Globally generated vector bundles on complete intersection CY threefolds (joint work with E.Ballico and F.Malaspina)

Sukmoon Huh

Department of Mathematics
Sungkyunkwan University

X : a smooth projective variety of dimension n over $\mathbb{C}$
$\mathcal{L}$: a line bundle on X
$Y \subset X$: locally complete intersection of codimension 2
$\mathcal{E}$: a vector bundle of rank $r \geq 2$ on X with

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E} \to \mathcal{I}_Y \otimes \mathcal{L} \to 0 \tag{1}$$

By tensoring (1) by $\mathcal{O}_Y$, we get

$$0 \to \wedge^2 N^\vee \otimes \mathcal{L}_{|Y} \to \mathcal{O}_Y^{\oplus(r-1)} \to \mathcal{E}_{|Y} \to N^\vee \otimes \mathcal{L}_{|Y} \to 0. \tag{2}$$

$\Rightarrow \wedge^2 N \otimes \mathcal{L}_{|Y}^\vee$ is generated by $(r-1)$ sections.

X : a smooth projective variety of dimension n over $\mathbb{C}$
$\mathcal{L}$: a line bundle on X
$Y \subset X$: locally complete intersection of codimension 2
$\mathcal{E}$: a vector bundle of rank $r \geq 2$ on X with

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E} \to \mathcal{I}_Y \otimes \mathcal{L} \to 0 \tag{1}$$

By tensoring (1) by $\mathcal{O}_Y$, we get

$$0 \to \wedge^2 N^\vee \otimes \mathcal{L}_{|Y} \to \mathcal{O}_Y^{\oplus(r-1)} \to \mathcal{E}_{|Y} \to N^\vee \otimes \mathcal{L}_{|Y} \to 0. \tag{2}$$

$\Rightarrow \wedge^2 N \otimes \mathcal{L}_{|Y}^\vee$ is generated by $(r-1)$ sections.

X : a smooth projective variety of dimension n over $\mathbb{C}$
$\mathcal{L}$: a line bundle on X
$Y \subset X$: locally complete intersection of codimension 2
$\mathcal{E}$: a vector bundle of rank $r \geq 2$ on X with

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E} \to \mathcal{I}_Y \otimes \mathcal{L} \to 0 \tag{1}$$

By tensoring (1) by $\mathcal{O}_Y$, we get

$$0 \to \wedge^2 N^\vee \otimes \mathcal{L}_{|Y} \to \mathcal{O}_Y^{\oplus(r-1)} \to \mathcal{E}_{|Y} \to N^\vee \otimes \mathcal{L}_{|Y} \to 0. \tag{2}$$

$\Rightarrow \wedge^2 N \otimes \mathcal{L}_{|Y}^\vee$ is generated by $(r-1)$ sections.

X : a smooth projective variety of dimension n over $\mathbb{C}$
$\mathcal{L}$: a line bundle on X
$Y \subset X$: locally complete intersection of codimension 2
$\mathcal{E}$: a vector bundle of rank $r \geq 2$ on X with

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E} \to \mathcal{I}_Y \otimes \mathcal{L} \to 0 \tag{1}$$

By tensoring (1) by $\mathcal{O}_Y$, we get

$$0 \to \wedge^2 N^\vee \otimes \mathcal{L}_{|Y} \to \mathcal{O}_Y^{\oplus(r-1)} \to \mathcal{E}_{|Y} \to N^\vee \otimes \mathcal{L}_{|Y} \to 0. \tag{2}$$

$\Rightarrow \wedge^2 N \otimes \mathcal{L}_{|Y}^\vee$ is generated by $(r-1)$ sections.

Conversely we have

Theorem (Hartshorne-Serre, Vogelaar)

Assume that

1. $\mathcal{L}$ *: a line bundle with* $H^i(X, \mathcal{L}^\vee) = 0$ *for* $i = 1, 2$
2. $\wedge^2 N \otimes \mathcal{L}^\vee_{|Y}$ *is generated by* $(r-1)$ *sections*

Then there exists a unique vector bundle $\mathcal{E}$ *of rank* r *fitting into (1).*

We will say that $\mathcal{E}$ and Y correspond if we have (1).

1. $\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$ and a line $L \subset \mathbb{P}^3$ correspond.
2. $T\mathbb{P}^3(-1)$ and a line $L \subset \mathbb{P}^3$ correspond.
3. $\mathcal{N}_{\mathbb{P}^3}(1)$ and two skew lines $L_1, L_2 \subset \mathbb{P}^3$ correspond.

Conversely we have

Theorem (Hartshorne-Serre, Vogelaar)

Assume that

1. $\mathcal{L}$ *: a line bundle with* $H^i(X, \mathcal{L}^\vee) = 0$ *for* $i = 1, 2$
2. $\wedge^2 N \otimes \mathcal{L}^\vee_{|Y}$ *is generated by* $(r-1)$ *sections*

Then there exists a unique vector bundle $\mathcal{E}$ *of rank* r *fitting into (1).*

We will say that $\mathcal{E}$ and Y correspond if we have (1).

1. $\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$ and a line $L \subset \mathbb{P}^3$ correspond.
2. $T\mathbb{P}^3(-1)$ and a line $L \subset \mathbb{P}^3$ correspond.
3. $\mathcal{N}_{\mathbb{P}^3}(1)$ and two skew lines $L_1, L_2 \subset \mathbb{P}^3$ correspond.

Conversely we have

Theorem (Hartshorne-Serre, Vogelaar)

Assume that

1. $\mathcal{L}$ *: a line bundle with* $H^i(X, \mathcal{L}^\vee) = 0$ *for* $i = 1, 2$
2. $\wedge^2 N \otimes \mathcal{L}^\vee_{|Y}$ *is generated by* $(r-1)$ *sections*

Then there exists a unique vector bundle $\mathcal{E}$ *of rank* r *fitting into (1).*

We will say that $\mathcal{E}$ and Y correspond if we have (1).

1. $\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$ and a line $L \subset \mathbb{P}^3$ correspond.
2. $T\mathbb{P}^3(-1)$ and a line $L \subset \mathbb{P}^3$ correspond.
3. $\mathcal{N}_{\mathbb{P}^3}(1)$ and two skew lines $L_1, L_2 \subset \mathbb{P}^3$ correspond.

