Equivariant quantum cohomology and puzzles

Anders Skovsted Buch

arXiv:1401.3065

Collaborators on subject:

A. Kresch, L. Mihalcea, K. Purbhoo, H. Tamvakis

Two-step flag varieties

Fix $0 \leq a \leq b \leq n$.

$$Y \;=\; \mathrm{Fl}(a, b; n) \;=\; \{(A, B) \mid A \subset B \subset \mathbb{C}^n \,;\, \dim(A) = a \,;\, \dim(B) = b\}$$

Two-step flag varieties

Fix $0 \leq a \leq b \leq n$.

$Y \;=\; \mathrm{Fl}(a,b;n) \;=\; \{(A,B) \mid A \subset B \subset \mathbb{C}^n\,;\, \dim(A) = a\,;\, \dim(B) = b\}$

Def: A **012-string** for Y is a permutation of $0^a\,1^{b-a}\,2^{n-b}$.

$\mathbb{C}^n$ has basis $\{e_1, e_2, \ldots, e_n\}$. $u = (u_1, u_2, \ldots, u_n)$ 012-string.

Def. $(A_u, B_u) \in Y$ by $A_u = \mathrm{Span}\{e_i : u_i = 0\}$ and $B_u = \mathrm{Span}\{e_i : u_i \leq 1\}$.

Example: $Y = \mathrm{Fl}(1,3;5)$. $u = 10212$. $(A_u, B_u) = (\mathbb{C}e_2\,,\, \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_4)$.

Two-step flag varieties

Fix $0 \leq a \leq b \leq n$.

$Y = \mathrm{Fl}(a, b; n) = \{(A, B) \mid A \subset B \subset \mathbb{C}^n ; \dim(A) = a ; \dim(B) = b\}$

Def: A **012-string** for Y is a permutation of $0^a\, 1^{b-a}\, 2^{n-b}$.

$\mathbb{C}^n$ has basis $\{e_1, e_2, \ldots, e_n\}$. $u = (u_1, u_2, \ldots, u_n)$ 012-string.

Def. $(A_u, B_u) \in Y$ by $A_u = \mathrm{Span}\{e_i : u_i = 0\}$ and $B_u = \mathrm{Span}\{e_i : u_i \leq 1\}$.

Example: $Y = \mathrm{Fl}(1, 3; 5)$. $u = 10212$. $(A_u, B_u) = (\mathbb{C}e_2\, , \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_4)$.

$\mathbf{B}^+ \subset \mathrm{GL}(\mathbb{C}^n)$ upper triangular ; $\mathbf{B}^- \subset \mathrm{GL}(\mathbb{C}^n)$ lower triangular matrices.

Schubert varieties: $Y_u = \overline{\mathbf{B}^+.(A_u, B_u)}$; $Y^u = \overline{\mathbf{B}^-.(A_u, B_u)} \subset Y$

$\dim(Y_u) = \mathrm{codim}(Y^u, Y) = \ell(u) = \#\{i < j \mid u_i > u_j\}$

Equivariant cohomology

$T \subset \mathrm{GL}(\mathbb{C}^n)$ max. torus of diagonal matrices.

$\Lambda := H_T^*(\text{point}; \mathbb{Z}) \; = \; \mathbb{Z}[y_1, \ldots, y_n]$, where $y_i = -c_1(\mathbb{C}e_i)$.

$$H_T^*(Y; \mathbb{Z}) \; = \; \bigoplus_u \Lambda \cdot [Y^u]$$ is an algebra over Λ.

Equivariant cohomology

$T \subset \mathrm{GL}(\mathbb{C}^n)$ max. torus of diagonal matrices.

$\Lambda := H_T^*(\text{point}; \mathbb{Z}) \;=\; \mathbb{Z}[y_1, \dots, y_n]$, where $y_i = -c_1(\mathbb{C}e_i)$.

$H_T^*(Y; \mathbb{Z}) \;=\; \bigoplus_u \Lambda \cdot [Y^u]$ is an algebra over Λ.

Equivariant Schubert structure constants $C_{u,v}^w \in \Lambda$:

$$[Y^u] \cdot [Y^v] \;=\; \sum_w C_{u,v}^w \, [Y^w]$$

Poincare duality: $C_{u,v}^w = \int_Y [Y^u] \cdot [Y^v] \cdot [Y_w]$

Equivariant cohomology

$T \subset \mathrm{GL}(\mathbb{C}^n)$ max. torus of diagonal matrices.

$\Lambda := H_T^*(\text{point};\mathbb{Z}) \;=\; \mathbb{Z}[y_1,\dots,y_n]$, where $y_i = -c_1(\mathbb{C}e_i)$.

$H_T^*(Y;\mathbb{Z}) \;=\; \bigoplus_u \Lambda\cdot[Y^u]$ is an algebra over Λ.

Equivariant Schubert structure constants $C_{u,v}^w \in \Lambda$ **:**

$$[Y^u]\cdot[Y^v] \;=\; \sum_w C_{u,v}^w\,[Y^w]$$

Poincare duality: $C_{u,v}^w = \int_Y [Y^u]\cdot[Y^v]\cdot[Y_w]$

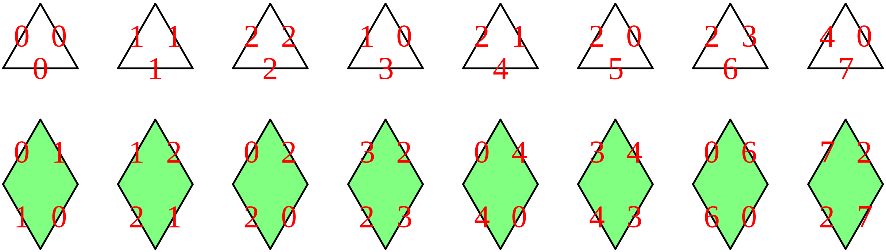
$C_{u,v}^w \in \mathbb{Z}[y_1,\dots,y_n]$ is homogeneous of degree $\ell(u)+\ell(v)-\ell(w)$.

$\ell(u)+\ell(v)=\ell(w) \;\Rightarrow$

$$C_{u,v}^w = \#(\,Y^u \cap g.Y^v \cap Y_w\,) \quad \text{for } g \in \mathrm{GL}(\mathbb{C}^n) \text{ general.}$$

Theorem (Graham) $\quad C_{u,v}^w \;\in\; \mathbb{Z}_{\geq 0}[y_2-y_1,\dots,y_n-y_{n-1}]$

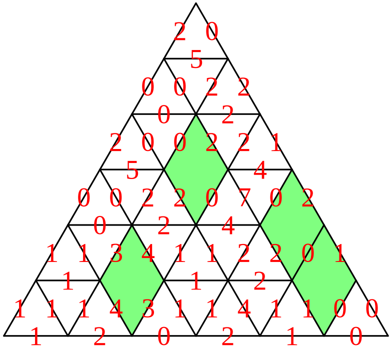
Puzzle pieces



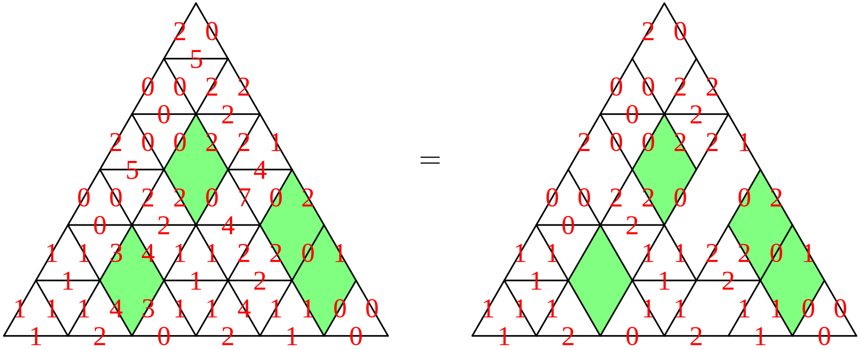
Simple labels: 0 , 1 , 2

Composed labels: $3 = 10$, $4 = 21$, $5 = 20$, $6 = 2(10)$, $7 = (21)0$

Equivariant puzzles



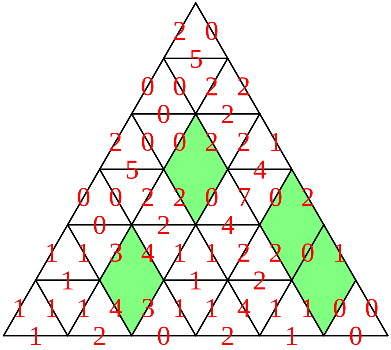
Equivariant puzzles



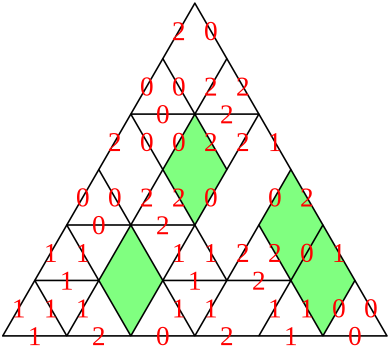
Note: The composed labels are uniquely determined by the simple labels.

Equivariant puzzles

$P =$



$=$



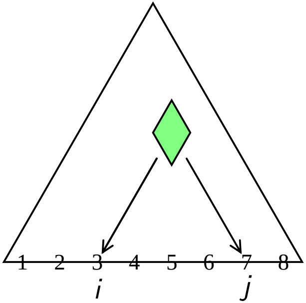
Note: The composed labels are uniquely determined by the simple labels.

Boundary: $\partial P = \triangle_w^{u,v}$ where $u = 110202$, $v = 021210$, $w = 120210$.

Equivariant puzzle formula

Theorem

$$C_{u,v}^{w} = \sum_{\partial P = \triangle_{w}^{u,v}} \prod_{\lozenge \in P} \mathrm{weight}(\lozenge)$$

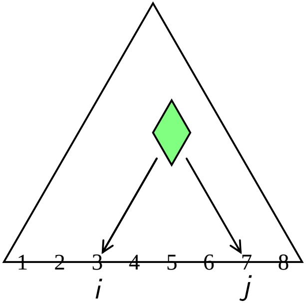


$$\mathrm{weight}(\lozenge) = y_j - y_i$$

Equivariant puzzle formula

Theorem

$$C_{u,v}^{w} = \sum_{\partial P = \triangle_w^{u,v}} \prod_{\Diamond \in P} \text{weight}(\Diamond)$$



$\text{weight}(\Diamond) = y_j - y_i$

Known cases:

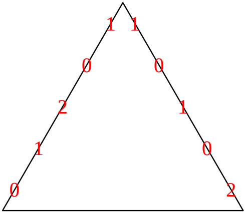
Puzzle rule for $H^*(\text{Gr}(m,n))$ (Knutson, Tao, Woodward)

Puzzle rule for $H_T^*(\text{Gr}(m,n))$ (Knutson, Tao)

Puzzle rule for $H^*(\text{Fl}(a,b;n))$ (conjectured by Knutson,
proof in [B-Kresch-Purbhoo-Tamvakis] ,
different positive formula by Coskun.)

