}(x + \lambda e_n)| \leq u \tag{2.9}$$

when x and $(x + \lambda e_n)$ belong to the same cell.

Let σ be a cell of $\widetilde{W}$ which is mapped into $\cup_{i=1}^r Y_i$. If $\pi_n(\sigma)$ is w-thin (2.9) follows from the induction hypothesis. Otherwise $\widetilde{h}(\sigma)$ must lie in the band delimited by the graphs of the restrictions of ξ_i and ξ_{i+1} for some $i \in \{1, \dots, s-1\}$ as described in (2.6). If $\widetilde{h}(\pi_n(\sigma))$ fails to be w-thin then, thanks to Lemma 2.2.2 (recall that $\widetilde{h}(\sigma)$ is w-thin) and the u-admissibility of the cell decomposition, we necessarily have:

$$|\xi_i(x) - \xi_{i+1}(x)| \leq u,$$

for any $x \in \pi_n(\sigma)$. This, together with definition of $\widetilde{h}$, implies that $\widetilde{h}$ satisfies (2.9) and yields that $\widetilde{h}$ is $\mathbb{N}u$-admissible. It remains to find the retraction r_E for each cell E.

Fix $E \in \mathcal{E}$ and observe that it follows from the definition of $\widetilde{h}$ and the induction hypothesis that $\widetilde{h}^{-1}(cl(E))$ is a union of cells of $\widetilde{W}$. If E is the graph of a function $\xi : E' \to k_v$ (where $E' := \pi_n(E)$), then the result directly follows from the induction hypothesis. Otherwise, since $\mathcal{E}$ is an L-cell decomposition, the cell E lies in the band delimited by the graphs of two consecutive functions, say ξ_i and ξ_{+1}. Let

$$\Gamma_i := \{(x; x_n) \in k_v^{n-1} \times k_v : i \leq x_n \leq i+1\}.$$

We first define first a retract:

$$r'_E : \widetilde{h}^{-1}(cl(E)) \times [0; \frac{1}{2}]_{k_v} \to \Gamma_i \cap \widetilde{h}^{-1}(cl(E)),$$

by setting for $x_n \geq i+1$:

$$r'_E(x; x_n; t) := (x; 2tx_n + (1-2t)(i+1)),$$

and for $x_n \leq i$:

$$r'_E(x; x_n; t) := (x; 2tx_n + (1-2t)i),$$

and of course $r'_E(x; x_n; t) := (x; x_n)$ when $i \leq x_n \leq i+1$.

Note that it follows from the definition of $\widetilde{h}$ that if $(x; x_n)$ belongs to $\widetilde{h}^{-1}(cl(E))$ then for any $i+1 \leq x'_n \leq x_n$ and any $x_n \leq x'_n \leq i$:

$$\widetilde{h}(x; x'_n) = \widetilde{h}(x; x_n).$$

This implies that r'_E preserves $\widetilde{h}^{-1}(cl(E))$.

On the other hand, thanks to the induction hypothesis, there exists a retract $r_{E'} : h^{-1}(cl(E')) \times [0;1]_{k_v} \to C_{E'}$. Let us extend this $r_{E'}$ into a retract:

$$r''_{E'} : \pi_n^{-1}(h^{-1}(cl(E'))) \times [\frac{1}{2}; 1]_{k_v} \to \pi_n^{-1}(C_{E'})$$

by

$$r'_E(x; x_n; t) := (r_{E'}(x; 2t-1); x_n).$$

Clearly, there exists a unique cell C_E of $\widetilde{W}$ which is included in Γ_i and which projects on $C_{E'}$. Now, these retracts give rise to a retract

$$\widetilde{r}_E : \widetilde{h}^{-1}(cl(E)) \times [0;1]_{k_v} \to C_E$$

defined by $\widetilde{r}_E(x;t) := r'_E(x;t)$ if $t \leq \frac{1}{2}$ and

$$\widetilde{r}_E(x;t) := r''_E(r'_E(x;\frac{1}{2});t)$$

if $t \geq \frac{1}{2}$. This yields $(\mathbf{H_n})$.

We return to the proof of the proposition. Apply Proposition 2.2.4 to $X_{1,v}, \dots, X_{m,v}$. This provides a bi-Lipschitz homeomorphism $g : k_v^n \to k_v^n$ such that we can find a u-admissible L-cell decomposition of k_v^n compatible with $g(X_{1,v}), \dots, g(X_{m,v})$. Note that, as the $g(X_{i,v})$'s are w-thin, each of them is the union of some w-thin cells. Then by $(\mathbf{H_n})$, there exists a $\mathbb{N}u$-admissible rectilinearization of these cells $h : k_v^n \to k_v^n$.

Composing with g, the mapping h gives rise to a $\mathbb{N}u$-admissible rectilinearization f of $X_{1,v}, \dots, X_{m,v}$. As the $X_{i,v}$ are extensions, there exist two families of rectilinearizations f_z and h_z for $z \in [a;b]$ with $a < u < b$ and $a, b \in R$. Let us check that these rectilinearizations are v-admissible for $z \in v$ large enough.

Note that each X_i is the union of the images by h_z of finitely many cells of W. Furthermore, as (2.9) is a first order formula we get that h_z satisfies on any given cell in the inverse image of the X_i's:

$$|h_z(x) - h_z(x + \lambda e_n)| \leq z,$$

when x and $(x + \lambda e_n)$ belong to this given cell.

This implies that f satisfies (since g is bi-Lipschitz):

$$|f_z(x) - f_z(x + \lambda e_n)| \leq Nz,$$

for some $N \in \mathbb{N}$ and any $z \in v$ large enough on any cell mapped into one of the X_i's. Thus, (2.8) holds and f_z is w-admissible. □

Remark 2.3.4. Actually, working a little more, we could have proved that the constructed rectilinearization induces an isomorphism in homology above any subset A of R^n. Namely, in the above proof, given a subset A of R^n, the induced mapping $\widetilde{h} : \widetilde{h}^{-1}(A) \to A$ induces an isomorphism in homology.