Conversely we have

Theorem (Hartshorne-Serre, Vogelaar)

Assume that

1. $\mathcal{L}$ *: a line bundle with* $H^i(X, \mathcal{L}^\vee) = 0$ *for* $i = 1, 2$
2. $\wedge^2 N \otimes \mathcal{L}^\vee_{|Y}$ *is generated by* $(r-1)$ *sections*

Then there exists a unique vector bundle $\mathcal{E}$ *of rank* r *fitting into (1).*

We will say that $\mathcal{E}$ and Y correspond if we have (1).

1. $\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$ and a line $L \subset \mathbb{P}^3$ correspond.
2. $T\mathbb{P}^3(-1)$ and a line $L \subset \mathbb{P}^3$ correspond.
3. $\mathcal{N}_{\mathbb{P}^3}(1)$ and two skew lines $L_1, L_2 \subset \mathbb{P}^3$ correspond.

Conversely we have

Theorem (Hartshorne-Serre, Vogelaar)

Assume that

1. $\mathcal{L}$ *: a line bundle with* $H^i(X, \mathcal{L}^\vee) = 0$ *for* $i = 1, 2$
2. $\wedge^2 N \otimes \mathcal{L}^\vee_{|Y}$ *is generated by* $(r-1)$ *sections*

Then there exists a unique vector bundle $\mathcal{E}$ *of rank* r *fitting into (1).*

We will say that $\mathcal{E}$ and Y correspond if we have (1).

1. $\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$ and a line $L \subset \mathbb{P}^3$ correspond.
2. $T\mathbb{P}^3(-1)$ and a line $L \subset \mathbb{P}^3$ correspond.
3. $\mathcal{N}_{\mathbb{P}^3}(1)$ and two skew lines $L_1, L_2 \subset \mathbb{P}^3$ correspond.

There are several well-known properties concerning globally generated vector bundles:

1. $\mathcal{E}$: globally generated of rank $r > n$
   $\Rightarrow \mathcal{E}$ has $\mathcal{O}_X^{\oplus(r-n)}$ [Serre]
2. $\mathcal{E}$: globally generated with $H^0(\mathcal{E}(-c_1)) \neq 0$
   $\Rightarrow \mathcal{E} \cong \mathcal{O}_X(c_1) \oplus \mathcal{O}_X^{\oplus(r-1)}$ [Sierra]
3. $\varphi : \mathcal{F} \to \mathcal{G}$: a general morphism with $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ globally generated
   $\Rightarrow$ dependency locus Y of φ is nonsingular outside codimension$\geq \mathrm{rk}(\mathcal{G}) - \mathrm{rk}(\mathcal{F}) + 1$. [Banica, Chang]
4. $\mathcal{E}$: globally generated with $(r-1)$ general sections in (1)
   $\Rightarrow Y$ is smooth outside 4-dimensional subset.
   In particular if $\dim(X) = 3$, we may have

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E} \to \mathcal{I}_C \otimes \mathcal{L} \to 0 \tag{3}$$

   with smooth C.

There are several well-known properties concerning globally generated vector bundles:

1. $\mathcal{E}$: globally generated of rank $r > n$
   $\Rightarrow \mathcal{E}$ has $\mathcal{O}_X^{\oplus(r-n)}$ [Serre]
2. $\mathcal{E}$: globally generated with $H^0(\mathcal{E}(-c_1)) \neq 0$
   $\Rightarrow \mathcal{E} \cong \mathcal{O}_X(c_1) \oplus \mathcal{O}_X^{\oplus(r-1)}$ [Sierra]
3. $\varphi : \mathcal{F} \to \mathcal{G}$: a general morphism with $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ globally generated
   $\Rightarrow$ dependency locus Y of φ is nonsingular outside codimension$\geq \mathrm{rk}(\mathcal{G}) - \mathrm{rk}(\mathcal{F}) + 1$. [Banica, Chang]
4. $\mathcal{E}$: globally generated with $(r-1)$ general sections in (1)
   $\Rightarrow Y$ is smooth outside 4-dimensional subset.
   In particular if $\dim(X) = 3$, we may have

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E} \to \mathcal{I}_C \otimes \mathcal{L} \to 0 \tag{3}$$

   with smooth C.

There are several well-known properties concerning globally generated vector bundles:

1. $\mathcal{E}$: globally generated of rank $r > n$
   $\Rightarrow \mathcal{E}$ has $\mathcal{O}_X^{\oplus(r-n)}$ [Serre]
2. $\mathcal{E}$: globally generated with $H^0(\mathcal{E}(-c_1)) \neq 0$
   $\Rightarrow \mathcal{E} \cong \mathcal{O}_X(c_1) \oplus \mathcal{O}_X^{\oplus(r-1)}$ [Sierra]
3. $\varphi : \mathcal{F} \to \mathcal{G}$: a general morphism with $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ globally generated
   $\Rightarrow$ dependency locus Y of φ is nonsingular outside codimension$\geq \mathrm{rk}(\mathcal{G}) - \mathrm{rk}(\mathcal{F}) + 1$. [Banica, Chang]
4. $\mathcal{E}$: globally generated with $(r-1)$ general sections in (1)
   $\Rightarrow Y$ is smooth outside 4-dimensional subset.
   In particular if $\dim(X) = 3$, we may have

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E} \to \mathcal{I}_C \otimes \mathcal{L} \to 0 \tag{3}$$

   with smooth C.

There are several well-known properties concerning globally generated vector bundles:

1. $\mathcal{E}$: globally generated of rank $r > n$
   $\Rightarrow \mathcal{E}$ has $\mathcal{O}_X^{\oplus(r-n)}$ [Serre]
2. $\mathcal{E}$: globally generated with $H^0(\mathcal{E}(-c_1)) \neq 0$
   $\Rightarrow \mathcal{E} \cong \mathcal{O}_X(c_1) \oplus \mathcal{O}_X^{\oplus(r-1)}$ [Sierra]
3. $\varphi : \mathcal{F} \to \mathcal{G}$: a general morphism with $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ globally generated
   $\Rightarrow$ dependency locus Y of φ is nonsingular outside codimension$\geq \mathrm{rk}(\mathcal{G}) - \mathrm{rk}(\mathcal{F}) + 1$. [Banica, Chang]
4. $\mathcal{E}$: globally generated with $(r-1)$ general sections in (1)
   $\Rightarrow Y$ is smooth outside 4-dimensional subset.
   In particular if $\dim(X) = 3$, we may have

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E} \to \mathcal{I}_C \otimes \mathcal{L} \to 0 \tag{3}$$

   with smooth C.