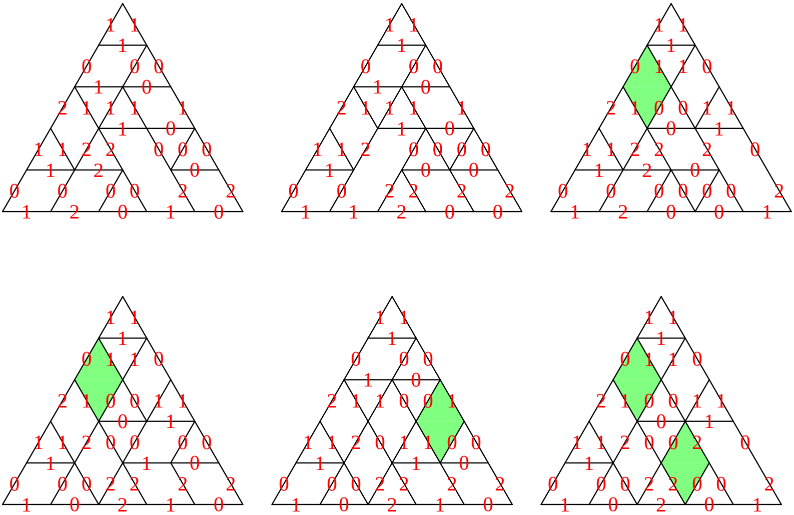
Example: Let $Y = \mathrm{Fl}(2,4;5)$. In $H_T^*(Y)$ we have:

$[Y^{01201}] \cdot [Y^{10102}] \ = \ $ **?**



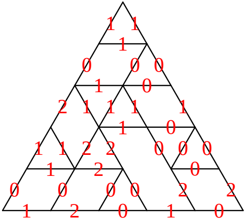
Example: Let $Y = \mathrm{Fl}(2,4;5)$. In $H_T^*(Y)$ we have:

$[Y^{01201}] \cdot [Y^{10102}] \;=\; ?$



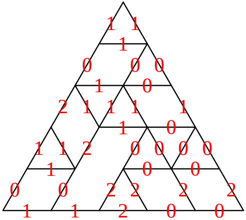
Example: Let $Y = \mathrm{Fl}(2,4;5)$. In $H_T^*(Y)$ we have:

$[Y^{01201}] \cdot [Y^{10102}] \ =$



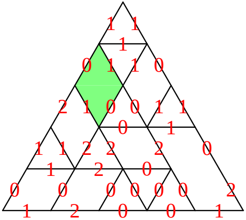
$[Y^{12010}]$

$+$

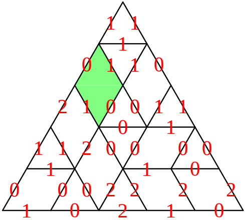


$[Y^{11200}]$

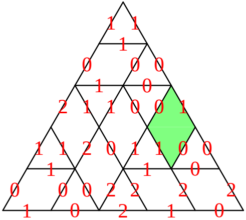
$+$

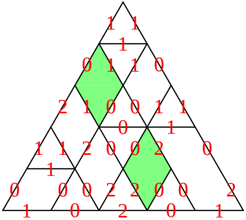


$(y_4 - y_1)[Y^{12001}]$



$+ \quad (y_5 + y_4 - y_3 - y_1)[Y^{10210}]$



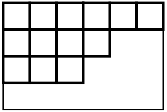


$+ \quad (y_4 - y_3)(y_4 - y_1)[Y^{10201}]$

Gromov-Witten invariants of Grassmannians

$$X \;=\; \mathrm{Gr}(m,n) \;=\; \{V \subset \mathbb{C}^n \mid \dim(V) = m\} \;=\; \mathrm{Fl}(m,m;n)$$

Schubert varieties $\longleftrightarrow$ **02-strings** $\longleftrightarrow$ **Young diagrams** λ

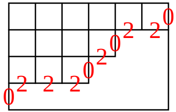
$X^{0222020220}$ $\longleftrightarrow$ 0222020220 $\longleftrightarrow$ 

Gromov-Witten invariants of Grassmannians

$$X \;=\; \mathrm{Gr}(m,n) \;=\; \{V \subset \mathbb{C}^n \mid \dim(V) = m\} \;=\; \mathrm{Fl}(m,m;n)$$

Schubert varieties $\longleftrightarrow$ **02-strings** $\longleftrightarrow$ **Young diagrams** λ

$X^{0222020220}$ $\longleftrightarrow$ 0222020220 $\longleftrightarrow$

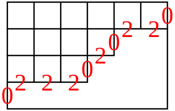


Gromov-Witten invariants of Grassmannians

$$X = \mathrm{Gr}(m,n) = \{V \subset \mathbb{C}^n \mid \dim(V) = m\} = \mathrm{Fl}(m,m;n)$$

Schubert varieties $\longleftrightarrow$ **02-strings** $\longleftrightarrow$ **Young diagrams** λ

$X^{0222020220}$ $\longleftrightarrow$ 0222020220 $\longleftrightarrow$



Def: A (rational) **curve** $C \subset X$ is any image of a polynomial map $\mathbb{P}^1 \to X$.

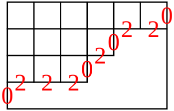
Degree: $\deg(C) = \#(C \cap g.X^{\square})$

Gromov-Witten invariants of Grassmannians

$X \;=\; \mathrm{Gr}(m,n) \;=\; \{V \subset \mathbb{C}^n \mid \dim(V) = m\} \;=\; \mathrm{Fl}(m,m;n)$

Schubert varieties $\longleftrightarrow$ **02-strings** $\longleftrightarrow$ **Young diagrams** λ

$X^{0222020220}$ $\longleftrightarrow$ 0222020220 $\longleftrightarrow$



Def: A (rational) **curve** $C \subset X$ is any image of a polynomial map $\mathbb{P}^1 \to X$.

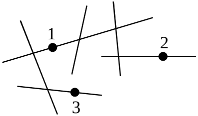
Degree: $\deg(C) \;=\; \#(C \cap g.X^{\square})$

Def: Given λ, μ, ν with $|\lambda| + |\mu| = |\nu| + nd$, define

$\langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d \;=\; \#$ curves $C \subset X$ of degree d meeting X^λ, $g.X^\mu$, and X_ν.

Kontsevich moduli space

$$\overline{\mathcal{M}}_{0,3}(X,d) = \{\text{stable } f : C \to X \mid f_*[C] = d\,[\text{line}]\}$$

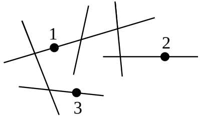
$C =$ 

Evaluation maps: $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,3}(X,d) \to X$

$$\mathrm{ev}_i(f) = f(i\text{-th marked point})$$

Kontsevich moduli space

$\overline{\mathcal{M}}_{0,3}(X,d) = \{\text{stable } f : C \to X \mid f_*[C] = d\,[\text{line}]\}$

$C =$ 

Evaluation maps: $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,3}(X,d) \to X$

$$\mathrm{ev}_i(f) = f(i\text{-th marked point})$$

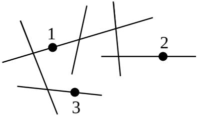
Equivariant Gromov-Witten invariant:

Given $\alpha_1, \alpha_2, \alpha_3 \in H_T^*(X;\mathbb{Z})$ define

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_d = \int_{\overline{\mathcal{M}}_{0,3}(X,d)} \mathrm{ev}_1^*(\alpha_1) \cdot \mathrm{ev}_2^*(\alpha_2) \cdot \mathrm{ev}_3^*(\alpha_3) \in \Lambda$$

Kontsevich moduli space

$\overline{\mathcal{M}}_{0,3}(X, d) = \{\text{stable } f : C \to X \mid f_*[C] = d\,[\text{line}]\}$

$C =$ 

Evaluation maps: $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,3}(X, d) \to X$

$\mathrm{ev}_i(f) = f(i\text{-th marked point})$

Equivariant Gromov-Witten invariant:

Given $\alpha_1, \alpha_2, \alpha_3 \in H_T^*(X; \mathbb{Z})$ define

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_d = \int_{\overline{\mathcal{M}}_{0,3}(X,d)} \mathrm{ev}_1^*(\alpha_1) \cdot \mathrm{ev}_2^*(\alpha_2) \cdot \mathrm{ev}_3^*(\alpha_3) \in \Lambda$$

Note: $\langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d \in \mathbb{Z}[y_1, \ldots, y_n]$ is homogeneous of degree $|\lambda| + |\mu| - |\nu| - nd$.

Note: $C_{\lambda,\mu}^{\nu} = \int_X [X^\lambda] \cdot [X^\mu] \cdot [X_\nu] = \langle [X^\lambda], [X^\mu], [X_\nu] \rangle_0$

Small equivariant quantum ring

$QH_T(X)$ is an algebra over $\Lambda[q]$.

As $\Lambda[q]$-module: $$QH_T(X) = H_T^*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q] = \bigoplus_{\lambda} \Lambda[q] \cdot [X^\lambda]$$

Quantum product: $$[X^\lambda] \star [X^\mu] = \sum_{\nu, d \geq 0} \langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d \, q^d \, [X^\nu]$$

Small equivariant quantum ring

$QH_T(X)$ is an algebra over $\Lambda[q]$.

As $\Lambda[q]$-module: $$QH_T(X) = H_T^*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q] = \bigoplus_{\lambda} \Lambda[q] \cdot [X^\lambda]$$

Quantum product: $$[X^\lambda] \star [X^\mu] = \sum_{\nu, d \geq 0} \langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d \, q^d \, [X^\nu]$$

Theorem (Ruan–Tian, Kontsevich–Manin, Kim):

$QH_T(X)$ is associative!

Small equivariant quantum ring

$QH_T(X)$ is an algebra over $\Lambda[q]$.

As $\Lambda[q]$-module: $QH_T(X) = H_T^*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q] = \bigoplus_\lambda \Lambda[q] \cdot [X^\lambda]$

Quantum product: $[X^\lambda] \star [X^\mu] = \sum_{\nu, d \geq 0} \langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d \, q^d \, [X^\nu]$

Theorem (Ruan–Tian, Kontsevich–Manin, Kim):

$QH_T(X)$ is associative!

Mihalcea: $\langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d \in \mathbb{Z}_{\geq 0}[y_2 - y_1, \ldots, y_n - y_{n-1}]$

Kernel and Span

Let $C \subset X = \mathrm{Gr}(m, n)$ be a curve.

Def: (B) $\mathrm{Ker}(C) = \bigcap_{V \in C} V \subset \mathbb{C}^n$ and $\mathrm{Span}(C) = \sum_{V \in C} V \subset \mathbb{C}^n$

Obs: $\dim \mathrm{Ker}(C) \geq m - \deg(C)$ and $\dim \mathrm{Span}(C) \leq m + \deg(C)$

Application (B) Simpler proofs of structure theorems for $QH(X)$ first proved by Bertram.

Quantum = classical theorem

Def: $Y_d = \mathrm{Fl}(m-d, m+d; n)$

$$= \{(A,B) \mid A \subset B \subset \mathbb{C}^n \ , \ \dim(A) = m-d \ , \ \dim(B) = m+d\}$$

Given subvariety $\Omega \subset X$, define

$$\widetilde{\Omega} = \{(A,B) \in Y_d \mid \exists V \in \Omega : A \subset V \subset B\}$$

Quantum = classical theorem

Def: $Y_d = \mathrm{Fl}(m-d, m+d; n)$

$$= \{(A,B) \mid A \subset B \subset \mathbb{C}^n\ ,\ \dim(A) = m-d\ ,\ \dim(B) = m+d\}$$

Given subvariety $\Omega \subset X$, define

$$\widetilde{\Omega} = \{(A,B) \in Y_d \mid \exists V \in \Omega : A \subset V \subset B\}$$

Theorem (B, Kresch, Tamvakis) Assume that $|\lambda| + |\mu| = |\nu| + nd$.

$$\left\{\begin{matrix}\text{curves } C \subset X \text{ of degree } d \\ \text{meeting } X^\lambda\ ,\ g.X^\mu\ ,\ X_\nu\end{matrix}\right\} \longleftrightarrow \widetilde{X}^\lambda \cap g.\widetilde{X}^\mu \cap \widetilde{X}_\nu \subset Y_d$$

$$C \mapsto (\mathrm{Ker}(C), \mathrm{Span}(C))$$

Cor: $$\langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d = \int_{Y_d} [\widetilde{X}^\lambda] \cdot [\widetilde{X}^\mu] \cdot [\widetilde{X}_\nu]$$

Generalized quantum = classical theorem

$$\begin{array}{ccc} Z_d & \xrightarrow{\;p\;} & X \\ \downarrow{\scriptstyle q} & & \\ Y_d & & \end{array}$$

$$X = \mathrm{Gr}(m, n) = \{V\}$$

$$Y_d = \mathrm{Fl}(m-d, m+d; n) = \{(A, B)\}$$

$$Z_d = \mathrm{Fl}(m-d, m, m+d; n) = \{(A, V, B)\}$$

Note: $\widetilde{\Omega} = q(p^{-1}(\Omega))$

Theorem (B, Mihalcea) Let $\alpha, \beta, \gamma \in H_T^*(X; \mathbb{Z})$. Then

$$\langle \alpha, \beta, \gamma \rangle_d \;=\; \int_{Y_d} q_*(p^*(\alpha)) \cdot q_*(p^*(\beta)) \cdot q_*(p^*(\gamma))$$