Observe also that the constructed mapping is a homeomorphism above a dense definable subset. If we take an algebraic hypersurface, the situation is fairly similar to the one which occurs with resolution of singularities in the sense that the inverse image of the set above which the map is not one-to-one (the "exceptional divisor") is constituted by finitely many coordinate hyperplanes normal to the hyperplanes lying above our given set. We could also have a more precise description of how the mapping h modifies the distances (like in [V1]). More precisely, it is possible to see that on each cell, we have

$$|h(x) - h(x')| \sim \sum_{i=1}^{n} \varphi_i(x)|x_i - x'_i|$$

where φ_i is a quotient of sums of product of powers of distances to cells of W. If we compare this result with Theorem 5.1.3 of [V1], we see that now the contractions (see [V1]) are expressed in the canonical basis. The inconvenient is that the map h is not a homeomorphism (contrarily as in [V1]), but since it induces an isomorphism between the homology groups, it will be enough for the purpose of the present paper.

3. The vanishing homology groups are finitely generated

3.1. **Some preliminary lemmas.** Every mapping $\sigma : T_j \to X$ may be extended to a mapping $\sigma_v : T_j(k_v) \to X_v$ (see subsection 1.1). Let $\Delta_j^{ext}(X_v)$ be the submodule of $C_j^w(X_v)$ generated by the simplices which are extensions of an element of $C_j^v(X)$. Clearly, for each j the mapping:

$$ext : C_j^v(X) \to \Delta_j^{ext}(X_v),$$

which assigns to every chain σ the chain σ_v, induces an isomorphism in homology.

The following Lemma says that the vanishing homology groups for the velocities $\mathbb{N}u$ and w coincide with the homology groups of Δ_j^{ext} when the considered set is $\mathcal{L}_R$-definable.

Lemma 3.1.1. *Let X be a definable subset of R^n. Then the maps induced by the inclusions $H_j(\Delta^{ext}(X_v)) \to H_j^u(X_v)$ and $H_j(\Delta^{ext}(X_v)) \to H_j^w(X_v)$ are isomorphisms for any j.*

Proof. We do the proof for u. To get the proof for w, just replace u by w. We first check that this map is onto. Let $\sigma = \sum_{i\in I} g_i c_i \in Z_j^u(X_v)$. By definition of u there exist finitely many $\mathcal{L}_R$-definable mappings, say $\tau_i : T_j(k_v) \times [a;u]_{k_v} \to X_v$, with $a \in v$ such that $c_i(x) = \tau_i(x;u)$ for any $x \in T_j(k_v)$. Define $\theta_i(x) := \tau_i(x;a)$ and $\theta := \sum_{i\in I} g_i\theta_i \in C_j^{ext}(X_v)$. Observe that τ_i gives rise to a $\mathbb{N}u$-admissible $(j+1)$-chain (after a subdivision of $T_j(k_v) \times [a;u]$). Moreover, as the property of admissibility may be expressed by a formula with parameters in R and with u, we know that the obtained chain is $\mathbb{N}u$-admissible if a is chosen large enough. Set $\tau := \sum_{i\in I} g_i\tau_i \in C_{j+1}^u(X_v)$ and note that since $\tau_i(x;u) = c_i(x)$ and $\tau_i(x;a) = \theta_i(x)$ we clearly have $\partial\tau = \sigma - \theta$. As θ belongs to $C_j^{ext}(X_v)$, this implies that the inclusion $C_j^{ext}(X_v) \to C_j^u(X_v)$ induces a surjection in homology.

We now check that this map is injective by applying a similar argument. Let $\alpha \in C_j^{ext}(X_v)$ with $\alpha = \partial\sigma$ where σ belongs to $C_{j+1}^u(X_v)$. The chain σ induces chains $\tau \in C_{j+2}^u(X_v)$ and $\theta \in C_{j+1}^{ext}(X_v)$ such that $\partial\tau = \sigma - \theta$ in the same way as in the previous paragraph. But this implies $\partial\theta = \alpha$ which means that $\alpha \in \partial C_{j+1}^{ext}(X_v)$, as required. $\square$

Given a definable family Y of $R^n \times R$ and $t \in R$, we denote by Y_t the **fiber at** t:

$$\{x \in R^n : (x;t) \in Y\}.$$

We also define the **restriction of the family** to $[a;b]$ as follows:

$$Y_{[a;b]} := \{(x;t) \in Y : a \leq t \leq b\}.$$

Lemma 3.1.2. *Let Y be a $\mathcal{L}_R(u)$-definable family of $k_v^n \times k_v$ such that Y_u is a $\mathbb{N}u$-thin subset of k_v^n and let $j = \dim Y_u$. Then there exists z in v such that for any $t \in v$ greater than z the map induced by inclusion:*

$$H_k^w(Y_t) \to H_k^u(Y_{[z;u]}),$$

is an isomorphism for $k = j$ and is one-to-one for $k = j - 1$.

Proof. As Y is $\mathcal{L}_R(u)$-definable and $\mathbb{N}u$-thin there exists z in v such that for any t in v greater than z, Y_t is w-thin. Thanks to Remark 1.3.1, this implies that the natural mapping $H_j^w(Y_t) \to H_j(Y_t)$ is an isomorphism.

Furthermore, since the family Y is topologically trivial if the interval $[z;u]$ is chosen small, the inclusion $H_j(Y_t) \to H_j(Y_{[z;u]})$ induces an isomorphism in homology as well.

We have the following commutative diagram for $t \in v$ greater than z:

$$\begin{tikzcd}
H_j^w(Y_t) \arrow[r, "1"] \arrow[d, "3"] & H_j(Y_t) \arrow[d, "2"] \\
H_j^u(Y_{[z;u]}) \arrow[r, "4"] & H_j(Y_{[z;u]})
\end{tikzcd}$$

By the above, the arrows 1 and 2 are isomorphisms. Moreover as Y_u is $\mathbb{N}u$-thin the family $Y_{[z;u]}$ is $\mathbb{N}u$-thin. Thus, the arrow 4 is an monomorphism (see the last sentence of Remark 1.3.1). This implies that the arrow 3 is an isomorphism and establishes the theorem in the case $k = j$.