There are several well-known properties concerning globally generated vector bundles:

1. $\mathcal{E}$: globally generated of rank $r > n$
   $\Rightarrow \mathcal{E}$ has $\mathcal{O}_X^{\oplus(r-n)}$ [Serre]
2. $\mathcal{E}$: globally generated with $H^0(\mathcal{E}(-c_1)) \neq 0$
   $\Rightarrow \mathcal{E} \cong \mathcal{O}_X(c_1) \oplus \mathcal{O}_X^{\oplus(r-1)}$ [Sierra]
3. $\varphi : \mathcal{F} \to \mathcal{G}$: a general morphism with $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ globally generated
   $\Rightarrow$ dependency locus Y of φ is nonsingular outside codimension$\geq \mathrm{rk}(\mathcal{G}) - \mathrm{rk}(\mathcal{F}) + 1$. [Banica, Chang]
4. $\mathcal{E}$: globally generated with $(r-1)$ general sections in (1)
   $\Rightarrow Y$ is smooth outside 4-dimensional subset.
   In particular if $\dim(X) = 3$, we may have

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E} \to \mathcal{I}_C \otimes \mathcal{L} \to 0 \tag{3}$$

with smooth C.

There have been several works on the classification of globally generated vector bundles with small first Chern classes over

- projective spaces [Anghel-Coanda-Manolache] [Sierra-Ugaglia]
- quadric hypersurfaces [Ballico-Malaspina-H]
- Segre threefolds [Ballico-Malaspina-H]

From the sequence

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E} \to \mathcal{I}_C(c_1) \to 0$$

- $\deg(C) = c_2(\mathcal{E}) \leq c_1^2 \deg(X)$
- $\omega_C \otimes \mathcal{O}_X(-c_1)$ is globally generated
- $\mathcal{I}_C(c_1)$ is globally generated

We may use

- Liaison theory for better bound of $c_2(\mathcal{E})$
- Smoothing of singular curves for the construction of C

There have been several works on the classification of globally generated vector bundles with small first Chern classes over

- projective spaces [Anghel-Coanda-Manolache] [Sierra-Ugaglia]
- quadric hypersurfaces [Ballico-Malaspina-H]
- Segre threefolds [Ballico-Malaspina-H]

From the sequence

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E} \to \mathcal{I}_C(c_1) \to 0$$

- $\deg(C) = c_2(\mathcal{E}) \leq c_1^2 \deg(X)$
- $\omega_C \otimes \mathcal{O}_X(-c_1)$ is globally generated
- $\mathcal{I}_C(c_1)$ is globally generated

We may use

- Liaison theory for better bound of $c_2(\mathcal{E})$
- Smoothing of singular curves for the construction of C

There have been several works on the classification of globally generated vector bundles with small first Chern classes over

- projective spaces [Anghel-Coanda-Manolache] [Sierra-Ugaglia]
- quadric hypersurfaces [Ballico-Malaspina-H]
- Segre threefolds [Ballico-Malaspina-H]

From the sequence

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E} \to \mathcal{I}_C(c_1) \to 0$$

- $\deg(C) = c_2(\mathcal{E}) \leq c_1^2 \deg(X)$
- $\omega_C \otimes \mathcal{O}_X(-c_1)$ is globally generated
- $\mathcal{I}_C(c_1)$ is globally generated

We may use

- Liaison theory for better bound of $c_2(\mathcal{E})$
- Smoothing of singular curves for the construction of C

Definition

1. A smooth 3-dimensional projective variety X is called a Calabi-Yau threefold if $\omega_X \cong \mathcal{O}_X$.
2. If a complete intersection $X = X_{r_1,\ldots,r_k} \subset \mathbb{P}^{k+3}$ is Calabi-Yau, then it is called a complete intersection Calabi-Yau (CICY).

There are 5 types of CICY 3-folds:

1. $X_5 \subset \mathbb{P}^4$
2. $X_{2,4} \subset \mathbb{P}^5$
3. $X_{3,3} \subset \mathbb{P}^5$
4. $X_{2,2,3} \subset \mathbb{P}^6$
5. $X_{2,2,2,2} \subset \mathbb{P}^7$

Goal : Classify GG bundles on CICY threefold with small c_1.

Definition

1. A smooth 3-dimensional projective variety X is called a Calabi-Yau threefold if $\omega_X \cong \mathcal{O}_X$.
2. If a complete intersection $X = X_{r_1,\dots,r_k} \subset \mathbb{P}^{k+3}$ is Calabi-Yau, then it is called a complete intersection Calabi-Yau (CICY).

There are 5 types of CICY 3-folds:

1. $X_5 \subset \mathbb{P}^4$
2. $X_{2,4} \subset \mathbb{P}^5$
3. $X_{3,3} \subset \mathbb{P}^5$
4. $X_{2,2,3} \subset \mathbb{P}^6$
5. $X_{2,2,2,2} \subset \mathbb{P}^7$

Goal : Classify GG bundles on CICY threefold with small c_1.

Definition

1. A smooth 3-dimensional projective variety X is called a Calabi-Yau threefold if $\omega_X \cong \mathcal{O}_X$.
2. If a complete intersection $X = X_{r_1,\ldots,r_k} \subset \mathbb{P}^{k+3}$ is Calabi-Yau, then it is called a complete intersection Calabi-Yau (CICY).

There are 5 types of CICY 3-folds:

1. $X_5 \subset \mathbb{P}^4$
2. $X_{2,4} \subset \mathbb{P}^5$
3. $X_{3,3} \subset \mathbb{P}^5$
4. $X_{2,2,3} \subset \mathbb{P}^6$
5. $X_{2,2,2,2} \subset \mathbb{P}^7$

Goal : Classify GG bundles on CICY threefold with small c_1.

Theorem (Ballico-Malaspina-H)

Let $\mathcal{E}$ be a globally generated bundle of rank $r \geq 2$ on $X = X_5$ with $c_1 \leq 2$ and no trivial factor. Then $\mathcal{E}$ is one of the following:

1. $T\mathbb{P}^4(-1)_{|X}$ *or* $\pi_p^* T\mathbb{P}^3(-1)$
2. $\mathcal{O}_X(1)^{\oplus 2}$ *or* $\pi_p^* \mathcal{N}_{\mathbb{P}^3}(1)$
3. $\pi_p^* \Omega_{\mathbb{P}^3}(2)$
4. $0 \to \mathcal{O}_X(-2) \to \mathcal{O}_X^{\oplus (r+1)} \to \mathcal{E} \to 0$ *with* $3 \leq r \leq 14$
5. $0 \to \mathcal{O}_X(-1)^{\oplus 2} \to \mathcal{O}_X^{\oplus (r+2)} \to \mathcal{E} \to 0$ *with* $3 \leq r \leq 8$
6. $0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X^{\oplus r} \oplus \mathcal{O}_X(1) \to \mathcal{E} \to 0$ *with* $3 \leq r \leq 5$

($\pi_p : X \to \mathbb{P}^3$ is a linear projection from $p \in \mathbb{P}^4 \setminus X$.)