Generalized quantum = classical theorem

$$\begin{array}{ccc} & & M_d \\ & & \downarrow {\scriptstyle \mathrm{ev}_i} \\ Z_d & \xrightarrow{p} & X \\ \downarrow {\scriptstyle q} & & \\ Y_d & & \end{array}$$

$$M_d = \overline{\mathcal{M}}_{0,3}(X, d)$$

$$X = \mathrm{Gr}(m, n) = \{V\}$$

$$Y_d = \mathrm{Fl}(m-d, m+d; n) = \{(A, B)\}$$

$$Z_d = \mathrm{Fl}(m-d, m, m+d; n) = \{(A, V, B)\}$$

Note: $\widetilde{\Omega} = q(p^{-1}(\Omega))$

Theorem (B, Mihalcea) Let $\alpha, \beta, \gamma \in H_T^*(X; \mathbb{Z})$. Then

$$\langle \alpha, \beta, \gamma \rangle_d = \int_{Y_d} q_*(p^*(\alpha)) \cdot q_*(p^*(\beta)) \cdot q_*(p^*(\gamma))$$

Generalized quantum = classical theorem

$$
\begin{array}{ccccc}
\mathrm{B}\ell_d & & \xrightarrow{\quad\pi\quad} & & M_d \\
\downarrow{\scriptstyle \phi} & & & & \downarrow{\scriptstyle \mathrm{ev}_i} \\
Z_d^{(3)} & \xrightarrow{e_i} & Z_d & \xrightarrow{p} & X \\
& & \downarrow{\scriptstyle q} & & \\
& & Y_d & &
\end{array}
$$

$$M_d = \overline{\mathcal{M}}_{0,3}(X, d)$$

$$X = \mathrm{Gr}(m, n) = \{V\}$$

$$Y_d = \mathrm{Fl}(m-d, m+d; n) = \{(A, B)\}$$

$$Z_d = \mathrm{Fl}(m-d, m, m+d; n) = \{(A, V, B)\}$$

$$\mathrm{B}\ell_d = \left\{ \begin{array}{c} (f, A, B) \in M_d \times Y_d \quad : \\ A \subset \mathrm{Ker}(f) \text{ and } \mathrm{Span}(f) \subset B \end{array} \right\}$$

$$Z_d^{(3)} = \left\{ \begin{array}{c} (V_1, V_2, V_3, A, B) \in X^3 \times Y_d \ : \\ A \subset V_i \subset B \end{array} \right\}$$

$$\pi(f, A, B) = f$$

$$\phi(f, A, B) = (\mathrm{ev}_1(f), \mathrm{ev}_2(f), \mathrm{ev}_3(f), A, B)$$

$$e_i(V_1, V_2, V_3, A, B) = (A, V_i, B)$$

Generalized quantum = classical theorem

$$\begin{array}{ccccc}
\mathrm{B}\ell_d & & \xrightarrow{\quad\pi\quad} & & M_d \\
\downarrow{\scriptstyle \phi} & & & & \downarrow{\scriptstyle \mathrm{ev}_i} \\
Z_d^{(3)} & \xrightarrow{e_i} & Z_d & \xrightarrow{p} & X \\
 & & \downarrow{\scriptstyle q} & & \\
 & & Y_d & &
\end{array}$$

$$\mathrm{B}\ell_d = \left\{ \begin{array}{c} (f, A, B) \in M_d \times Y_d \quad : \\ A \subset \mathsf{Ker}(f) \text{ and } \mathsf{Span}(f) \subset B \end{array} \right\}$$

$$Z_d^{(3)} = \left\{ \begin{array}{c} (V_1, V_2, V_3, A, B) \in X^3 \times Y_d \; : \\ A \subset V_i \subset B \end{array} \right\}$$

Facts:

(1) π is birational. (A general curve has kernel and span of expected dimensions.)

(2) ϕ is birational. (A general curve is determined by 3 points for $d \leq \min(m, n-m)$.)

(3) $Z_d^{(3)} = Z_d \times_{Y_d} Z_d \times_{Y_d} Z_d$

Generalized quantum = classical theorem

$$
\begin{array}{ccccc}
\mathrm{B}\ell_d & & \xrightarrow{\pi} & & M_d \\
\big\downarrow{\scriptstyle \phi} & & & & \big\downarrow{\scriptstyle \mathrm{ev}_i} \\
Z_d^{(3)} & \xrightarrow{e_i} & Z_d & \xrightarrow{p} & X \\
 & & \big\downarrow{\scriptstyle q} & & \\
 & & Y_d & &
\end{array}
$$

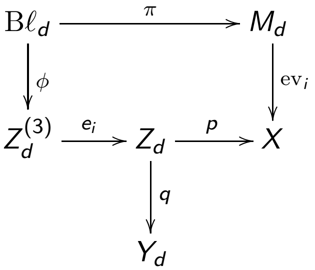
$$\mathrm{B}\ell_d = \left\{ \begin{array}{c} (f, A, B) \in M_d \times Y_d \quad : \\ A \subset \mathrm{Ker}(f) \text{ and } \mathrm{Span}(f) \subset B \end{array} \right\}$$

$$Z_d^{(3)} = \left\{ \begin{array}{c} (V_1, V_2, V_3, A, B) \in X^3 \times Y_d \ : \\ A \subset V_i \subset B \end{array} \right\}$$

Let $\alpha, \beta, \gamma \in H_T^*(X; \mathbb{Z})$.

$$\int_{M_d} \mathrm{ev}_1^*(\alpha) \cdot \mathrm{ev}_2^*(\beta) \cdot \mathrm{ev}_3^*(\gamma) \ = \ \int_{\mathrm{B}\ell_d} (\mathrm{ev}_1\, \pi)^*(\alpha) \cdot (\mathrm{ev}_2\, \pi)^*(\beta) \cdot (\mathrm{ev}_3\, \pi)^*(\gamma)$$

Generalized quantum = classical theorem



$$\mathrm{B}\ell_d = \left\{ \begin{array}{c} (f, A, B) \in M_d \times Y_d \quad : \\ A \subset \mathrm{Ker}(f) \text{ and } \mathrm{Span}(f) \subset B \end{array} \right\}$$

$$Z_d^{(3)} = \left\{ \begin{array}{c} (V_1, V_2, V_3, A, B) \in X^3 \times Y_d \quad : \\ A \subset V_i \subset B \end{array} \right\}$$

Let $\alpha, \beta, \gamma \in H_T^*(X; \mathbb{Z})$.

$$\int_{M_d} \mathrm{ev}_1^*(\alpha) \cdot \mathrm{ev}_2^*(\beta) \cdot \mathrm{ev}_3^*(\gamma) = \int_{\mathrm{B}\ell_d} (\mathrm{ev}_1 \pi)^*(\alpha) \cdot (\mathrm{ev}_2 \pi)^*(\beta) \cdot (\mathrm{ev}_3 \pi)^*(\gamma)$$

$$= \int_{Z_d^{(3)}} e_1^*(p^*\alpha) \cdot e_2^*(p^*\beta) \cdot e_3^*(p^*\gamma)$$

Generalized quantum = classical theorem

$$
\begin{array}{ccccc}
\mathrm{B}\ell_d & \xrightarrow{\quad\pi\quad} & & & M_d \\
\downarrow{\scriptstyle\phi} & & & & \downarrow{\scriptstyle\mathrm{ev}_i} \\
Z_d^{(3)} & \xrightarrow{e_i} & Z_d & \xrightarrow{p} & X \\
 & & \downarrow{\scriptstyle q} & & \\
 & & Y_d & &
\end{array}
$$

$$\mathrm{B}\ell_d = \left\{ \begin{array}{c} (f, A, B) \in M_d \times Y_d \quad : \\ A \subset \mathrm{Ker}(f) \text{ and } \mathrm{Span}(f) \subset B \end{array} \right\}$$

$$Z_d^{(3)} = \left\{ \begin{array}{c} (V_1, V_2, V_3, A, B) \in X^3 \times Y_d \ : \\ A \subset V_i \subset B \end{array} \right\}$$

Let $\alpha, \beta, \gamma \in H_T^*(X; \mathbb{Z})$.

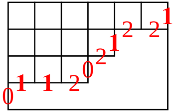
$$
\begin{aligned}
\int_{M_d} \mathrm{ev}_1^*(\alpha) \cdot \mathrm{ev}_2^*(\beta) \cdot \mathrm{ev}_3^*(\gamma) \ &= \ \int_{\mathrm{B}\ell_d} (\mathrm{ev}_1 \, \pi)^*(\alpha) \cdot (\mathrm{ev}_2 \, \pi)^*(\beta) \cdot (\mathrm{ev}_3 \, \pi)^*(\gamma) \\
&= \ \int_{Z_d^{(3)}} e_1^*(p^*\alpha) \cdot e_2^*(p^*\beta) \cdot e_3^*(p^*\gamma) \\
&= \ \int_{Y_d} q_* p^*(\alpha) \cdot q_* p^*(\beta) \cdot q_* p^*(\gamma)
\end{aligned}
$$

Littlewood-Richardson rule for $QH_T(X)$

Let $\lambda(d)$ is the 012-string obtained from λ by replacing the first d occurrences of 2 and the last d occurrences of 0 with 1.

Then $\widetilde{X}^{\lambda} = Y_d^{\lambda(d)}$.

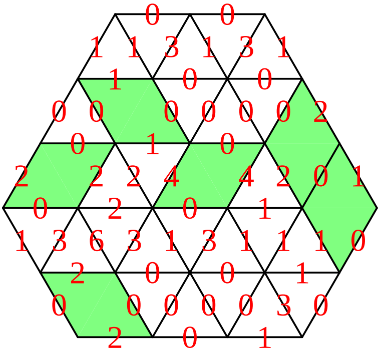
$\lambda =$0**22**202**0**22**0** and $d = 2$ gives $\lambda(d) =$ 0**11**202**1**22**1**.



Corollary:

$$\langle [X^{\lambda}], [X^{\mu}], [X_{\nu^{\vee}}] \rangle_d = C^{\nu(d)^{\vee}}_{\lambda(d),\mu(d)} = \sum_{\partial P = \triangle^{\lambda(d),\mu(d)}_{\nu(d)^{\vee}}} \prod_{\lozenge \in P} \text{weight}(\lozenge)$$

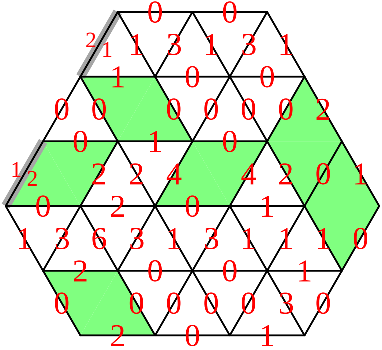
The mutation algorithm



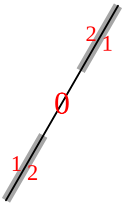
Puzzle:

- Shape is a hexagon.
- All pieces may be rotated.
- Boundary labels are simple.

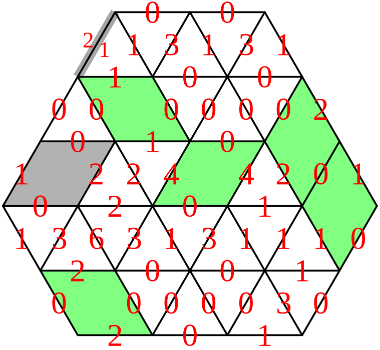
The mutation algorithm



Flawed puzzle containing the **gash pair**:

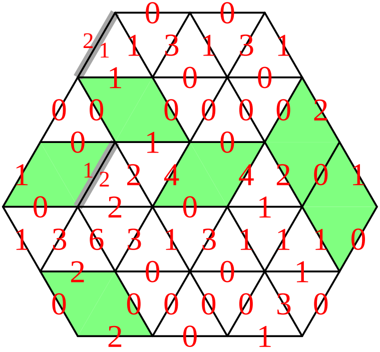


The mutation algorithm

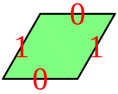


Remove problematic piece.