Now, in the case where $k = j - 1$ we can write the same diagram for H_{j-1}. The arrows 1 and 2 (of the obtained diagram) are still one-to-one (again thanks to Remark 1.3.1 and the topological triviality of $Y_{[z;u]}$), so that the arrow 3 is clearly one-to-one. $\square$

The following lemma is a consequence of existence of v-admissible rectilinearizations.

Lemma 3.1.3. *Given $X \subset k_v^n$ $\mathcal{L}_R(u)$-definable and $\mathcal{F}$ finite family of closed $\mathcal{L}_R(u)$-definable subsets of X, the map $\widehat{H}_j^w(X;\mathcal{F}) \to H_j^w(X)$, induced by the inclusion, is onto.*

Proof. Let $\sigma \in C_j^w(X)$. If the support of σ is of dimension $< j$ then the class of σ is 0 in $H_j^w(X)$. Thus, we may assume that $dim|\sigma| = j$.

Let $h : k_v^n \to k_v^n$ be a w-admissible rectilinearization of $|\sigma|$ and of all the elements of $\mathcal{F}$. There exists a simplicial chain τ (see Remark 2.3.2), which is strongly w-admissible since h is w-admissible, such that $\sigma = \tau$ in $H_j(|\sigma|) = H_j^w(|\sigma|)$ (see Remark 1.3.1). But this means that the class of τ is that of σ also in $H_j^w(X)$. This yields that the inclusion induces an onto map in homology. $\square$

It is unclear for the author whether the inclusion of the above lemma is one-to-one. Actually, it is even unclear whether $\widehat{H}_j^v(X)$ is finitely generated.

3.2. **The main result.** It is very hard to construct homotopies which are Lipschitz mappings. To compute the homology, we actually just need to find a homotopy that carries a chain σ to the cells of a given cell decomposition, and which preserves the v-admissibility of the chain σ. We prove something even weaker: given a strongly w-admissible chain, we may construct a homotopy which carries the chain σ to a strongly $\mathbb{N}u$-admissible chain of the cells of dimension j. This is enough since we have seen that we had isomorphisms between the theories defined by w and $\mathbb{N}u$. This technical step is performed in the following proposition.

Proposition 3.2.1. *Let X be a closed $\mathcal{L}_R(u)$-definable subset of k_v^n and let $\mathcal{E}$ be a u-admissible L-cell decomposition compatible with X. Let $\mathcal{F}$ be the family constituted by the closed cells of $\mathcal{E}$ and let Y_j be the union of the closures of the $(\mathbb{N}u; j)$-thin elements of the barycentric subdivision of $\mathcal{E}$ which lie in X. Then, there exists a map*

$$\varphi : \widehat{C}_j^w(X; \mathcal{F}) \to \widehat{C}_j^u(Y_j)$$

such that:

- *(i)* $\quad \varphi\partial - \partial\varphi = 0$
- *(ii)* *For any $\sigma \in \widehat{Z}_j^w(X; \mathcal{F})$ we have: $\varphi_\sigma = \sigma$, in $H_j^u(X)$,*
- *(iii)* *If Y is the union of some elements of $\mathcal{F}$, then for any $\sigma \in \widehat{Z}_j^w(X; \mathcal{F})$ with $|\sigma| \subset Y$ we have: $\varphi_\sigma = \sigma$ in $H_j^u(Y)$.*

Proof. We are going to prove the following statements:
Claim. Given $\sigma \in C_j(X; \mathcal{F})$, there exists a definable homotopy

$$h_\sigma : T_j(k_v) \times [0; 1]_{k_v} \to X,$$

such that:

1. For each x the path $t \mapsto h_\sigma(x; t)$ stays in the same closed cell,
2. For each t the map $x \mapsto h_\sigma(x; t)$ is a strongly $\mathbb{N}u$-admissible simplex if σ is a strongly w-admissible simples,
3. If σ is strongly w-admissible, the support of the simplex $\varphi_\sigma : T_j(k_v) \to X$ defined by $\varphi_\sigma(x) = h_\sigma(x; 1)$ entirely lies in Y_j

(4) We have
$$\partial h_*(\sigma) - h_*(\partial\sigma) = \varphi_\sigma - \sigma$$
for any $\sigma \in C_j(X;\mathcal{F})$ where (as usual) $h_* : C_j(X;\mathcal{F}) \to C_{j+1}(X;\mathcal{F})$ is the mapping induced by h on the chain complexes.

Note that φ is defined by (3). Observe that (4) implies (i), together with (2) implies (ii), and together with (1) yields (iii).

We prove that it is possible to construct such a homotopy by induction on n (the dimension of the ambient space). Let $\mathcal{E}'$ be the cell decomposition of k_v^{n-1} constituted by all the cells of $\mathcal{E}$ lying in k_v^{n-1}. Let σ in $C_j(X;\mathcal{F})$ and write $\sigma := (\widetilde{\sigma};\sigma_n) \in k_v^{n-1} \times k_v$. Apply the induction hypothesis to $\widetilde{\sigma}$ and $\mathcal{E}'$ to get a homotopy $h_{\widetilde{\sigma}} : T_j(k_v) \times [0;1]_{k_v} \to k_v^{n-1}$.

By definition, the union of the cells of $\mathcal{E}$ on which π_n is one-to-one is given by the graphs of finitely many Lipschitz functions $\xi_1 \le \cdots \le \xi_s$. Note that we may retract the cells above (resp. below) the graph of ξ_s (resp. ξ_1) onto the graph of ξ_s (resp. ξ_1) so that we may assume that X entirely lies between these two graphs.