In particular we have $c_2(\mathcal{E}) \in \{0, 5, 10, 15, 20\}$.

Theorem (Ballico-Malaspina-H)

Let $\mathcal{E}$ be a globally generated bundle of rank $r \geq 2$ on $X = X_5$ with $c_1 \leq 2$ and no trivial factor. Then $\mathcal{E}$ is one of the following:

1. $T\mathbb{P}^4(-1)_{|X}$ *or* $\pi_p^* T\mathbb{P}^3(-1)$
2. $\mathcal{O}_X(1)^{\oplus 2}$ *or* $\pi_p^* \mathcal{N}_{\mathbb{P}^3}(1)$
3. $\pi_p^* \Omega_{\mathbb{P}^3}(2)$
4. $0 \to \mathcal{O}_X(-2) \to \mathcal{O}_X^{\oplus(r+1)} \to \mathcal{E} \to 0$ *with* $3 \leq r \leq 14$
5. $0 \to \mathcal{O}_X(-1)^{\oplus 2} \to \mathcal{O}_X^{\oplus(r+2)} \to \mathcal{E} \to 0$ *with* $3 \leq r \leq 8$
6. $0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X^{\oplus r} \oplus \mathcal{O}_X(1) \to \mathcal{E} \to 0$ *with* $3 \leq r \leq 5$

($\pi_p : X \to \mathbb{P}^3$ is a linear projection from $p \in \mathbb{P}^4 \setminus X$.)

In particular we have $c_2(\mathcal{E}) \in \{0, 5, 10, 15, 20\}$.

Example

U_1, U_2 : planes in $\mathbb{P}^4$ with $\langle U_1 \cup U_2 \rangle = \mathbb{P}^4$
Assume $\{p\} = U_1 \cap U_2 \not\subset X$.
Set $U = U_1 \cup U_2$ and $C = U \cap X = C_1 \sqcup C_2$ with $C_i = U_i \cap X$
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$.

It is easy to check that $\mathcal{I}_C(2)$ is globally generated.
$\Rightarrow$ There exists a globally generated bundle $\mathcal{E}$ of rank 2 fitting into (1) with $\mathcal{L} \cong \mathcal{O}_X(2)$.

Letting $L_i = \pi_p(C_i)$, we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{N}_{\mathbb{P}^3}(1) \longrightarrow \mathcal{I}_{L_1 \cup L_2}(2) \longrightarrow 0.$$

Thus the example gives $\pi_p^* \mathcal{N}_{\mathbb{P}^3}(1)$.

Example

U_1, U_2 : planes in $\mathbb{P}^4$ with $\langle U_1 \cup U_2 \rangle = \mathbb{P}^4$
Assume $\{p\} = U_1 \cap U_2 \not\subset X$.
Set $U = U_1 \cup U_2$ and $C = U \cap X = C_1 \sqcup C_2$ with $C_i = U_i \cap X$
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$.

It is easy to check that $\mathcal{I}_C(2)$ is globally generated.
$\Rightarrow$ There exists a globally generated bundle $\mathcal{E}$ of rank 2 fitting into (1) with $\mathcal{L} \cong \mathcal{O}_X(2)$.

Letting $L_i = \pi_p(C_i)$, we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{N}_{\mathbb{P}^3}(1) \longrightarrow \mathcal{I}_{L_1 \cup L_2}(2) \longrightarrow 0.$$

Thus the example gives $\pi_p^* \mathcal{N}_{\mathbb{P}^3}(1)$.

Example

U_1, U_2 : planes in $\mathbb{P}^4$ with $\langle U_1 \cup U_2 \rangle = \mathbb{P}^4$
Assume $\{p\} = U_1 \cap U_2 \not\subset X$.
Set $U = U_1 \cup U_2$ and $C = U \cap X = C_1 \sqcup C_2$ with $C_i = U_i \cap X$
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$.

It is easy to check that $\mathcal{I}_C(2)$ is globally generated.
$\Rightarrow$ There exists a globally generated bundle $\mathcal{E}$ of rank 2 fitting into (1) with $\mathcal{L} \cong \mathcal{O}_X(2)$.

Letting $L_i = \pi_p(C_i)$, we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{N}_{\mathbb{P}^3}(1) \longrightarrow \mathcal{I}_{L_1 \cup L_2}(2) \longrightarrow 0.$$

Thus the example gives $\pi_p^* \mathcal{N}_{\mathbb{P}^3}(1)$.

Example

U_1, U_2 : planes in $\mathbb{P}^4$ with $\langle U_1 \cup U_2 \rangle = \mathbb{P}^4$
Assume $\{p\} = U_1 \cap U_2 \not\subset X$.
Set $U = U_1 \cup U_2$ and $C = U \cap X = C_1 \sqcup C_2$ with $C_i = U_i \cap X$
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$.

It is easy to check that $\mathcal{I}_C(2)$ is globally generated.
$\Rightarrow$ There exists a globally generated bundle $\mathcal{E}$ of rank 2 fitting into (1) with $\mathcal{L} \cong \mathcal{O}_X(2)$.

Letting $L_i = \pi_p(C_i)$, we have

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{N}_{\mathbb{P}^3}(1) \to \mathcal{I}_{L_1 \cup L_2}(2) \to 0.$$

Thus the example gives $\pi_p^* \mathcal{N}_{\mathbb{P}^3}(1)$.

Example

U_1, U_2 : planes in $\mathbb{P}^4$ with $\langle U_1 \cup U_2 \rangle = \mathbb{P}^4$
Assume $\{p\} = U_1 \cap U_2 \not\subset X$.
Set $U = U_1 \cup U_2$ and $C = U \cap X = C_1 \sqcup C_2$ with $C_i = U_i \cap X$
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$.

It is easy to check that $\mathcal{I}_C(2)$ is globally generated.
$\Rightarrow$ There exists a globally generated bundle $\mathcal{E}$ of rank 2 fitting into (1) with $\mathcal{L} \cong \mathcal{O}_X(2)$.