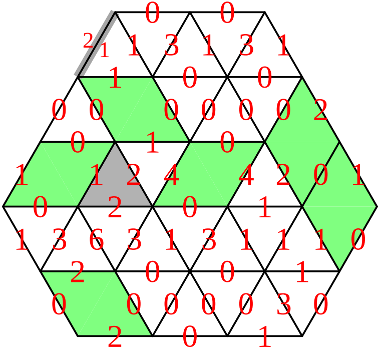
The mutation algorithm

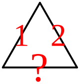


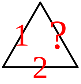
Replace with:



The mutation algorithm

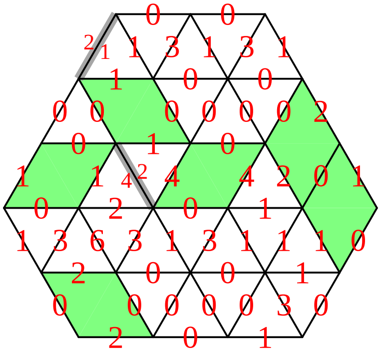


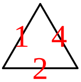
Replace with 

OR 

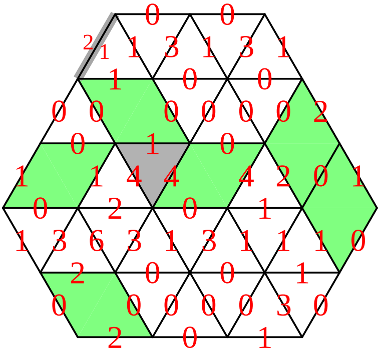
?

The mutation algorithm



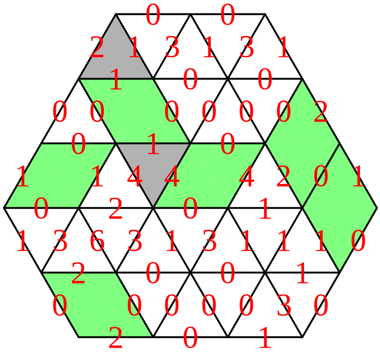
The piece  fits. Always at most one choice !!!

The mutation algorithm

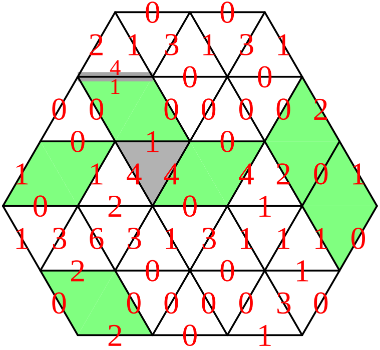


But no puzzle piece fits this time.

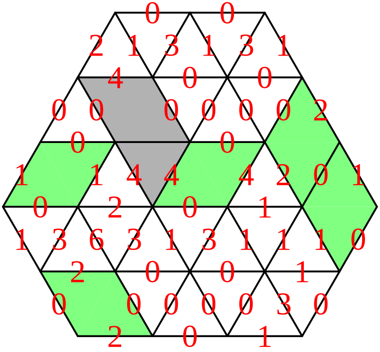
The mutation algorithm



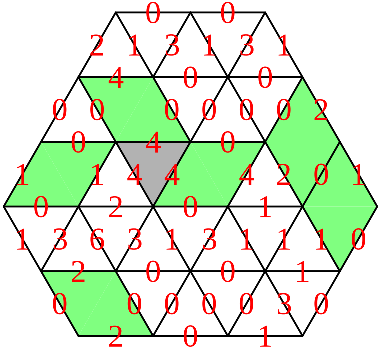
The mutation algorithm



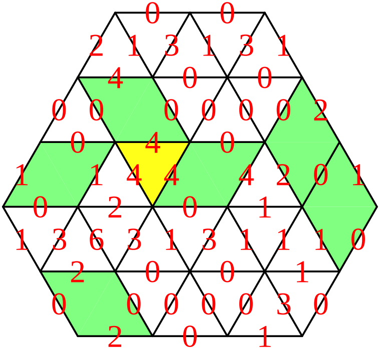
The mutation algorithm

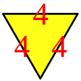


The mutation algorithm

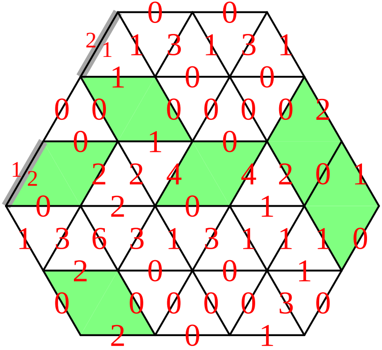


The mutation algorithm

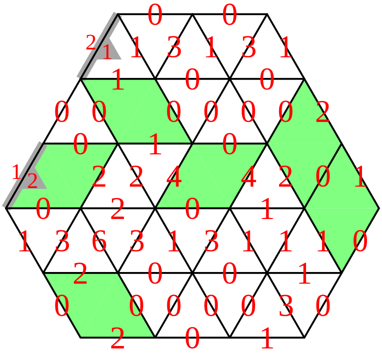


Flawed puzzle containing the **temporary puzzle piece**: 

The mutation algorithm

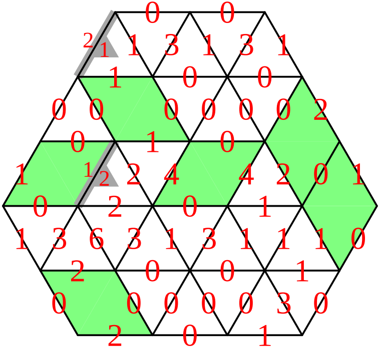


The mutation algorithm

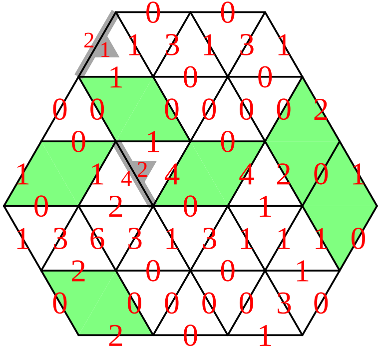


Use **directed gashes**.

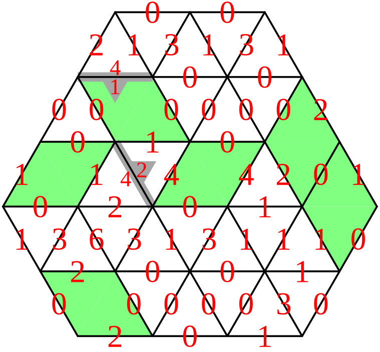
The mutation algorithm



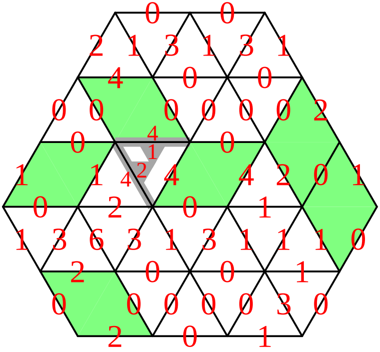
The mutation algorithm



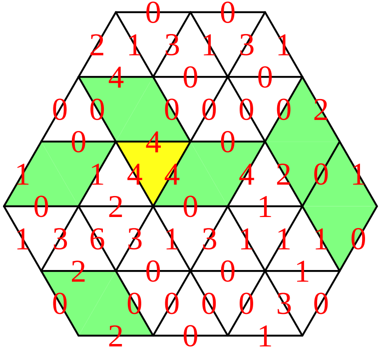
The mutation algorithm



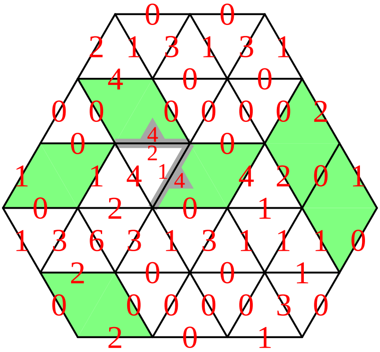
The mutation algorithm



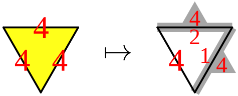
The mutation algorithm



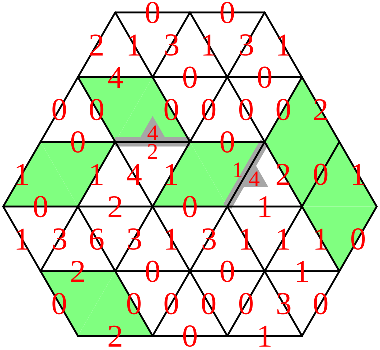
The mutation algorithm



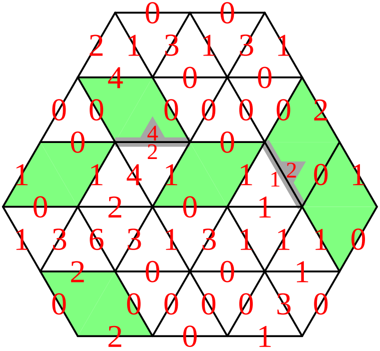
Resolution:



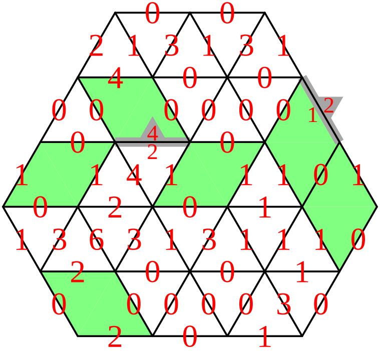
The mutation algorithm



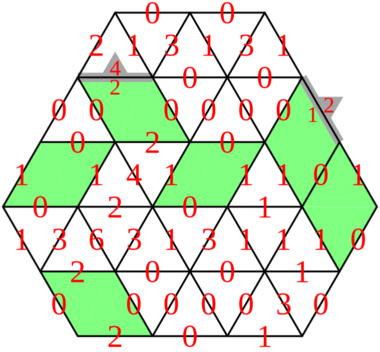
The mutation algorithm



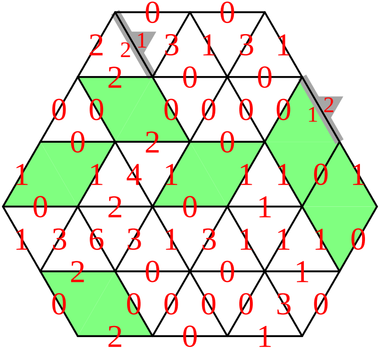
The mutation algorithm



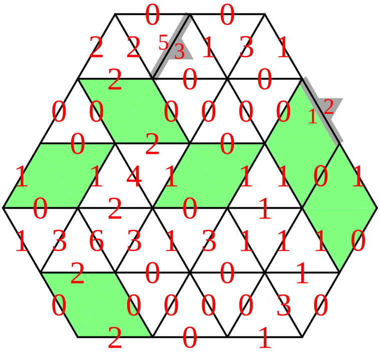
The mutation algorithm



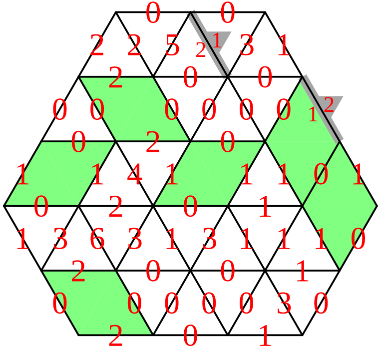
The mutation algorithm



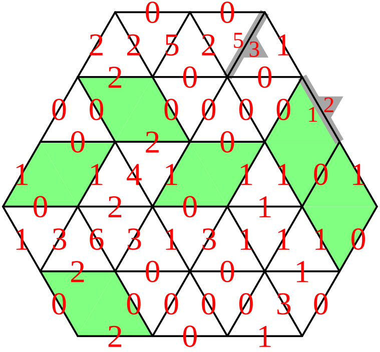
The mutation algorithm



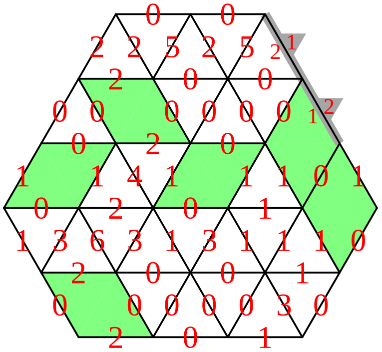
The mutation algorithm



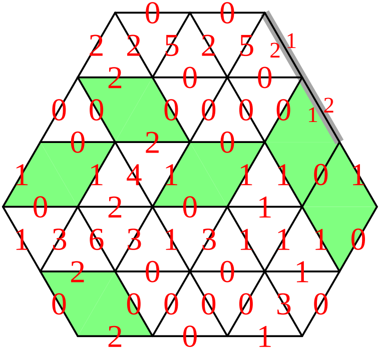
The mutation algorithm



The mutation algorithm

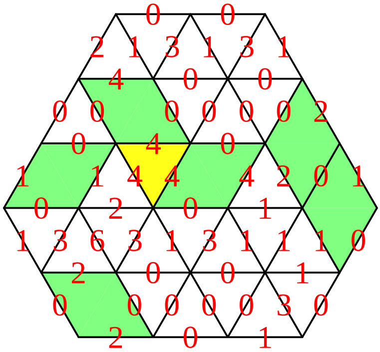


The mutation algorithm

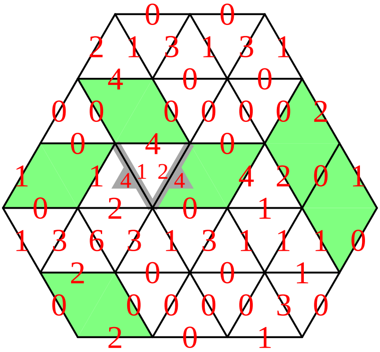


Flawed puzzle containing a gash pair.