By compatibility with $\mathcal{F}$ we know that the support of σ entirely lies in one single cell $E \in \mathcal{E}$ which is either the graph of a Lipschitz function ξ or a band which is delimited by the graph of the restriction to $E' := \pi_n(E)$ of two consecutive functions ξ_i and ξ_{i+1}, with $\xi_i < \xi_{i+1}$ on E'. In the latter case, we may define a function $\nu_\sigma : T_j(k_v) \to [0;1]_{k_v}$ by setting for $x \in T_j(k_v)$
$$\nu_\sigma(x) := \frac{\sigma_n(x) - \xi_i(\widetilde{\sigma}(x))}{\xi_{i+1}(\widetilde{\sigma}(x)) - \xi_i(\widetilde{\sigma}(x))}.$$

To deal with both cases simultaneously it is convenient to set $\nu_\sigma(x) \equiv 0$ and $\xi_i = \xi_{i+1} = \xi$, if the cell is described by the graph of a single function ξ. To define h_σ we first define a function $s_\sigma : T_j(k_v) \to [0;1]_{k_v}$. We set:
$$s_\sigma(e_i) = 0 \quad \text{if} \quad \sigma_n(e_i) - \xi_i(\widetilde{\sigma}(e_i)) \in w \quad \text{and} \quad \xi_{i+1}(\widetilde{\sigma}(e_i)) - \sigma_n(e_i) \neq 0$$
$$\text{and} \quad s_\sigma(e_i) = 1 \quad \text{otherwise.}$$

Then we extend s_σ over $T_j(k_v)$ linearly.

Now we can set for $(x;t) \in T_j(k_v) \times [0;\frac{1}{2}]_{k_v}$:
$$\theta(x;t) = 2ts_\sigma(x) + (2t-1)\nu_\sigma(x).$$

Set for simplicity: $\xi' = \xi_{i+1} - \xi_i$ and, for $x = (\widetilde{x};x_n) \in k_v^{n-1} \times k_v$ and $t \in [0;1]_{k_v}$, let:
$$h_\sigma(x;t) := (\widetilde{\sigma}(x); \xi_i(\widetilde{\sigma}(x)) + \theta(x;t)\xi'(\widetilde{\sigma}(x))) \qquad \text{if} \quad t \le \frac{1}{2}$$
$$h_\sigma(x;t) := (\, h_{\widetilde{\sigma}}(\widetilde{x};2t-1)\,;\, \xi_i(h_{\widetilde{\sigma}}(\widetilde{x};2t-1)) + s_\sigma(x)\,\xi'(\, h_{\widetilde{\sigma}}(\widetilde{x};2t-1))\,) \qquad \text{if} \quad t \ge \frac{1}{2}.$$

Note that as s_σ (resp. ν_σ) satisfies:
$$s_{\partial\sigma} = \partial s_\sigma$$

(resp. $\nu_{\partial\sigma} = \partial\nu_\sigma$), we see that the map induced by h_σ is a chain homotopy. Moreover, it is clear from the definition of h_σ that the path $t \mapsto h_\sigma(x;t)$ remains in the same closed cells. Therefore (1) and (4) hold.

To check (2), fix a strongly admissible simplex σ. We have to check that there exists $q \in \{1,\dots,n\}$ such that:

$$(h_\sigma(x+\lambda e_q;t) - h_\sigma(x;t)) \in \mathbb{N}u \tag{3.10}$$

for any $(x;\lambda) \in T_j^q(k_v)$ and any t in $[0;1]_{k_v}$.

If σ is the graph of one single function ξ then the result is immediate for $t \le \frac{1}{2}$ and follows from the induction hypothesis for $t \ge \frac{1}{2}$.

By definition of strongly admissible simplices there exists a vector of the canonical basis, say e_q, such that:

$$(\sigma(x) - \sigma(x+\lambda e_q)) \in w, \tag{3.11}$$

for any $(x;\lambda) \in T_j^q(k_v)$. This implies that

$$(\sigma(0) - \sigma(e_q)) \in w. \tag{3.12}$$

We distinguish two cases:

<u>*First case:*</u> $s_\sigma(0) = s_\sigma(e_q)$. This implies that for any $(x;\lambda) \in T_j^q(k_v)$ we have

$$s_\sigma(x) = s_\sigma(x+\lambda e_q),$$

and therefore

$$|\theta(x) - \theta(x+\lambda e_q)| \le |\nu_\sigma(x) - \nu_\sigma(x+\lambda e_q)|. \tag{3.13}$$

Note that if $\xi'(\widetilde{\sigma}(x)) \in w$ then $\xi'(\widetilde{\sigma}(x+\lambda e_q)) \in w$, which means that in this case (3.10) follows immediately from (3.11) for $t \le \frac{1}{2}$. Otherwise $\xi'(\widetilde{\sigma}(x)) \notin w$ and then by (3.11):

$$\frac{1}{2}\xi'(\widetilde{\sigma}(x)) \le \xi'(\widetilde{\sigma}(x+\lambda e_q)) \le 2\,\xi'(\widetilde{\sigma}(x)). \tag{3.14}$$

Recall that the functions ξ_i and ξ_{i+1} are both Lipschitz functions. Hence, if σ is strongly admissible, for $t \le \frac{1}{2}$ a straightforward computation shows that thanks to (3.13) and (3.14) we have for any $(x;\lambda) \in T_j^q(k_v)$:

$$(h_\sigma(x+\lambda e_q;t) - h_\sigma(x;t)) \in w. \tag{3.15}$$

For $t \ge \frac{1}{2}$, (3.10) still holds thanks to the induction hypothesis and the Lipschitzness of ξ_i and ξ_{i+1}.