Letting $L_i = \pi_p(C_i)$, we have

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{N}_{\mathbb{P}^3}(1) \to \mathcal{I}_{L_1 \cup L_2}(2) \to 0.$$

Thus the example gives $\pi_p^* \mathcal{N}_{\mathbb{P}^3}(1)$.

$\mathcal{E}$: globally generated of rank 2 with $c_1 = 2$ and $H^0(\mathcal{E}(-1)) = 0$.
C : a corresponding smooth curve to $\mathcal{E}$.
Let $C := C_1 \sqcup \cdots \sqcup C_s$, C_i irreducible component
$\omega_C \cong \mathcal{O}_C(2) \Rightarrow d_i = g_i - 1$.

Definition

$\pi(d, n)$: the upper bound on the genus for non-degenerate curves of degree d in $\mathbb{P}^n$

e.g. $\pi(6,3) = 4$, $\pi(7,3) = 6$, $\pi(7,4) = 3$, $\cdots$

Choose general $A_1, A_2, A_3 \in |\mathcal{I}_C(2)|$ with $B_i \subset \mathbb{P}^4$ quadrics such that $B_i \cap X = A_i$ and $B_i \cap B_j$ is a reduced surface of degree 4.

Case1 : $B_1 \cap B_2 \cap B_3$ contains no surface

$Y := B_1 \cap B_2 \cap B_3$ is a curve of degree 8 with $\omega_Y \cong \mathcal{O}_Y(1)$.
$\Rightarrow C$ is connected and contained in $\mathbb{P}^2$, a contradiction.

$\mathcal{E}$: globally generated of rank 2 with $c_1 = 2$ and $H^0(\mathcal{E}(-1)) = 0$.
C : a corresponding smooth curve to $\mathcal{E}$.
Let $C := C_1 \sqcup \cdots \sqcup C_s$, C_i irreducible component
$\omega_C \cong \mathcal{O}_C(2) \Rightarrow d_i = g_i - 1$.

Definition

$\pi(d, n)$: the upper bound on the genus for non-degenerate curves of degree d in $\mathbb{P}^n$

e.g. $\pi(6,3) = 4$, $\pi(7,3) = 6$, $\pi(7,4) = 3$, $\cdots$

Choose general $A_1, A_2, A_3 \in |\mathcal{I}_C(2)|$ with $B_i \subset \mathbb{P}^4$ quadrics such that $B_i \cap X = A_i$ and $B_i \cap B_j$ is a reduced surface of degree 4.

Case1 : $B_1 \cap B_2 \cap B_3$ contains no surface

$Y := B_1 \cap B_2 \cap B_3$ is a curve of degree 8 with $\omega_Y \cong \mathcal{O}_Y(1)$.
$\Rightarrow C$ is connected and contained in $\mathbb{P}^2$, a contradiction.

$\mathcal{E}$: globally generated of rank 2 with $c_1 = 2$ and $H^0(\mathcal{E}(-1)) = 0$.
C : a corresponding smooth curve to $\mathcal{E}$.
Let $C := C_1 \sqcup \cdots \sqcup C_s$, C_i irreducible component
$\omega_C \cong \mathcal{O}_C(2) \Rightarrow d_i = g_i - 1$.

Definition

$\pi(d, n)$: the upper bound on the genus for non-degenerate curves of degree d in $\mathbb{P}^n$

e.g. $\pi(6,3) = 4$, $\pi(7,3) = 6$, $\pi(7,4) = 3$, $\cdots$

Choose general $A_1, A_2, A_3 \in |\mathcal{I}_C(2)|$ with $B_i \subset \mathbb{P}^4$ quadrics such that $B_i \cap X = A_i$ and $B_i \cap B_j$ is a reduced surface of degree 4.

Case1 : $B_1 \cap B_2 \cap B_3$ contains no surface

$Y := B_1 \cap B_2 \cap B_3$ is a curve of degree 8 with $\omega_Y \cong \mathcal{O}_Y(1)$.
$\Rightarrow C$ is connected and contained in $\mathbb{P}^2$, a contradiction.

$\mathcal{E}$: globally generated of rank 2 with $c_1 = 2$ and $H^0(\mathcal{E}(-1)) = 0$.
C : a corresponding smooth curve to $\mathcal{E}$.
Let $C := C_1 \sqcup \cdots \sqcup C_s$, C_i irreducible component
$\omega_C \cong \mathcal{O}_C(2) \Rightarrow d_i = g_i - 1$.

Definition

$\pi(d, n)$: the upper bound on the genus for non-degenerate curves of degree d in $\mathbb{P}^n$

e.g. $\pi(6, 3) = 4$, $\pi(7, 3) = 6$, $\pi(7, 4) = 3$, $\cdots$

Choose general $A_1, A_2, A_3 \in |\mathcal{I}_C(2)|$ with $B_i \subset \mathbb{P}^4$ quadrics such that $B_i \cap X = A_i$ and $B_i \cap B_j$ is a reduced surface of degree 4.

Case1 : $B_1 \cap B_2 \cap B_3$ contains no surface

$Y := B_1 \cap B_2 \cap B_3$ is a curve of degree 8 with $\omega_Y \cong \mathcal{O}_Y(1)$.
$\Rightarrow C$ is connected and contained in $\mathbb{P}^2$, a contradiction.

$\mathcal{E}$: globally generated of rank 2 with $c_1 = 2$ and $H^0(\mathcal{E}(-1)) = 0$.
C : a corresponding smooth curve to $\mathcal{E}$.
Let $C := C_1 \sqcup \cdots \sqcup C_s$, C_i irreducible component
$\omega_C \cong \mathcal{O}_C(2) \Rightarrow d_i = g_i - 1$.

Definition

$\pi(d,n)$: the upper bound on the genus for non-degenerate curves of degree d in $\mathbb{P}^n$

e.g. $\pi(6,3) = 4$, $\pi(7,3) = 6$, $\pi(7,4) = 3$, $\cdots$

Choose general $A_1, A_2, A_3 \in |\mathcal{I}_C(2)|$ with $B_i \subset \mathbb{P}^4$ quadrics such that $B_i \cap X = A_i$ and $B_i \cap B_j$ is a reduced surface of degree 4.

Case1 : $B_1 \cap B_2 \cap B_3$ contains no surface

$Y := B_1 \cap B_2 \cap B_3$ is a curve of degree 8 with $\omega_Y \cong \mathcal{O}_Y(1)$.
$\Rightarrow C$ is connected and contained in $\mathbb{P}^2$, a contradiction.