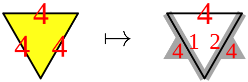
The mutation algorithm



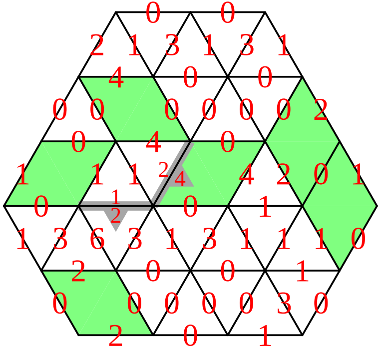
The mutation algorithm



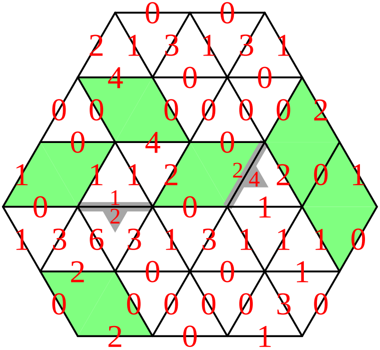
Resolution:



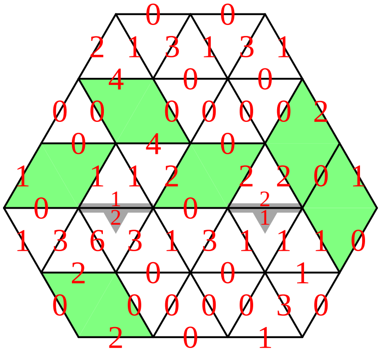
The mutation algorithm



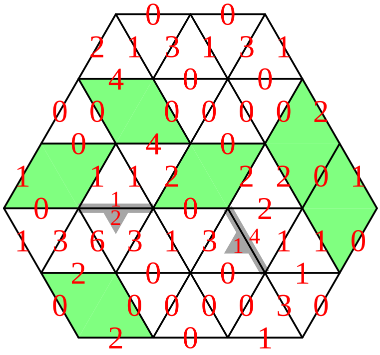
The mutation algorithm



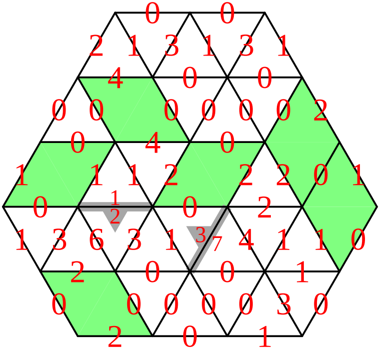
The mutation algorithm



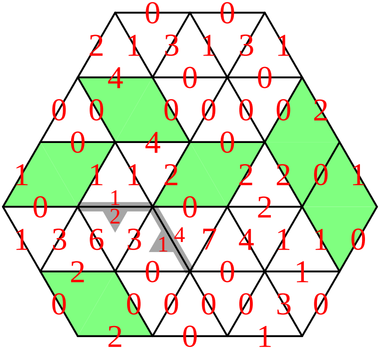
The mutation algorithm



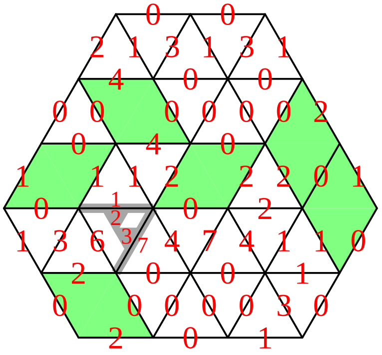
The mutation algorithm



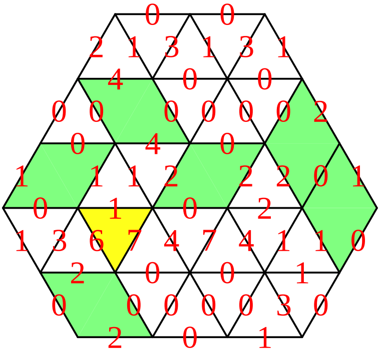
The mutation algorithm



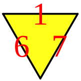
The mutation algorithm



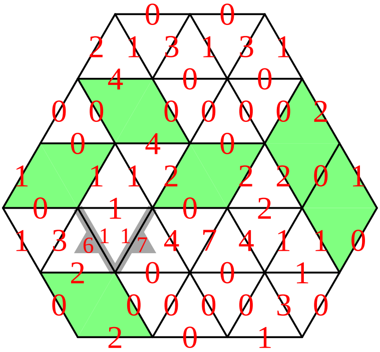
The mutation algorithm



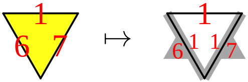
Flawed puzzle containing the **temporary puzzle piece**:



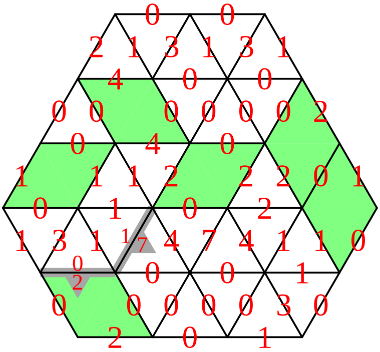
The mutation algorithm



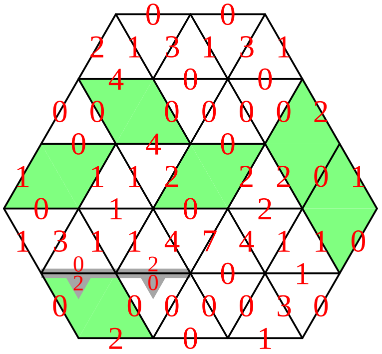
Resolution:



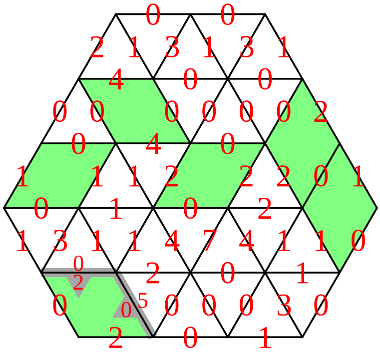
The mutation algorithm



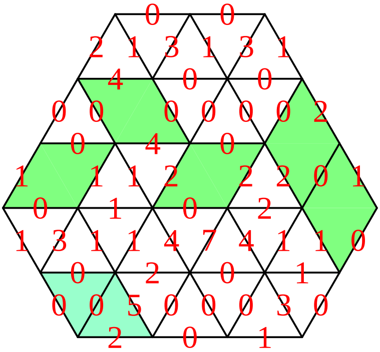
The mutation algorithm



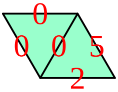
The mutation algorithm



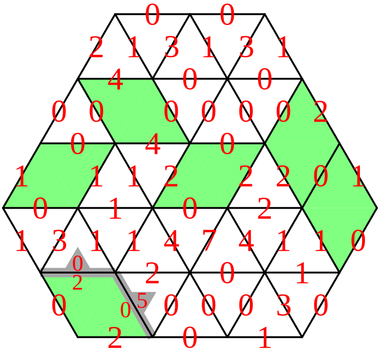
The mutation algorithm



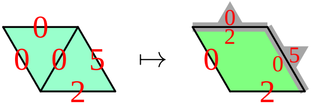
Flawed puzzle containing the **marked scab**:



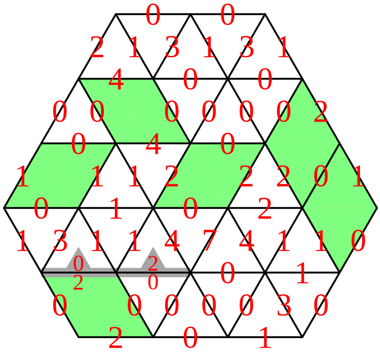
The mutation algorithm



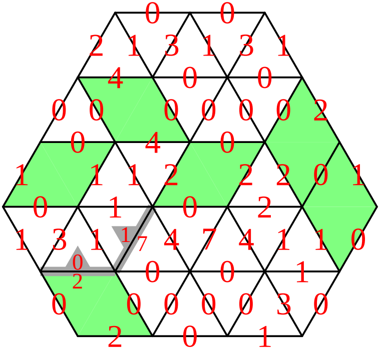
Resolution:



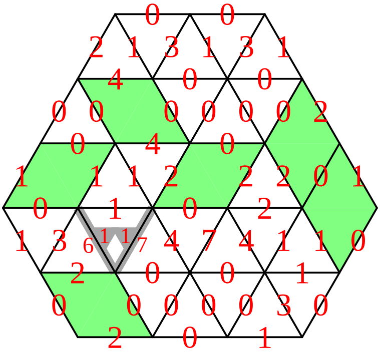
The mutation algorithm



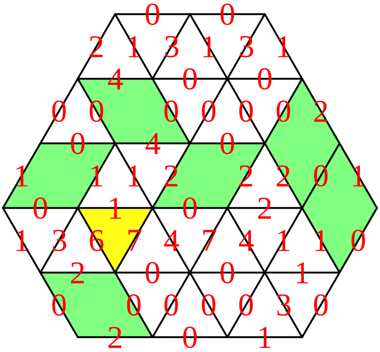
The mutation algorithm



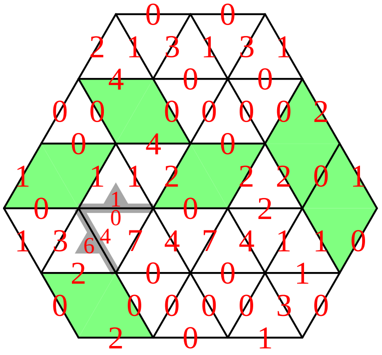
The mutation algorithm



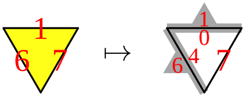
The mutation algorithm



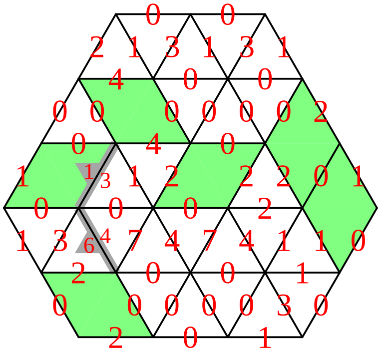
The mutation algorithm



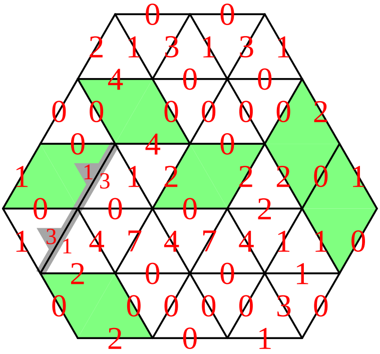
Resolution:



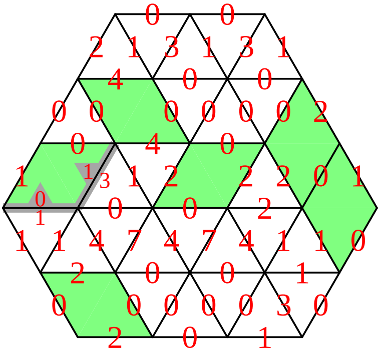
The mutation algorithm



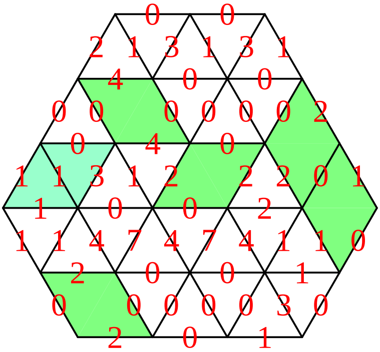
The mutation algorithm



The mutation algorithm

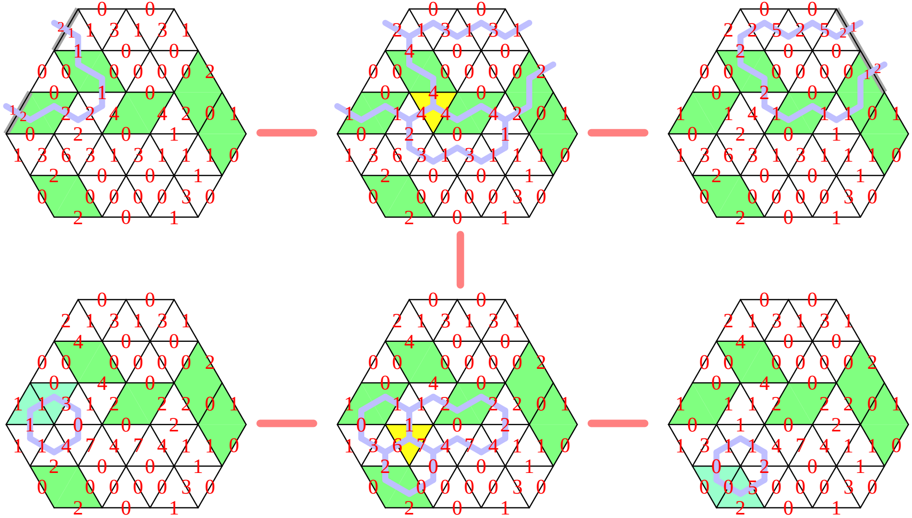


The mutation algorithm

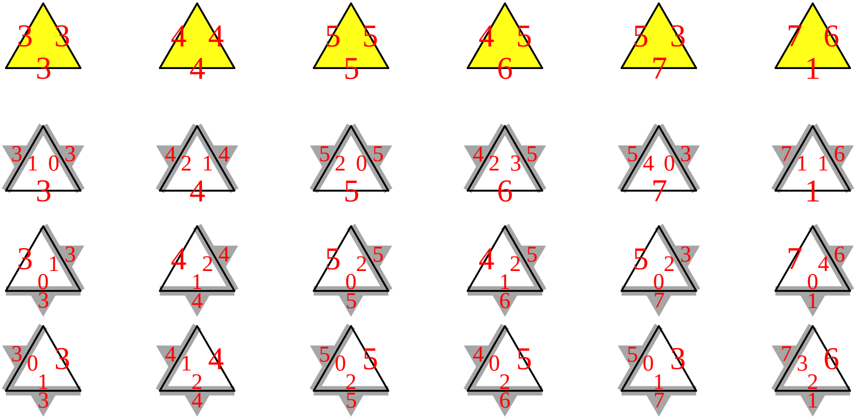


Flawed puzzle containing a marked scab.

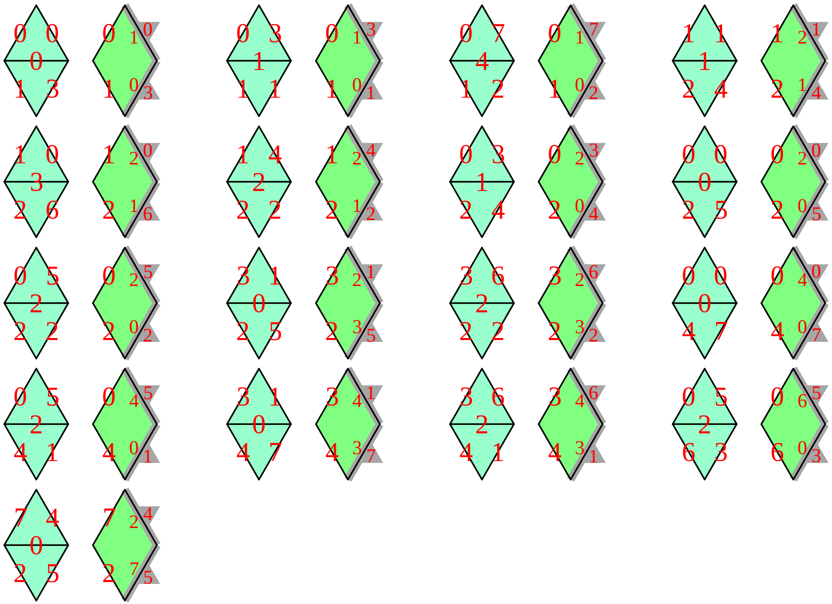
Component of the mutation graph:



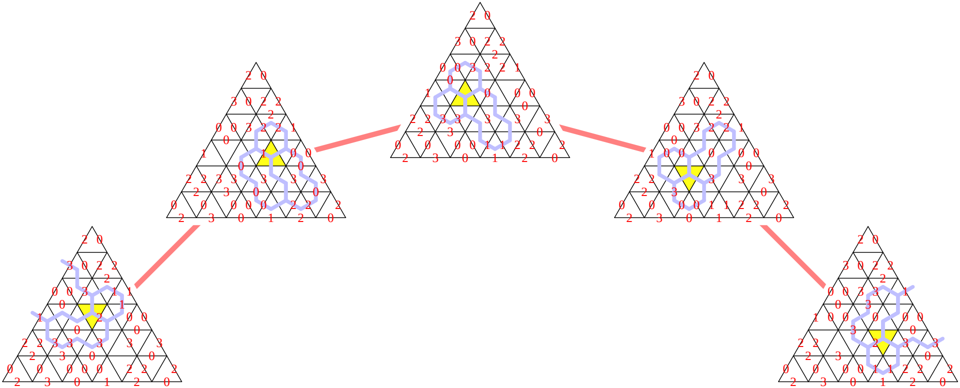
Resolutions of temporary puzzle pieces:

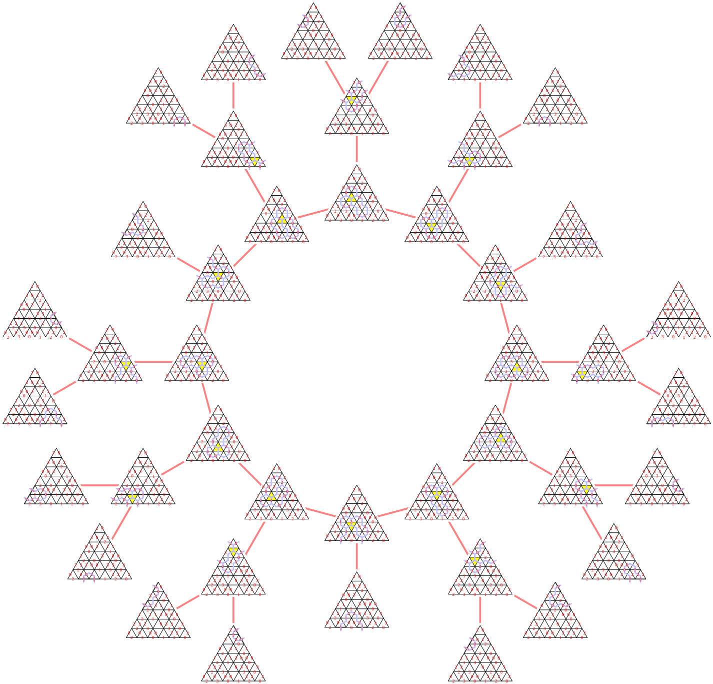


Resolutions of marked scabs:

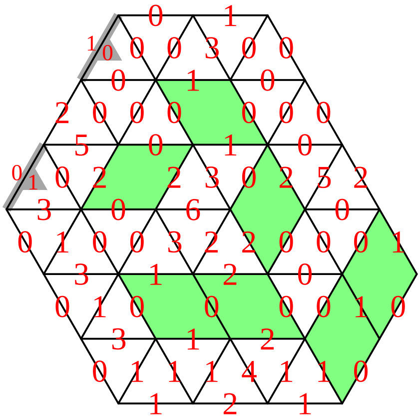


Example:

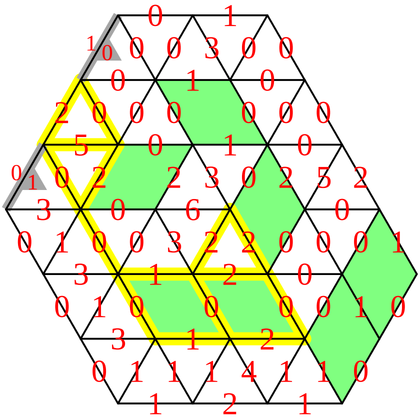




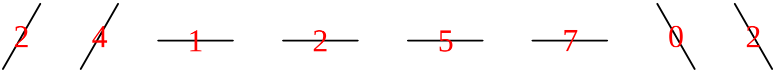
Proof that mutation algorithm works:



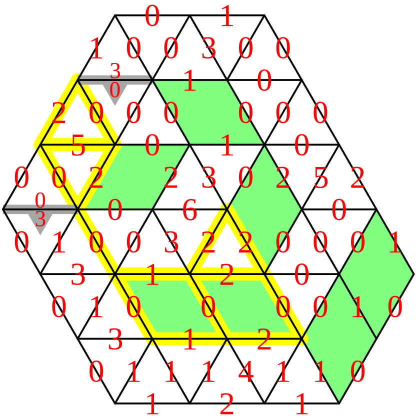
Proof that mutation algorithm works:



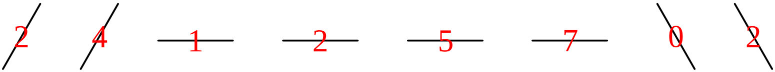
Consider connected component of the edges:



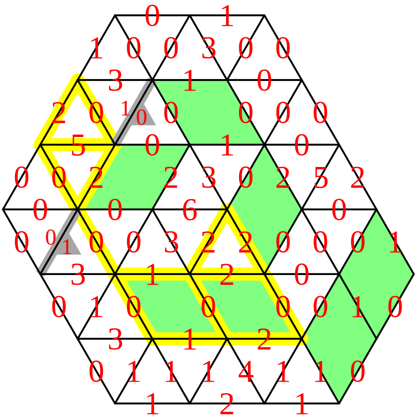
Proof that mutation algorithm works:



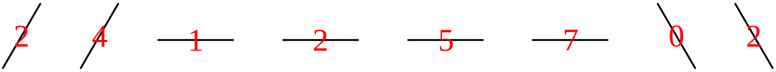
Consider connected component of the edges:



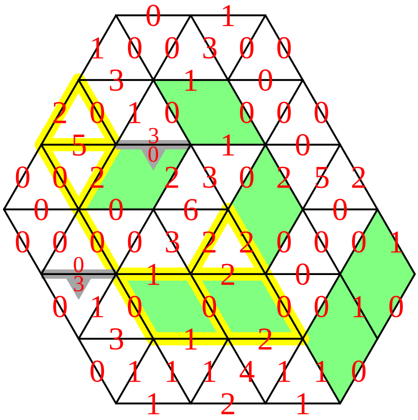
Proof that mutation algorithm works:



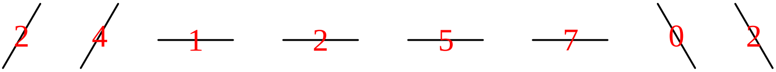
Consider connected component of the edges:



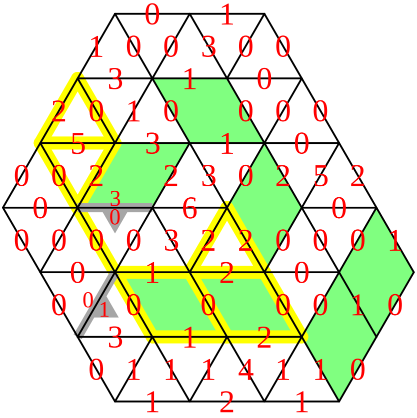
Proof that mutation algorithm works:



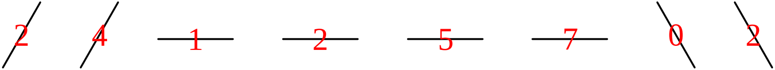
Consider connected component of the edges:



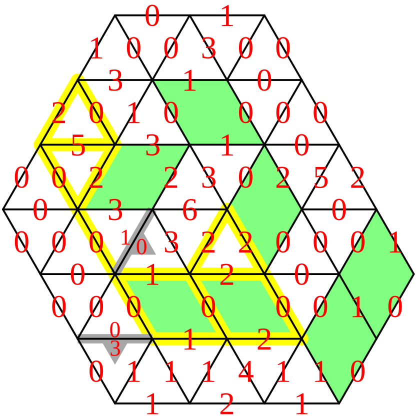
Proof that mutation algorithm works:



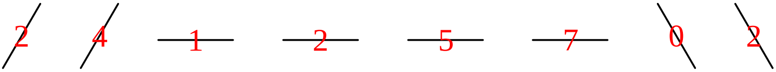
Consider connected component of the edges:



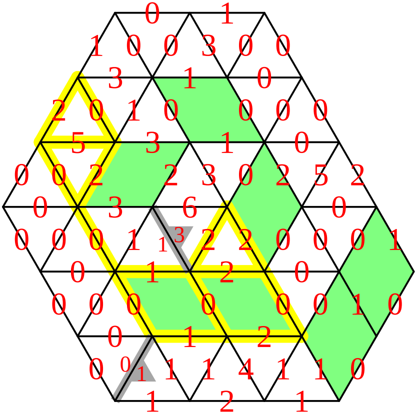
Proof that mutation algorithm works:



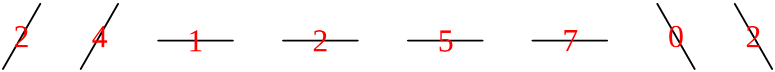
Consider connected component of the edges:



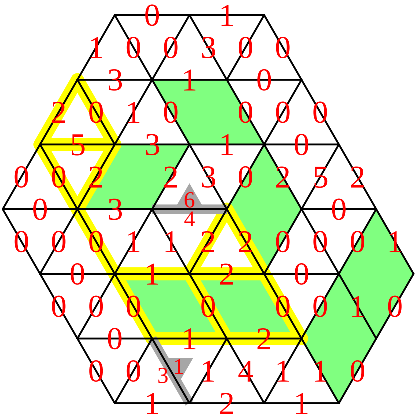
Proof that mutation algorithm works:



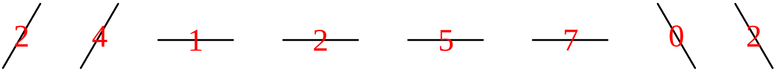
Consider connected component of the edges:



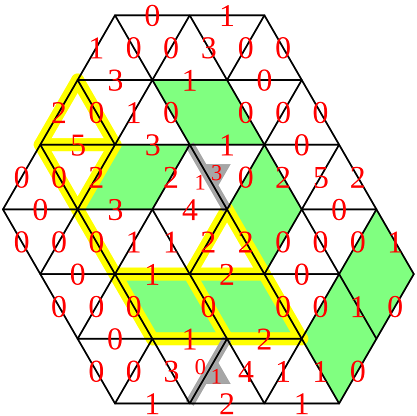
Proof that mutation algorithm works:



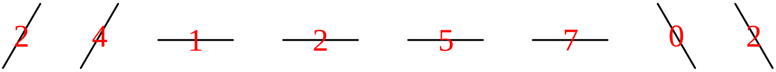
Consider connected component of the edges:



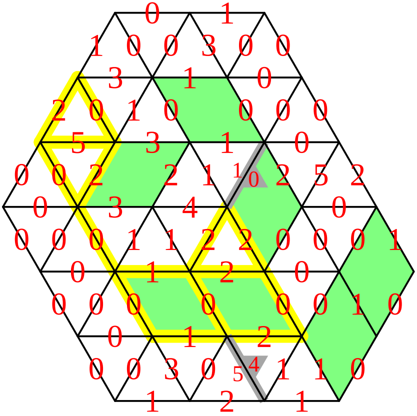
Proof that mutation algorithm works:



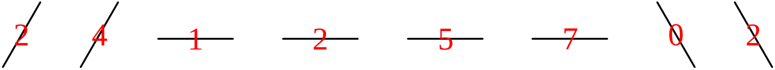
Consider connected component of the edges:



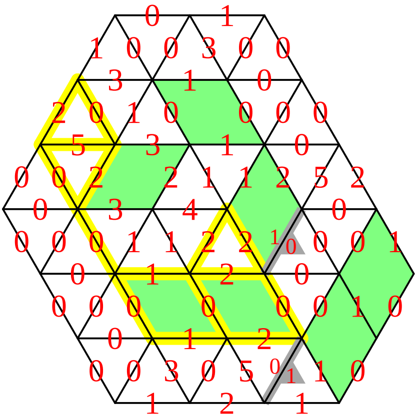
Proof that mutation algorithm works:



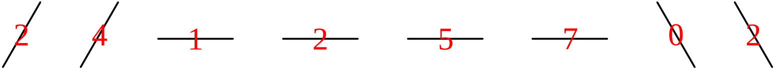
Consider connected component of the edges:



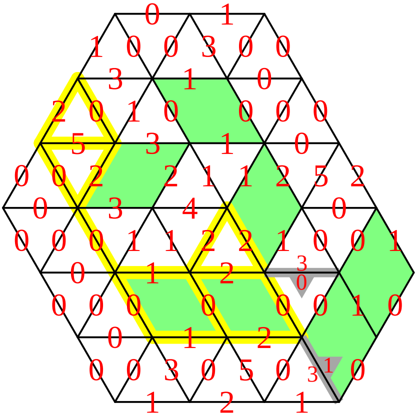
Proof that mutation algorithm works:



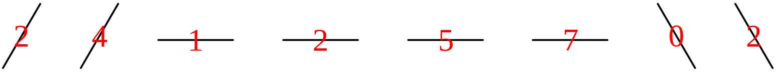
Consider connected component of the edges:



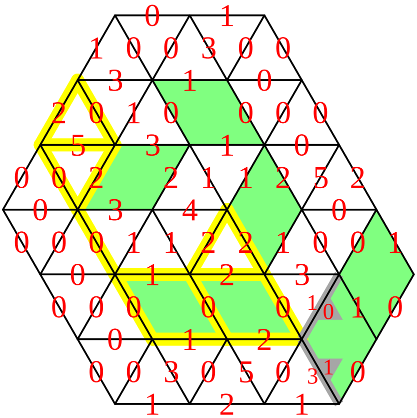
Proof that mutation algorithm works:



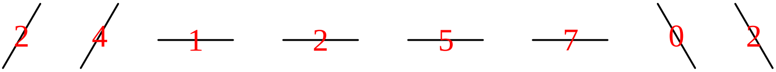
Consider connected component of the edges:



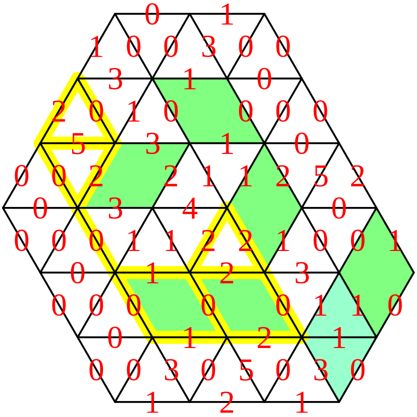
Proof that mutation algorithm works:



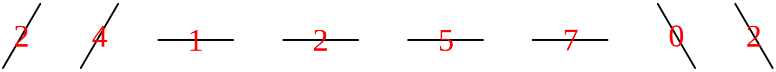
Consider connected component of the edges:



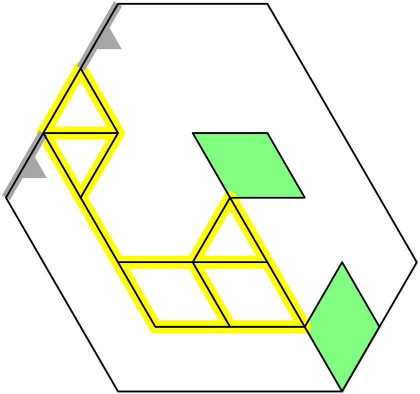
Proof that mutation algorithm works:



Consider connected component of the edges:



Proof that mutation algorithm works:



Technical result: The two gashes will propagate to the same location.

In particular, the above situation is impossible !!

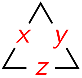
Aura of semi-labeled edges

An **aura** is a linear form in $R = \mathbb{C}[\delta_0, \delta_1, \delta_2]$. $\uparrow \in \mathbb{C}$ is a unit vector.

Aura of semi-labeled edges

An **aura** is a linear form in $R = \mathbb{C}[\delta_0, \delta_1, \delta_2]$. $\uparrow \in \mathbb{C}$ is a unit vector.

Def: $\mathcal{A}(\overset{0}{\text{—}}) = \overset{\delta_0}{\uparrow}$ $\mathcal{A}(\overset{1}{\text{—}}) = \overset{\delta_1}{\uparrow}$ $\mathcal{A}(\overset{2}{\text{—}}) = \overset{\delta_2}{\uparrow}$

If 

is a puzzle piece, then $\mathcal{A}(\diagup x) + \mathcal{A}(y \diagdown) + \mathcal{A}(\overset{z}{\text{—}}) = 0$.

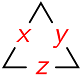
Aura of semi-labeled edges

An **aura** is a linear form in $R = \mathbb{C}[\delta_0, \delta_1, \delta_2]$.  $\in \mathbb{C}$ is a unit vector.

Def: $\mathcal{A}(\underline{\ 0\ }) =$ 

$\mathcal{A}(\underline{\ 1\ }) =$ 

$\mathcal{A}(\underline{\ 2\ }) =$ 

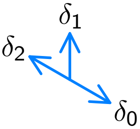
If 

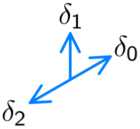
is a puzzle piece, then $\mathcal{A}(/x) + \mathcal{A}(y\backslash) + \mathcal{A}(\underline{\ z\ }) = 0$.

$\mathcal{A}(\underline{\ 3\ }) =$ 

$\mathcal{A}(\underline{\ 4\ }) =$ 

$\mathcal{A}(\underline{\ 5\ }) =$ 

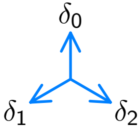
$\mathcal{A}(\underline{\ 6\ }) =$ 

$\mathcal{A}(\underline{\ 7\ }) =$ 

Aura of gashes

Definition: $\mathcal{A}(\frac{x}{y}) = \mathcal{A}(\frac{x}{}) + \mathcal{A}(\frac{}{y})$

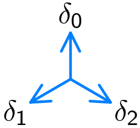
Example: $\mathcal{A}(\frac{0}{4}) = \mathcal{A}(\frac{0}{}) + \mathcal{A}(\frac{}{4}) =$



Aura of gashes

Definition: $\mathcal{A}\left(\frac{x}{y}\right) = \mathcal{A}\left(\frac{x}{}\right) + \mathcal{A}\left(\frac{}{y}\right)$

Example: $\mathcal{A}\left(\frac{0}{4}\right) = \mathcal{A}\left(\frac{0}{}\right) + \mathcal{A}\left(\frac{}{4}\right) =$



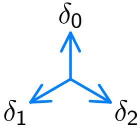
Properties:

- The aura of a gash is invariant under propagations.

Aura of gashes

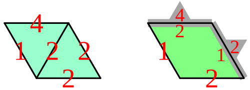
Definition: $$\mathcal{A}\left(\frac{x}{y}\right) = \mathcal{A}\left(\frac{x}{}\right) + \mathcal{A}\left(\frac{}{y}\right)$$

Example: $$\mathcal{A}\left(\frac{0}{4}\right) = \mathcal{A}\left(\frac{0}{}\right) + \mathcal{A}\left(\frac{}{4}\right) =$$



Properties:

- The aura of a gash is invariant under propagations.
- Sum of auras of gashes of any resolution is zero.

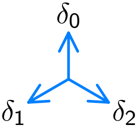


$$\mathcal{A}\left(\frac{4}{2}\right) + \mathcal{A}\left({}_1\backslash^2\right) = 0$$

Aura of gashes

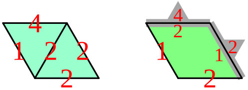
Definition: $\mathcal{A}\left(\frac{x}{y}\right) = \mathcal{A}\left(\frac{x}{}\right) + \mathcal{A}\left(\frac{}{y}\right)$

Example: $\mathcal{A}\left(\frac{0}{4}\right) = \mathcal{A}\left(\frac{0}{}\right) + \mathcal{A}\left(\frac{}{4}\right) =$



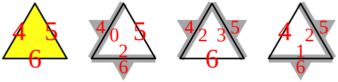
Properties:

- The aura of a gash is invariant under propagations.
- Sum of auras of gashes of any resolution is zero.



$\mathcal{A}\left(\frac{4}{2}\right) + \mathcal{A}\left({}_1\diagdown^2\right) = 0$

- Sum of auras of right gashes of resolutions of temporary piece is zero.