<u>*Second case:*</u> $s_\sigma(0) \ne s_\sigma(e_q)$. In this case we observe that if $s_\sigma(0)$ is 0 then

$$(\sigma_n(0) - \xi_i(\widetilde{\sigma}(0))) \in w$$

which amounts to

$$d(\sigma(0);\Gamma_{\xi_i}) \in w,$$

(where Γ_{ξ_i} denotes the graph of ξ_i). By (3.12), this implies that $d(\sigma(e_q);\Gamma_{\xi_i})$ belongs to w and so

$$(\sigma_n(e_q) - \xi_i(\widetilde{\sigma}(e_q))) \in w.$$

As $s_\sigma(e_q)$ is necessarily equal to 1 we see that

$$\sigma_n(e_q) - \xi_{i+1}(\widetilde{\sigma}(e_q)) = 0$$

so that

$$\xi'(\widetilde{\sigma}(e_q)) \in w.$$

But, as the cell E is u-admissible this implies that for *any* $x \in E'$:

$$\xi'(x) \leq u.$$

This, together with the induction hypothesis, implies that h_σ satisfies (3.10). This completes the proof of (2).

It remains to prove (3). First observe that all the e_j's are sent by φ_σ onto vertices of E. Note also that

$$\varphi_\sigma(x) = (\varphi_{\widetilde{\sigma}}(x); \xi_i(\varphi_{\widetilde{\sigma}}(x)) + s_\sigma(x)\xi'(\varphi_{\widetilde{\sigma}}(x)))$$

and so, by the definition of the cells, the support of φ_σ lies in cells of dimension at most j of $\mathcal{F}$. Moreover we just checked that (3.10) holds in any case. This implies that φ_σ is strongly admissible and therefore its support must lie in Y_j. This completes the proof of the claim. □

We are now able to express the v-vanishing homology groups in terms of the (usual) homology groups of some v-thin subsets constituted by the v-thin cells of the barycentric subdivision of some L-cell decompositions.

Theorem 3.2.2. *For any $X \subset R^n$ closed definable, there exist some definable subsets of X:*

$$X_0 \subset \cdots \subset X_{d+1} = X_d$$

such that:

$$H_j^v(X) \simeq Im(H_j(X_j) \to H_j(X_{j+1}))$$

(where the arrow is induced by inclusion and Im stands for image).

Proof. We start by defining inductively the subsets X_j 's. Set $X_0 = \emptyset$ and assume that $X_0, \ldots, X_{j-1}$ have already been defined. According to Proposition 2.2.4, up to a bi-Lipschitz homeomorphism, we can assume that we have a u-admissible L-cell decomposition compatible with X_v and $X_{j-1,v}$. Let $\mathcal{E}_j$ be the barycentric subdivision of this cell decomposition and define Θ_j as the union of all the $(j; \mathbb{N}u)$-thin cells. There exists a $\mathcal{L}_R$-definable family Y_j such that $Y_{j,u} = \Theta_j$. Now, thanks to Lemma 3.1.2, there exists z in v, such that for any t in v greater than z:

$$H_j^w(Y_{j,t}) \simeq H_j^u(Y_{j,u}).$$

Now define X_j as the subset of R^n defined by a $\mathcal{L}_R$-formula defining $Y_{j,t}$ for some $t \geq z$ in v. If t is chosen large enough, X_j is v-thin. As bi-Lipschitz homeomorphisms induce isomorphisms between the vanishing homology groups, we identify subsets with their image so that, for instance, we consider below the $X_{j,v}$'s and $Y_{j,u}$ as subsets of X_v.

Consider the following diagram:

$$Im\{H_j(X_j) \to H_j(X_{j+1})\} \overset{a}{\leftarrow} Im\{H_j^v(X_j) \to H_j^v(X_{j+1})\} \overset{b}{\to} H_j^v(X),$$

where again a and b are induced by the inclusions of the corresponding chain complexes. We shall show that a and b are both isomorphisms.

a is an isomorphism: We have the following commutative diagram:

$$\begin{tikzcd}
H_j^v(X_j) \arrow[r] \arrow[d] & H_j^v(X_{j+1}) \arrow[d] \\
H_j(X_j) \arrow[r] & H_j(X_{j+1})
\end{tikzcd}$$

where all the maps are induced by inclusion. By Remark 1.3.1, the first vertical arrow is an isomorphism and the second is one-to-one. This proves that a is an isomorphism.

b is onto: Note that it is enough to prove that the inclusion $X_j \to X$ induces an onto map between the v-vanishing homology groups.

We have the following commutative diagram:

$$\begin{tikzcd}
H_j^v(X_j) \arrow[r, "ext"] \arrow[d] & H_j(\Delta^{ext}(X_{j,v})) \arrow[r] & H_j^w(X_{j,v}) \arrow[d] \\
H_j^v(X) \arrow[r, "ext"] & H_{j+1}(\Delta^{ext}(X_v)) \arrow[r] & H_j^w(X_v)
\end{tikzcd} \quad \text{diag. 1.}$$

where the mapping ext, provided by extension of chains, is an isomorphism (see section 3.1).

By Lemma 3.1.1 the latter horizontal arrows are isomorphisms as well. Therefore, it is enough to prove that the map induced by inclusion $H_j^w(X_{j,v}) \to H_j^w(X_v)$ (the last vertical arrow) is onto.