$\mathcal{E}$: globally generated of rank 2 with $c_1 = 2$ and $H^0(\mathcal{E}(-1)) = 0$.
C : a corresponding smooth curve to $\mathcal{E}$.
Let $C := C_1 \sqcup \cdots \sqcup C_s$, C_i irreducible component
$\omega_C \cong \mathcal{O}_C(2) \Rightarrow d_i = g_i - 1$.

Definition

$\pi(d, n)$: the upper bound on the genus for non-degenerate curves of degree d in $\mathbb{P}^n$

e.g. $\pi(6, 3) = 4$, $\pi(7, 3) = 6$, $\pi(7, 4) = 3$, $\cdots$

Choose general $A_1, A_2, A_3 \in |\mathcal{I}_C(2)|$ with $B_i \subset \mathbb{P}^4$ quadrics such that $B_i \cap X = A_i$ and $B_i \cap B_j$ is a reduced surface of degree 4.

Case1 : $B_1 \cap B_2 \cap B_3$ contains no surface

$Y := B_1 \cap B_2 \cap B_3$ is a curve of degree 8 with $\omega_Y \cong \mathcal{O}_Y(1)$.
$\Rightarrow C$ is connected and contained in $\mathbb{P}^2$, a contradiction.

$\mathcal{E}$: globally generated of rank 2 with $c_1 = 2$ and $H^0(\mathcal{E}(-1)) = 0$.
C : a corresponding smooth curve to $\mathcal{E}$.
Let $C := C_1 \sqcup \cdots \sqcup C_s$, C_i irreducible component
$\omega_C \cong \mathcal{O}_C(2) \Rightarrow d_i = g_i - 1$.

Definition

$\pi(d,n)$: the upper bound on the genus for non-degenerate curves of degree d in $\mathbb{P}^n$

e.g. $\pi(6,3) = 4$, $\pi(7,3) = 6$, $\pi(7,4) = 3$, $\cdots$

Choose general $A_1, A_2, A_3 \in |\mathcal{I}_C(2)|$ with $B_i \subset \mathbb{P}^4$ quadrics such that $B_i \cap X = A_i$ and $B_i \cap B_j$ is a reduced surface of degree 4.

Case1 : $B_1 \cap B_2 \cap B_3$ contains no surface

$Y := B_1 \cap B_2 \cap B_3$ is a curve of degree 8 with $\omega_Y \cong \mathcal{O}_Y(1)$.
$\Rightarrow C$ is connected and contained in $\mathbb{P}^2$, a contradiction.

Case2 : $B_1 \cap B_2 \cap B_3$ contains a surface
$= S \cup$ (lower dimensional part) with $(S \cap X)_{\text{red}} = C$.

S is one of the following

- $S = U_1 \cup U_2$ the union of two planes with $U_1 \cap U_2 = \{p\}$
- $S = U_1 \cup U_2 \cup U_3$ spanning $\mathbb{P}^4$
- $S = Q \cup U$ with $U \not\subset \langle Q \rangle$
- S is an integral non-degenerated surface of degree 3 in $\mathbb{P}^4$.

In the last case, S is either

(i) a cubic scroll

(ii) a cone over a rational normal curve in $\mathbb{P}^3$

$\Rightarrow$ In each case except the first, we get contradictions.
Similarly we may deal with higher rank case.

Case2 : $B_1 \cap B_2 \cap B_3$ contains a surface
$= S \cup$ (lower dimensional part) with $(S \cap X)_{\text{red}} = C$.

S is one of the following

- $S = U_1 \cup U_2$ the union of two planes with $U_1 \cap U_2 = \{p\}$
- $S = U_1 \cup U_2 \cup U_3$ spanning $\mathbb{P}^4$
- $S = Q \cup U$ with $U \not\subset \langle Q \rangle$
- S is an integral non-degenerated surface of degree 3 in $\mathbb{P}^4$.

In the last case, S is either

(i) a cubic scroll

(ii) a cone over a rational normal curve in $\mathbb{P}^3$

$\Rightarrow$ In each case except the first, we get contradictions.
Similarly we may deal with higher rank case.

Case2 : $B_1 \cap B_2 \cap B_3$ contains a surface
$= S \cup$ (lower dimensional part) with $(S \cap X)_{\text{red}} = C$.

S is one of the following

- $S = U_1 \cup U_2$ the union of two planes with $U_1 \cap U_2 = \{p\}$
- $S = U_1 \cup U_2 \cup U_3$ spanning $\mathbb{P}^4$
- $S = Q \cup U$ with $U \not\subset \langle Q \rangle$
- S is an integral non-degenerated surface of degree 3 in $\mathbb{P}^4$.

In the last case, S is either

(i) a cubic scroll

(ii) a cone over a rational normal curve in $\mathbb{P}^3$

$\Rightarrow$ In each case except the first, we get contradictions.
Similarly we may deal with higher rank case.

Case2 : $B_1 \cap B_2 \cap B_3$ contains a surface
$= S \cup$ (lower dimensional part) with $(S \cap X)_{\text{red}} = C$.

S is one of the following

- $S = U_1 \cup U_2$ the union of two planes with $U_1 \cap U_2 = \{p\}$
- $S = U_1 \cup U_2 \cup U_3$ spanning $\mathbb{P}^4$
- $S = Q \cup U$ with $U \not\subset \langle Q \rangle$
- S is an integral non-degenerated surface of degree 3 in $\mathbb{P}^4$.

In the last case, S is either

(i) a cubic scroll

(ii) a cone over a rational normal curve in $\mathbb{P}^3$

$\Rightarrow$ In each case except the first, we get contradictions.
Similarly we may deal with higher rank case.

Case2 : $B_1 \cap B_2 \cap B_3$ contains a surface
$= S\cup$ (lower dimensional part) with $(S \cap X)_{\text{red}} = C$.

S is one of the following

- $S = U_1 \cup U_2$ the union of two planes with $U_1 \cap U_2 = \{p\}$
- $S = U_1 \cup U_2 \cup U_3$ spanning $\mathbb{P}^4$
- $S = Q \cup U$ with $U \not\subset \langle Q \rangle$
- S is an integral non-degenerated surface of degree 3 in $\mathbb{P}^4$.

In the last case, S is either

(i) a cubic scroll

(ii) a cone over a rational normal curve in $\mathbb{P}^3$

$\Rightarrow$ In each case except the first, we get contradictions.
Similarly we may deal with higher rank case.

Let us assume $X = X_{2,4}$ or $X_{3,3}$.