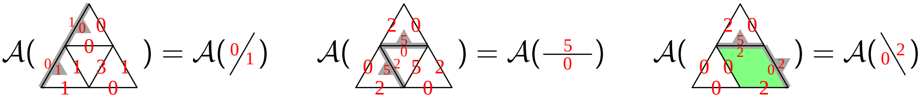


$\mathcal{A}\left({}^4\diagup_0\right) + \mathcal{A}\left({}_3\diagdown^5\right) + \mathcal{A}\left(\frac{1}{6}\right) = 0$

Aura of puzzles

Let $\widetilde{P}$ be a resolution of a flawed puzzle P.

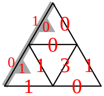
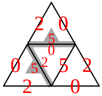
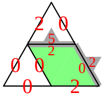
Def: $\mathcal{A}(\widetilde{P}) \;=\; \mathcal{A}($ right gash in $\widetilde{P}$ $)$



Aura of puzzles

Let $\widetilde{P}$ be a resolution of a flawed puzzle P.

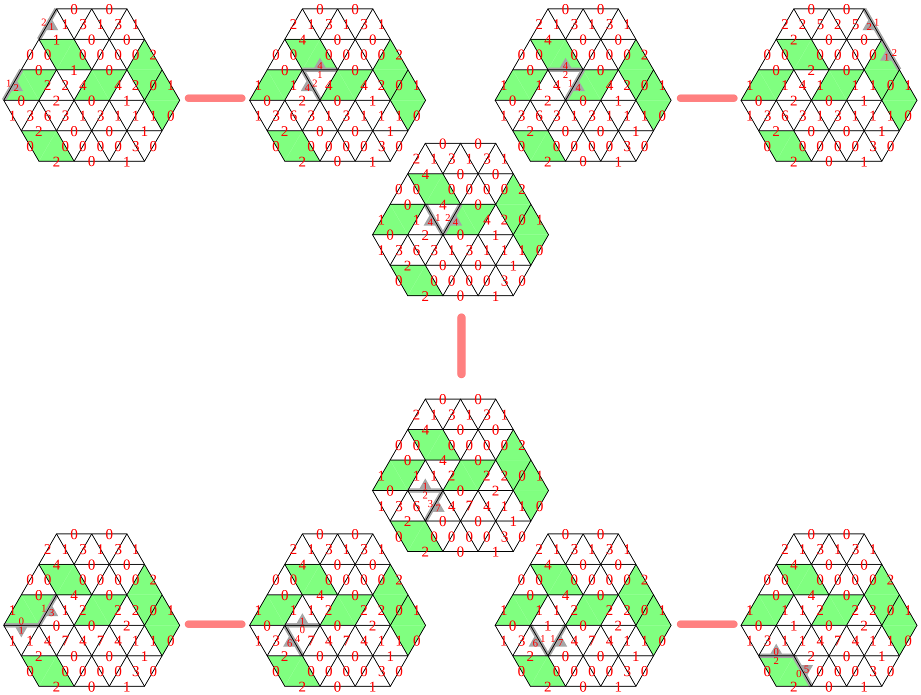
Def: $\mathcal{A}(\widetilde{P}) \;=\; \mathcal{A}($ right gash in $\widetilde{P}$ $)$

$\mathcal{A}($  $) = \mathcal{A}($  $)$ $\quad$ $\mathcal{A}($  $) = \mathcal{A}($  $)$ $\quad$ $\mathcal{A}($  $) = \mathcal{A}($  $)$

If $\widetilde{P}$ is the only resolution of P, then set $\mathcal{A}(P) = \mathcal{A}(\widetilde{P})$.

Key identity: Let S be any finite set of flawed puzzles that is closed under mutations. Then

$$\sum_{P \in S_{\text{scab}}} \mathcal{A}(P) \;+\; \sum_{P \in S_{\text{gash}}} \mathcal{A}(P) \;=\; 0$$



From now on:

- All puzzles are triangles.
- All equivariant puzzle pieces and scabs are vertical.

Def: For any 012-string $u = (u_1, u_2, \ldots, u_n)$ we set

$$C_u := \sum_{i=1}^{n} \delta_{u_i}\, y_i \in R[y_1, \ldots, y_n]$$

Exercise: $\partial P = \triangle_w^{u,v} \Rightarrow$

$$\sum_{s \text{ scab in } P} -\operatorname{weight}(s)\, \mathcal{A}(s) = C_u \cdot \searrow + C_v \cdot \swarrow + C_w \cdot \uparrow$$

From now on:

- All puzzles are triangles.
- All equivariant puzzle pieces and scabs are vertical.

Def: For any 012-string $u = (u_1, u_2, \ldots, u_n)$ we set

$$C_u := \sum_{i=1}^{n} \delta_{u_i} \, y_i \in R[y_1, \ldots, y_n]$$

Exercise: $\partial P = \triangle_w^{u,v} \Rightarrow$

$$\sum_{s \text{ scab in } P} -\text{weight}(s)\, \mathcal{A}(s) = C_u \cdot \searrow + C_v \cdot \swarrow + C_w \cdot \uparrow$$

Write $u \to u'$ if $u \leq u'$ in Bruhat order and $\ell(u) + 1 = \ell(u')$.

Examples: $022221 \to 122220$; $02 \to 20$; $100002 \to 200001$

Set $\delta(\frac{u}{u'}) = \delta_{u_i} - \delta_{u'_i}$ where i is minimal with $u_i \neq u'_i$.

From now on:

- All puzzles are triangles.
- All equivariant puzzle pieces and scabs are vertical.

Def: For any 012-string $u = (u_1, u_2, \ldots, u_n)$ we set

$$C_u \;:=\; \sum_{i=1}^{n} \delta_{u_i}\, y_i \;\in\; R[y_1, \ldots, y_n]$$

Exercise: $\partial P = \triangle_w^{u,v} \;\Rightarrow$

$$\sum_{s \text{ scab in } P} -\operatorname{weight}(s)\,\mathcal{A}(s) \;=\; C_u \cdot \searrow \;+\; C_v \cdot \swarrow \;+\; C_w \cdot \uparrow$$

Write $u \to u'$ if $u \leq u'$ in Bruhat order and $\ell(u) + 1 = \ell(u')$.

Examples: $022221 \to 122220$; $02 \to 20$; $100002 \to 200001$

Set $\delta(\frac{u}{u'}) \;=\; \delta_{u_i} - \delta_{u'_i}$ where i is minimal with $u_i \neq u'_i$.

Def: $$\widehat{C}_{u,v}^{w} \;=\; \sum_{\partial P = \triangle_w^{u,v}} \;\prod_{\lozenge \in P} \operatorname{weight}(\lozenge)$$

Molev–Sagan type recursion:

$$\left(C_u \cdot \searrow \; + \; C_v \cdot \swarrow \; + \; C_w \cdot \uparrow \right) \cdot \widehat{C}_{u,v}^{w}$$

$$= \sum_{\partial P = \triangle_{w}^{u,v}} \; \sum_{s \text{ scab in } P} -\mathcal{A}(s) \, \text{weight}(s) \prod_{\lozenge \in P} \text{weight}(\lozenge)$$

Molev–Sagan type recursion:

$$\left(C_u \cdot \searrow \; + \; C_v \cdot \swarrow \; + \; C_w \cdot \uparrow \right) \cdot \widehat{C}_{u,v}^{w}$$

$$= \sum_{\partial P = \triangle_w^{u,v}} \; \sum_{s \text{ scab in } P} -\mathcal{A}(s) \, \mathrm{weight}(s) \prod_{\lozenge \in P} \mathrm{weight}(\lozenge)$$

$$= \sum_{\substack{\partial P = \triangle_w^{u,v} \\ P \text{ has marked scab } s}} -\mathcal{A}(P) \, \mathrm{weight}(s) \prod_{\lozenge \in P} \mathrm{weight}(\lozenge)$$

Molev–Sagan type recursion:

$$\left(C_u \cdot \searrow \; + \; C_v \cdot \swarrow \; + \; C_w \cdot \uparrow \right) \cdot \widehat{C}^{w}_{u,v}$$

$$= \sum_{\partial P = \triangle^{u,v}_{w}} \; \sum_{s \text{ scab in } P} -\mathcal{A}(s)\,\text{weight}(s) \prod_{\blacklozenge \in P} \text{weight}(\blacklozenge)$$

$$= \sum_{\substack{\partial P = \triangle^{u,v}_{w} \\ P \text{ has marked scab } s}} -\mathcal{A}(P)\,\text{weight}(s) \prod_{\blacklozenge \in P} \text{weight}(\blacklozenge)$$

$$= \sum_{\substack{\partial P = \triangle^{u,v}_{w} \\ P \text{ has gash pair}}} \mathcal{A}(P) \prod_{\blacklozenge \in P} \text{weight}(\blacklozenge)$$

Molev–Sagan type recursion:

$$\left(C_u \cdot \searrow \; + \; C_v \cdot \swarrow \; + \; C_w \cdot \uparrow \right) \cdot \widehat{C}_{u,v}^{w}$$

$$= \sum_{\partial P = \triangle_w^{u,v}} \; \sum_{s \text{ scab in } P} -\mathcal{A}(s)\, \mathrm{weight}(s) \prod_{\blacklozenge \in P} \mathrm{weight}(\blacklozenge)$$

$$= \sum_{\substack{\partial P = \triangle_w^{u,v} \\ P \text{ has marked scab } s}} -\mathcal{A}(P)\, \mathrm{weight}(s) \prod_{\blacklozenge \in P} \mathrm{weight}(\blacklozenge)$$

$$= \sum_{\substack{\partial P = \triangle_w^{u,v} \\ P \text{ has gash pair}}} \mathcal{A}(P) \prod_{\blacklozenge \in P} \mathrm{weight}(\blacklozenge)$$

$$= \nwarrow \cdot \sum_{u \to u'} \delta\!\left(\frac{u}{u'}\right) \widehat{C}_{u',v}^{w} \; + \; \nearrow \cdot \sum_{v \to v'} \delta\!\left(\frac{v}{v'}\right) \widehat{C}_{u,v'}^{w} \; + \; \downarrow \cdot \sum_{w' \to w} \delta\!\left(\frac{w'}{w}\right) \widehat{C}_{u,v}^{w'}$$

Theorem (Method first applied by Molev and Sagan.)

The equivariant Schubert structure constants $C_{u,v}^{w} \in \mathbb{Z}[y_1, \ldots, y_n]$ of $Y = \mathrm{Fl}(a, b; n)$ are uniquely determined by

(1) $C_{w,w}^{w} = \prod_{i<j\,:\,w_i>w_j} (y_j - y_i)$ (Kostant-Kumar)

(2) $\left(C_u \cdot \searrow + C_v \cdot \swarrow + C_w \cdot \uparrow \right) \cdot C_{u,v}^{w}$

$$= \nwarrow \cdot \sum_{u \to u'} \delta\left(\frac{u}{u'}\right) C_{u',v}^{w} + \nearrow \cdot \sum_{v \to v'} \delta\left(\frac{v}{v'}\right) C_{u,v'}^{w} + \downarrow \cdot \sum_{w' \to w} \delta\left(\frac{w'}{w}\right) C_{u,v}^{w'}$$

Theorem (Method first applied by Molev and Sagan.)

The equivariant Schubert structure constants $C_{u,v}^{w} \in \mathbb{Z}[y_1, \ldots, y_n]$ of $Y = \mathrm{Fl}(a, b; n)$ are uniquely determined by

(1) $C_{w,w}^{w} = \prod_{i<j \,:\, w_i > w_j} (y_j - y_i)$ (Kostant-Kumar)

(2) $$\left(C_u \cdot \searrow \; + \; C_v \cdot \swarrow \; + \; C_w \cdot \uparrow \right) \cdot C_{u,v}^{w}$$

$$= \nwarrow \cdot \sum_{u \to u'} \delta\!\left(\frac{u}{u'}\right) C_{u',v}^{w} + \nearrow \cdot \sum_{v \to v'} \delta\!\left(\frac{v}{v'}\right) C_{u,v'}^{w} + \downarrow \cdot \sum_{w' \to w} \delta\!\left(\frac{w'}{w}\right) C_{u,v}^{w'}$$

Consequence:

$$C_{u,v}^{w} \;=\; \widehat{C}_{u,v}^{w} \;=\; \sum_{\partial P = \triangle_{w}^{u,v}} \; \prod_{\lozenge \in P} \mathrm{weight}(\lozenge)$$