For $t \geq z$ in v, let α and β be the maps defined by inclusion:

$$H_j^w(Y_{j,t}) \overset{\alpha}{\to} H_j(Y_{j,[z;u]}) \overset{\beta}{\leftarrow} H_j^u(Y_{j,u}).$$

By Lemma 3.1.2, α and β are isomorphisms so that $\gamma := \beta^{-1}\alpha$ provides the following commutative diagram:

$$\begin{tikzcd}
H_j^w(Y_{j,t}) \arrow[r] \arrow[d, "\gamma"'] & H_j^w(X_v) \arrow[d] \\
H_j^u(Y_{j,u}) \arrow[r] & H_j^u(X_v)
\end{tikzcd}$$

By Lemma 3.1.1 the second vertical arrow is onto. Thus, it is enough to show that $H_j^u(Y_{j,u}) \to H_j^u(X_v)$ is onto. By construction, $Y_{j,u}$ is the union of all the $(j;\mathbb{N}u)$-thin closed cells of the barycentric subdivision of $\mathcal{E}_j$. Note that it is enough to consider a chain $\sigma \in \widehat{Z}_j^w(X_v;\mathcal{F})$ where $\mathcal{F}$ is the family constituted by all the closure of the cells of $\mathcal{E}_j$ (since the inclusion $\widehat{H}_j^w(X_v;\mathcal{F}) \to H_j^u(X_v)$ is onto, thanks to Lemmas 3.1.1 and 3.1.3). By (ii) of Proposition 3.2.1, there exists $\varphi_\sigma \in C_j^u(Y_{j,u})$ such that $\sigma = \varphi_\sigma$ in $H_j^u(X_v)$, as required.

b is one-to-one: Note that as diag. 1. holds for X_{j+1} as well (and the horizontal arrows are isomorphisms as well), it is enough to show that the map induced by inclusion

$$b' : Im(H_j^w(X_{j,v}) \to H_j^w(X_{j+1,v})) \to H_j^w(X_v)$$

is one-to-one. Recall that by definition $X_{j+1,v}$ is $Y_{j+1,t}$, for some t and consider the following commutative diagram:

$$\begin{tikzcd}
H_j^w(X_{j,v}) \arrow[r] \arrow[d] & H_j^w(Y_{j+1,t}) \arrow[d, "\nu_t"] \\
H_j^u(Y_{j+1,u}) \arrow[r, "\nu_u"] & H_j^u(Y_{j+1,[z;u]})
\end{tikzcd}$$

where again ν_u and ν_t are induced by the respective inclusions. By Lemma 3.1.2 these maps are one-to-one.

This implies that we have the following commutative diagram:

$$\begin{tikzcd}
Im(H_j^w(X_{j,v}) \to H_j^w(Y_{j+1,t})) \arrow[r, "b'"] \arrow[d, "\mu"'] & H_j^w(X_v) \arrow[d] \\
Im(H_j^w(X_{j,v}) \to H_j^u(Y_{j+1,u})) \arrow[r, "b''"] & H_j^u(X_v)
\end{tikzcd}$$

where all the horizontal arrows are induced by the corresponding inclusions and μ is induced by the restriction of $\nu_u^{-1}\nu_t$. Since μ is one-to-one, it is enough to show that b'' is one-to-one.

To check that b'' is one-to-one, take σ in $Z_j^w(X_{j,v})$ which bounds a chain of $C_{j+1}^u(X_v)$. As the inclusion $H_j^w(X_v) \to H_j^u(X_v)$ is an isomorphism, there exists τ in $C_{j+1}^w(X_v)$ such that $\sigma = \partial\tau$. Consider a w-admissible rectilinearization of $|\tau|$, $|\sigma|$ and $\mathcal{F}$ where $\mathcal{F}$ is the family constituted by the closure the cells of the barycentric subdivision of $\mathcal{E}_{j+1}$. The chain σ is equal in $H_j(X_{j,v}) \simeq H_j^w(X_{j,v})$ (for $X_{j,v}$ is $(j;w)$-thin, see Remark 1.3.1) to a simplicial chain σ' which is strongly w-admissible and compatible with $\mathcal{F}$ (see Remark 2.3.2). The class of the chain σ is zero in $H_j(|\tau|)$ and therefore σ' bounds a simplicial chain τ' which is also strongly w-admissible (again by Remarks 1.3.1 and 2.3.2).

By construction, $X_{j,v}$ is a union of cells of $\mathcal{E}_{j+1}$ and the union of all the closure of the cells of dimension $(j+1)$ of the barycentric subdivision of $\mathcal{E}_{j+1}$ which are $(j+1;\mathbb{N}u)$-thin is precisely $Y_{j+1,u}$. Therefore we may apply Proposition 3.2.1 to X_v. This provides a map $\varphi : \widehat{C}_j^w(X_v;\mathcal{F}) \to \widehat{C}_j^u(Y_{j+1,u};\mathcal{F})$ such that

$$\partial\varphi_{\tau'} = \varphi_{\partial\tau'} = \varphi_{\sigma'}.$$

As by (iii) of this proposition $\sigma' = \varphi_{\sigma'}$ in $H_j^u(X_{j,v})$, this implies that the class of σ is zero in $H_j^u(Y_{j+1,u})$ and yields that b'' is one-to-one. □

Corollary 3.2.3. *For any closed definable subset X, the vanishing homology groups $H_j^v(X)$ are finitely generated.*

Note that the above corollary enables us to define an Euler characteristic which is a definable metric invariant by setting:

$$\chi_v(X) := \sum_{i=1}^{\infty}(-1)^i \dim H_i^v(X). \tag{3.16}$$

This invariant for definable subsets of R^n gives rise to a metric for definable families or for germs of definable sets (see example 1.3.2 and Corollary 0.0.2).

Remark 3.2.4. The hypothesis closed is assumed for convenience. We could shrink an open tubular neighborhood of radius $z \in v$ of the points lying in the closure but not in X so that we would have a deformation retract of our set onto the complement of this neighborhood which is very close to the identity, and hence which preserves thin subsets, identifying the vanishing homology groups of our given set with those of a closed subset.

4. Some examples.

We give some examples of computations of the homology groups. It is convenient to develop ad hoc techniques to compute the homology groups such as the excision property. Let us take $\mathbb{Q}$ as our coefficient group.

4.1. **The excision property.**

It follows from the definition that we may have $c+c'$ in C_j^v although neither c nor c' belong to this set. This is embarrassing since it makes it impossible the splitting of a chain of X into a chain of $X \setminus A$ plus a chain of A, which is crucial for the excision property. To overcome this difficulty we are going to consider more chains. This will *not* affect the resulting homology groups.

We defined the vanishing homology groups by requiring for a chain σ that $|\sigma|$ and $|\partial\sigma|$ to be both $(j;v)$-thin. We may work with another chain complex.