Example (1)

$U_1, U_2 \cong \mathbb{P}^3$ in $\mathbb{P}^5$
$U := U_1 \cup U_2$ spans $\mathbb{P}^5$, transversal to X with $U_1 \cap U_2 \cap X = \emptyset$
$C = U \cap X$
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$ and $\mathcal{I}_C(2)$ is globally generated.

Example (2)

$C = Q_1 \cap Q_2 \cap Q_3 \cap Q_4 \subset \mathbb{P}^5$
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$ and $\mathcal{I}_C(2)$ is globally generated.

There exist some $X_{2,4}$ and $X_{3,3}$ containing such C.

Let us assume $X = X_{2,4}$ or $X_{3,3}$.

Example (1)

$U_1, U_2 \cong \mathbb{P}^3$ in $\mathbb{P}^5$
$U := U_1 \cup U_2$ spans $\mathbb{P}^5$, transversal to X with $U_1 \cap U_2 \cap X = \emptyset$
$C = U \cap X$
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$ and $\mathcal{I}_C(2)$ is globally generated.

Example (2)

$C = Q_1 \cap Q_2 \cap Q_3 \cap Q_4 \subset \mathbb{P}^5$
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$ and $\mathcal{I}_C(2)$ is globally generated.

There exist some $X_{2,4}$ and $X_{3,3}$ containing such C.

Let us assume $X = X_{2,4}$ or $X_{3,3}$.

Example (1)

$U_1, U_2 \cong \mathbb{P}^3$ in $\mathbb{P}^5$
$U := U_1 \cup U_2$ spans $\mathbb{P}^5$, transversal to X with $U_1 \cap U_2 \cap X = \emptyset$
$C = U \cap X$
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$ and $\mathcal{I}_C(2)$ is globally generated.

Example (2)

$C = Q_1 \cap Q_2 \cap Q_3 \cap Q_4 \subset \mathbb{P}^5$
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$ and $\mathcal{I}_C(2)$ is globally generated.

There exist some $X_{2,4}$ and $X_{3,3}$ containing such C.

Let us assume $X = X_{2,4}$ or $X_{3,3}$.

Example (1)

$U_1, U_2 \cong \mathbb{P}^3$ in $\mathbb{P}^5$
$U := U_1 \cup U_2$ spans $\mathbb{P}^5$, transversal to X with $U_1 \cap U_2 \cap X = \emptyset$
$C = U \cap X$
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$ and $\mathcal{I}_C(2)$ is globally generated.

Example (2)

$C = Q_1 \cap Q_2 \cap Q_3 \cap Q_4 \subset \mathbb{P}^5$
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$ and $\mathcal{I}_C(2)$ is globally generated.

There exist some $X_{2,4}$ and $X_{3,3}$ containing such C.

Let us assume $X = X_{2,4}$ or $X_{3,3}$.

Example (1)

$U_1, U_2 \cong \mathbb{P}^3$ in $\mathbb{P}^5$
$U := U_1 \cup U_2$ spans $\mathbb{P}^5$, transversal to X with $U_1 \cap U_2 \cap X = \emptyset$
$C = U \cap X$
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$ and $\mathcal{I}_C(2)$ is globally generated.

Example (2)

$C = Q_1 \cap Q_2 \cap Q_3 \cap Q_4 \subset \mathbb{P}^5$
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$ and $\mathcal{I}_C(2)$ is globally generated.

There exist some $X_{2,4}$ and $X_{3,3}$ containing such C.

Example (3)

$S \subset \mathbb{P}^5$: a surface of degree 5 with $\omega_S \cong \mathcal{O}_S(-1)$.
$C := S \cap U_3$, U_3: a cubic
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$
For $X := U_3 \cap U_3'$, we get that $\mathcal{I}_C(2)$ is globally generated.

Example (4)

$Y = Q_1 \cap Q_2 \cap Q_3 \cap U_3$: $\deg(Y) = 24$ and $\omega_Y \cong \mathcal{O}_Y(3)$
Assume $Y = C \cup D$ with $\deg(C) = d$ and D smooth outside $C \cap D$
$\Rightarrow \omega_{Y|C} \cong \omega_C(C \cap D)$ and so $\deg(C \cap D) = d$

If C is cut out by U_3 and U_3' inside $S := Q_1 \cap Q_2 \cap Q_3$, then we have $d = \deg(U_3' \cap D) = 3(24 - d)$, i.e. $d = 18$.

$\Rightarrow C \subset U_3 \cap U_3' =: X_{3,3}$ with globally generated $\mathcal{I}_C(2)$

Example (3)

$S \subset \mathbb{P}^5$: a surface of degree 5 with $\omega_S \cong \mathcal{O}_S(-1)$.
$C := S \cap U_3$, U_3: a cubic
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$
For $X := U_3 \cap U_3'$, we get that $\mathcal{I}_C(2)$ is globally generated.

Example (4)

$Y = Q_1 \cap Q_2 \cap Q_3 \cap U_3$: $\deg(Y) = 24$ and $\omega_Y \cong \mathcal{O}_Y(3)$
Assume $Y = C \cup D$ with $\deg(C) = d$ and D smooth outside $C \cap D$
$\Rightarrow \omega_{Y|C} \cong \omega_C(C \cap D)$ and so $\deg(C \cap D) = d$

If C is cut out by U_3 and U_3' inside $S := Q_1 \cap Q_2 \cap Q_3$, then we have $d = \deg(U_3' \cap D) = 3(24 - d)$, i.e. $d = 18$.

$\Rightarrow C \subset U_3 \cap U_3' =: X_{3,3}$ with globally generated $\mathcal{I}_C(2)$

Example (3)

$S \subset \mathbb{P}^5$: a surface of degree 5 with $\omega_S \cong \mathcal{O}_S(-1)$.
$C := S \cap U_3$, U_3: a cubic
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$
For $X := U_3 \cap U_3'$, we get that $\mathcal{I}_C(2)$ is globally generated.

Example (4)

$Y = Q_1 \cap Q_2 \cap Q_3 \cap U_3$: $\deg(Y) = 24$ and $\omega_Y \cong \mathcal{O}_Y(3)$
Assume $Y = C \cup D$ with $\deg(C) = d$ and D smooth outside $C \cap D$
$\Rightarrow \omega_{Y|C} \cong \omega_C(C \cap D)$ and so $\deg(C \cap D) = d$

If C is cut out by U_3 and U_3' inside $S := Q_1 \cap Q_2 \cap Q_3$, then we have $d = \deg(U_3' \cap D) = 3(24 - d)$, i.e. $d = 18$.