Let A and X be closed definable subsets of R^n with $A \subset X$ and denote by $\mathcal{F}$ the pair $\{X \setminus Int(A); A\}$ where $Int(A)$ is the interior of A. Let $\Delta_j^v(X)$ the subset of $C_j^v(X;\mathcal{F})$ constituted by the j-chains having a $(j;v)$-thin support. Of course, such a family of modules is not preserved by the boundary operator but, if we want to have a chain complex, we may add the boundaries by setting:

$$\Delta_j'^v(X) := \Delta_j^v(X) + \partial\Delta_{j+1}^v(X).$$

This provides a chain complex with obviously $H_j(\Delta'^v(X)) = H_j^v(X)$.

The inconvenient is that we are going to work with non admissible chains but the advantage is that we have now more freedom to work since we have more chains. For instance if $(c_1 + c_2) \in \Delta_j'^v(X)$ then c_1 and c_2 both belong to $\Delta_j'^v(X)$.

To state the excision property we need to introduce the homology groups of a pair. For this purpose, we first set:

$$\Delta_j^v(X/A) := \{c \in \Delta_j^v(X) : (\partial c - \partial_A c) \in \Delta_{j-1}^v(X)\},$$

where ∂_A takes the boundary and projects it onto $C_j(A)$.

Define also

$$\Delta_j^{v,X}(A) := \Delta_j^v(A) + \partial_A\Delta_{j+1}^v(X/A).$$

First observe that by definition if $c \in \Delta_{j+1}^v(X/A)$ then

$$\partial_A c \in \Delta_j^v(X) + \partial\Delta_{j+1}^v(X).$$

Therefore, by definition of $\Delta_j^{v,X}$ we get

$$\Delta_j^{v,X}(A) \subset \Delta_j^v(X) + \partial\Delta_{j+1}^v(X) = \Delta_j'^v(X).$$

Thus, we may set

$$\Delta_j^v(X; A) := \frac{\Delta_j'^v(X)}{\Delta_j^{v,X}(A)},$$

and

$$H_j^v(X; A) := H_j(\Delta^v(X; A)).$$

Remark 4.1.1. If X is a v-thin set *of dimension* j then $H_j^v(X; A) = H_j(X; A)$ (see Remark 1.3.1).

Let $i : \Delta_j^{v,X}(A) \to \Delta_j'^v(X)$ be the inclusion. Clearly, we have the following exact sequences:

$$0 \to \Delta_j^{v,X}(A) \xrightarrow{i} \Delta_j'^v(X) \xrightarrow{q} \Delta_j'^v(X; A) \to 0, \tag{4.17}$$

(where q is the quotient map) and therefore we get the following long exact sequence:

$$\dots \to H_j^v(\delta_X A) \to H_j^v(X) \to H_j^v(X; A) \to H_{j-1}^v(\delta_X A) \to \dots$$

Remark 4.1.2. We could have defined the homology groups of a pair by $H_j^v(X; A) := H_j^v(C^v(X; A))$ where $C^v(X; A) := \frac{C_j^v(X)}{C_j^v(A)}$, and of course the latter exact sequence would hold for $H_j^v(A)$ (instead of $H_j^v(\delta_X A)$). However the excision property would not hold.

As we said, if $(c+c')$ belongs to $\Delta_j^v(X)$ then c and c' both belong to $C_j^v(X)$. Therefore, the excision property holds for $H_j^v(X;A)$. Let $(X;A)$ and W be definable such that W lies in the interior of A. Then for any j:

$$H_j^v(X;A) = H_j^v(X \setminus W; A \setminus W). \tag{4.18}$$

4.2. **Basic examples.** The easiest example is the link of a set at a nonsingular point.

Example 4.2.1. In the case where the set is smooth near the origin, as the vanishing homology is invariant under definable bi-Lipschitz homeomorphisms, we may identify our set with open neighborhood of zero in $\mathbb{R}^n$. If X denotes such a set then L_X is $S^{n-1}(0;T)$, i. e. the sphere of radius T in $k(0_+)^n$. Using some very basic techniques (by constructing homotopies between $S^{n-1}(T)\setminus\{pt\}$ and a point) we can easily see that $H_j^v(L_X)$ is zero if $j < n-1$ for any velocity v.

Moreover if T belongs to v then the sphere is v-thin and by Remark 1.3.1 we immediately get

$$H_{n-1}^v(L_X) \simeq \mathbb{Q}.$$

On the other hand, if $T \notin v$, L_X is v-thick and then the support of a 2-cycle σ may not cover the whole sphere and thus may be retracted onto a point. Therefore if $T \notin v$ then

$$H_{n-1}^v(L_X) \simeq 0.$$

In the following example we deal with the constant family. In this case the result is also very natural. This example will be useful for the next one.

Example 4.2.2. Let N be a compact semialgebraic manifold of $\mathbb{R}^n$. Consider then the extension $N_{k(0_+)}$ of N to $k(0_+)^n$, i. e. the submanifold of $k(0_+)^n$ obtained by regarding the equations of N in $k(0_+)^n$. This is the generic fiber of the constant family $N \times \mathbb{R}$. Let us fix a velocity v. If $1 \in v$ then all the simplices are admissible and

$$H_j^v(N_{k(0_+)}) \simeq H_j(N),$$

for any j.

Otherwise, the support any $(j;v)$-thin chain σ is collapsing onto a subset of dimension less than j in N. This means that σ bounds a chain τ is $N_{k(0_+)}$ which is also v-admissible. Therefore, if $1 \notin v$ then $H_j^v(N_{k(0_+)}) = 0$, for any j.

The next case that we will consider is the case of conical singularities. In this case, the Lipschitz geometry of the set is completely determined by the topology of the germ and the vanishing homology may be easily computed.