$\Rightarrow C \subset U_3 \cap U_3' =: X_{3,3}$ with globally generated $\mathcal{I}_C(2)$

Example (3)

$S \subset \mathbb{P}^5$: a surface of degree 5 with $\omega_S \cong \mathcal{O}_S(-1)$.
$C := S \cap U_3$, U_3: a cubic
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$
For $X := U_3 \cap U_3'$, we get that $\mathcal{I}_C(2)$ is globally generated.

Example (4)

$Y = Q_1 \cap Q_2 \cap Q_3 \cap U_3$: $\deg(Y) = 24$ and $\omega_Y \cong \mathcal{O}_Y(3)$
Assume $Y = C \cup D$ with $\deg(C) = d$ and D smooth outside $C \cap D$
$\Rightarrow \omega_{Y|C} \cong \omega_C(C \cap D)$ and so $\deg(C \cap D) = d$

If C is cut out by U_3 and U_3' inside $S := Q_1 \cap Q_2 \cap Q_3$, then we have $d = \deg(U_3' \cap D) = 3(24 - d)$, i.e. $d = 18$.

$\Rightarrow C \subset U_3 \cap U_3' =: X_{3,3}$ with globally generated $\mathcal{I}_C(2)$

Example (3)

$S \subset \mathbb{P}^5$: a surface of degree 5 with $\omega_S \cong \mathcal{O}_S(-1)$.
$C := S \cap U_3$, U_3: a cubic
$\Rightarrow \omega_C \cong \mathcal{O}_C(2)$
For $X := U_3 \cap U_3'$, we get that $\mathcal{I}_C(2)$ is globally generated.

Example (4)

$Y = Q_1 \cap Q_2 \cap Q_3 \cap U_3$: $\deg(Y) = 24$ and $\omega_Y \cong \mathcal{O}_Y(3)$
Assume $Y = C \cup D$ with $\deg(C) = d$ and D smooth outside $C \cap D$
$\Rightarrow \omega_{Y|C} \cong \omega_C(C \cap D)$ and so $\deg(C \cap D) = d$

If C is cut out by U_3 and U_3' inside $S := Q_1 \cap Q_2 \cap Q_3$, then we have $d = \deg(U_3' \cap D) = 3(24 - d)$, i.e. $d = 18$.

$\Rightarrow C \subset U_3 \cap U_3' =: X_{3,3}$ with globally generated $\mathcal{I}_C(2)$

Theorem

Let $\mathcal{E}$ be globally generated of rank 2 with $c_1 = 2$ and $h^0(\mathcal{E}(-1)) = 0$.

1. *On $X_{2,4}$, we have Example (1), (2)*
2. *On $X_{3,3}$, we have Example (1), (2), (3), (4)*

except the case of $c_2 = 16$.

Corollary

$\mathcal{E}$: globally generated of rank 2 on X with $c_1 \leq 2$

1. $X_{2,4} : c_2 \in \{0, 4, 8, 11, 16\}$
2. $X_{3,3} : c_2 \in \{0, 9, 12, 15, 16, 18\}$

Theorem

Let $\mathcal{E}$ be globally generated of rank 2 with $c_1 = 2$ and $h^0(\mathcal{E}(-1)) = 0$.

1. *On $X_{2,4}$, we have Example (1), (2)*
2. *On $X_{3,3}$, we have Example (1), (2), (3), (4)*

except the case of $c_2 = 16$.

Corollary

$\mathcal{E}$: globally generated of rank 2 on X with $c_1 \leq 2$

1. $X_{2,4}$: $c_2 \in \{0, 4, 8, 11, 16\}$
2. $X_{3,3}$: $c_2 \in \{0, 9, 12, 15, 16, 18\}$

Ψ: the scheme-theoretic base locus of $H^0(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(2))$
Φ: the union of the irreducible components of Ψ_{red} containing C

- $\Psi \cap X = C$ as schemes
- $\deg(\Phi) \leq 2^{5-\dim(\Phi)}$ and the equality holds iff $\Phi = \Psi$ is equidimensional and complete intersection of hyperquadrics.

$S_i :=$ a fixed reduced and irreducible component $S_i \subset \Psi$ containing C_i.

- $s = 1$, i.e. set $S = S_1 \Rightarrow \dim(S) \in \{1, 2\}$.
  $\Rightarrow$ Example (2)-(4).
- $s = 2$:
  1. $i \neq j \Rightarrow S_i \neq S_j$
  2. $\dim(S_i) \geq 2$ for all i
  3. There exists i with $\dim(S_i) = 3$
  4. $\dim(S_i) = 3$ for all i
  5. get contradiction except Example(1).

Ψ: the scheme-theoretic base locus of $H^0(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(2))$
Φ: the union of the irreducible components of Ψ_{red} containing C

- $\Psi \cap X = C$ as schemes
- $\deg(\Phi) \leq 2^{5-\dim(\Phi)}$ and the equality holds iff $\Phi = \Psi$ is equidimensional and complete intersection of hyperquadrics.

$S_i :=$ a fixed reduced and irreducible component $S_i \subset \Psi$ containing C_i.

- $s = 1$, i.e. set $S = S_1 \Rightarrow \dim(S) \in \{1, 2\}$.
  $\Rightarrow$ Example (2)-(4).
- $s = 2$:
  1. $i \neq j \Rightarrow S_i \neq S_j$
  2. $\dim(S_i) \geq 2$ for all i
  3. There exists i with $\dim(S_i) = 3$
  4. $\dim(S_i) = 3$ for all i
  5. get contradiction except Example(1).

Ψ: the scheme-theoretic base locus of $H^0(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(2))$
Φ: the union of the irreducible components of Ψ_{red} containing C

- $\Psi \cap X = C$ as schemes
- $\deg(\Phi) \leq 2^{5-\dim(\Phi)}$ and the equality holds iff $\Phi = \Psi$ is equidimensional and complete intersection of hyperquadrics.

$S_i :=$ a fixed reduced and irreducible component $S_i \subset \Psi$ containing C_i.

- $s = 1$, i.e. set $S = S_1 \Rightarrow \dim(S) \in \{1, 2\}$.
  $\Rightarrow$ Example (2)-(4).
- $s = 2$:
  1. $i \neq j \Rightarrow S_i \neq S_j$
  2. $\dim(S_i) \geq 2$ for all i
  3. There exists i with $\dim(S_i) = 3$
  4. $\dim(S_i) = 3$ for all i
  5. get contradiction except Example(1).

Thank You Very Much !!!