Example 4.2.3. *(Conical singularities)* Let $X \subset \mathbb{R}^n$ be the germ at 0 of a definable set having an isolated singularity at 0. Given a definable set N, we denote by C_N the cone over N. We say that X has a **conical singularity at** 0 if there exists a germ of definable homeomorphism $h : (C_N;0) \to (X;0)$, with N definable manifold, bi-Lipschitz with respect to the inner metric.

If X denotes such a set, observe that L_X is nothing that L_X is nothing but the image of $N_{k(0_+)}$ under the homothety $\iota : k(0_+)^n \to k(0_+)^n$, $x \mapsto T \cdot x$.

Let us now fix a velocity v and compute the corresponding v-vanishing homology groups. Define

$$v' := \frac{1}{T} \cdot v = \{x \in k(0_+) : Tx \in v\}.$$

As $\iota(N_{k(0_+)}) = L_X$, by definition of the v-vanishing homology groups we have:

$$H_j^v(L_X) = H_j^{v'}(N_{k(0_+)}).$$

But, by the preceding example, this means that we have, if $T \in v$ then $1 \in v'$ and

$$H_j^v(L_X) \simeq H_j(L_X).$$

Similarly if $T \in v$ then $1 \notin v'$ and $H_j^v(L_X) = 0$.

4.3. **Further examples.** We give two extra examples. The first one is similar to the one given by L. Birbrair and A. Fernandes ([BF] example 4.2) for the metric homology and which looks like the one sketched on fig 1. It is characterized by the fact that there is a vanishing 1-cycle σ which bounds a chain τ too big for being admissible. This creates homology. This situation may occur with the metric homology groups as well and these cycles are referred in [BF] by the authors, as *Cheeger's cycles*.

Example 4.3.1. We consider two spheres from which we shrink a little disk which collapses into a point and which intersects along the boundaries of these disks. This is the generic fiber of a family of sets collapsing to zero.

Let

$$X(\varepsilon) := \{(x; y; z) \in k(0_+)^3 : (x - \varepsilon(T^2 - T^6))^2 + y^2 + z^2 = T^4, \varepsilon x \geq 0\}$$

for $\varepsilon = \pm 1$. Then let $X := X(1) \cup X(-1)$ and $A = X(1) \cap X(-1)$.

Let us simply consider the velocity T^2. The computation could actually be carried out for any velocity. Since the set A is $\mathbb{N}T^4$-thin we have:

$$H_1^{T^4}(\delta_X A) = H_1^{T^4}(A) = H_1(A) = \mathbb{Q},$$

and $H_0^{T^4}(\delta_X A) = 0$.

Note that, thanks to the excision property, we have:

$$H_1^v(X; A) \simeq H_1^v(X(1); A) \oplus H_1^v(X(-1); A).$$

If we add the disk

$$D = \{(x; y; z) \in k(0_+)^3 : (x - \varepsilon(T^2 - T^6))^2 + y^2 + z^2 = T^4, \varepsilon x \leq 0\}$$

to $X(1)$, we get the sphere S^2. Thus, by the excision property,

$$H_1^{T^4}(X(1); A) \simeq H_1^{T^4}(S^2; D) = 0,$$

and so $H_1^{T^4}(X; A) = 0$. Examining the exact sequence of the pair $(X; A)$ we see that:

$$H_1^{T^4}(X) \simeq H_1^{T^4}(\delta_X A) \simeq \mathbb{Q}.$$

Observe also that we have: $H_2^{T^4}(X) \simeq 0$ and $H_0^{T^4}(X) \simeq 0$.

It is interesting to look at the metric homology [BB1] [BB2] on this example. The volume of the two half-spheres is proportional to T^4, which is precisely the rate of the volume of the vanishing cycle in the middle. Hence, the above vanishing cycle is not captured by the metric homology.

We end by computing the vanishing homology groups of Birbrair-Goldshtein examples. The result is actually similar to the one found by Birbrair and Brasselet in [BB1] section 7 for the metric homology.

Example 4.3.2. Let X be the set defined by (0.2) assume that $p < q$. Let us compute for instance the vanishing homology groups for the velocity T^q. We could use here the excision property and follow the classical methods for computing the homology groups of the torus but it is actually simpler to derive it from the classical homology groups of X since it is $\mathbb{N}T^q$-thin. This implies that the inclusion $H_2^{T^q}(X) \to H_2(X)$ is an isomorphism and that the inclusion $H_1^{T^q}(X) \to H_1(X)$ is one-to-one. Therefore

$$H_2^{T^q}(X) \simeq \mathbb{Q}$$

and $\dim H_1^{T^q}(X) \leq 2$. Actually, one generator of $H_1(X)$ has a representant with T^q-thin support and every 1-chain representing a different class has a support whose length is clearly bigger than T^p. This proves that $\dim H_1^{T^q}(X) = 1$.

4.4. **On Corollary 0.0.2.** We assumed in Corollary 0.0.2 that X is a semialgebraic set because this was the setting of [V2]. Nevertheless, the main ingredient of the proof of Theorem 0.0.1 is Theorem 5.1.3 of [V1]. As this theorem holds over any polynomially bounded o-minimal structure, Corollary 0.0.2 is still true in this setting as well. However, the metric type of the link L_X may fail to be a metric invariant of the singularity when the set is definable in a non-polynomially bounded o-minimal structure as it is shown by the following example.

Example 4.4.1. Let $X := \{(x; y) \in \mathbb{R}^2 : |y| = e^{\frac{-1}{x^2}}\}$ and $Y = \{(x; y) \in \mathbb{R}^2 : |y| = e^{\frac{-2}{x^2}}\}$. Note that X and Y are both definable in the $ln - exp$ structure (see [vDS], [LR], [W]). Furthermore X and Y are definably bi-Lipschitz homeomorphic. However the links of X and Y are constituted by two points of $k_{0_+}^2$ (where k_{0_+} is the corresponding residue field) whose respective distances are clearly not equivalent.

Note that a revolution of these subsets about the x-axis provides two subsets whose links have different vanishing homology groups (for a suitable velocity).

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