3. Project summary

Study of spaces and structures built on them (like bundles of vectors, measures of distances) is one of the main aims of contemporary mathematics and mathematical physics. Classification of structures is usually achieved by means of computing invariants: they are designed to distinguish the inequivalent classes of structures. They are indispensable but hard to compute. For this reasons, new methods and new techniques are desired and required. One of the main lines of approach is based on symmetries, which were always fundamental in solving problems both in mathematics and in physics. In the studies of quantum spaces, new types of symmetries arise and serve as a crucial organizing principle allowing one to handle highly complicated situations.

Our project, in precise mathematical terms, concentrates around noncommutative index theory, Connes' spectral geometry, and appropriate adaptations to coalgebraic or Hopf-algebraic setting of cyclic (co)homology and the Chern-Connes character as our general tools. As a specific venue of investigation, we plan Hopf-algebra equivariant fiber products of algebras, where Mayer-Vietoris methods can be applied. A key technical feature of this project is a standing assumption that the algebra describing a compact quantum fibration corresponds to the algebra of functions continuous along the base space and bounded-degree polynomial along the fibers. This is a strategy to take the best of two different worlds: our fixed-point algebras are C*-algebras offering advantages of functional calculus and computability of K-theory, whereas the total space algebras are comodule algebras allowing us to take advantage of tools and methods of the well-developed noncommutative Galois theory of ring extensions.

The goal of this project is to develop and apply verifiable criteria to distinguish different isomorphism classes of principal extensions of noncommutative algebras and K_0-classes of modules associated to them. On the geometric side, we aim to achieve a new guiding principle in gluing together smooth noncommutative geometries along boundaries.

PART B

4. Quality of the Exchange Programme[3][4]

4.1 Objectives and relevance of the joint research exchange programme

Table 1- List of Work Packages

Work package No	Work package title	Start month	End month
1	Fibre-product approach to index computations for principal extensions	1	48
2	Compact and locally compact quantum principal bundles	1	48
3	Non-crossed-product examples of principal extensions of C*-algebras	1	48
4	From non-contractibility of compact quantum groups to non-triviality of principal extensions	1	48
5	Principal actions of locally compact quantum groups	1	48
6	K-theory and homology theory for principal extensions:	1	48
7	The Chern-Galois character and Hopf-cyclic homology	1	48
8	Spectral geometry and applications of the local index formula	1	48
9	Fibre product of spectral triples	1	48
10	Towards a geometric realisation of Hopf-cyclic cohomology	1	48
	TOTAL		

[3] The researchers identified by this Annex I can be replaced by other researchers in accordance with the definitions contained in Art. III.1.
[4] The exchange plan foreseen by this Annex I is indicative and can be modified subject to the approval of the Commission. However the maximum number of researcher months in the project shall not be increased throughout the duration of the grant agreement.

Work package number	**1**		
Work package title	Fibre-product approach to index computations for principal extensions		
Partner short name	IMPAN, PENN STATE, SWANSEA, SISSA, UNB, UMSNH,UOG, SU		

Researchers involved: P.Baum, S.Brain, P.M.Hajac, N.Higson, U.Krähmer, D.Kucerovsky, W.Szymanski, E.Wagner. B.Zielinski

Objectives: As a guiding example, we study noncommutative 2n-spheres constructed via gluings (fiber products of algebras) of noncommutative 2n-balls over their boundaries. Using the Mayer-Vietoris map from the K_1 of the boundary noncommutative (2n-1)-sphere algebra to the K_0 of the 2n-sphere algebra, we obtain a family of finitely generated projective modules indexed by integers (the winding number). By the compatibility of the boundary maps in the Mayer-Vietoris six-term exact sequence of KK-groups with the index pairing (Kasparov product), the index of these projective modules can be computed via the index pairing of the defining unitaries over the (2n-1)-sphere algebra. This gives us a significant simplification of the index problem.

All already studied examples (like the noncommutative Hopf fibrations of the Heegaard-type quantum 3-sphere [2] over the generic Podles 2-sphere [21] or the mirror 2-sphere [14]) strongly encourage the fiber-product approach to index computations. On the other hand, the ideal structure of noncommutative C*-algebras is far more rigid than of commutative ones. This makes fiber-product decompositions of noncommutative C*-algebras more difficult. For example, the minimal unitization of compact operators, which corresponds to the standard Podles sphere, cannot be decomposed if we assume both defining homomorphisms of the fiber product to be surjective. Therefore, since the Mayer-Vietoris sequence in K-theory remains valid if we give up the surjectivity of one of the defining maps, it seems desirable to:

Apply the fiber-product approach to index computations assuming the surjectivity of only one of the defining morphisms. This should allow for a significantly bigger scope of examples, including examples such as the standard Hopf fibration of SUq(2).

Work package number	2		
Work package title	Compact and locally compact quantum principal bundles		
Partner short name	IMPAN, UW, SU, SISSA, PENN STATE		
Researchers involved: P.Baum, L.Dabrowski, P.M.Hajac, P.Kasprzak, P.Soltan, P.Stachura, W.Szymanski			

Objectives: Much as Hopf algebras do not capture the whole nature of topological quantum groups, Galois-type extensions describe only the algebraic part of topological quantum principal bundles [8]. A functional-analytic treatment of non-compact quantum principal bundles is evidently needed but largely unexplored. First, we focus on the far better understood compact case and a very concrete research problem.

A key feature of compact quantum groups [24] is the existence of Peter-Weyl theory due to Woronowicz. Thus we can speak of regular or polynomial functions on compact quantum groups: they are represented by dense *-Hopf-algebras generated by irreducible representations. This immediately allows us to define, for any unital C*-algebra on which a compact quantum group acts, the algebra of functions “polynomial along the fibers and continuous along the quotient space". The algebra defined this way automatically becomes a comodule algebra coacted by the aforementioned Hopf algebra, and the invariant subalgebra (functions on the ``quotient space") is a C*-algebra.

In case this coaction is Galois, it is clear that the action of a compact quantum group is principal in the sense of Ellwood [8]. Proving the reverse implication is a problem even if groups and bundles are classical. Using Fourier analysis with coefficients in a C*-algebra, we proved (cf. [14]) that it is true for the group U(1). It is very likely that, using results of Quigg [22], our proof has a straightforward extension to any abelian compact quantum group, i.e., a compact quantum group whose Pontryagin dual is given by a commutative C*-algebra. The main difficulty is in the fact that, when the dual quantum group algebra is no longer commutative, its irreducible representations are not necessarily one-dimensional, so that we are forced to work with general noncommutative vector bundles rather than noncommutative line bundles. Now, our research task can be formulated as follows.

Prove that the actions of compact quantum groups on unital C-algebras are principal if and only if the induced coactions of their polynomial Hopf algebras are Galois.*

Work package number	**3**		
Work package title	Non-crossed-product examples of principal extensions of C*-algebras		
Partner short name	ANU, IMPAN, SU, PENN STATE		
Researchers involved: P.Baum, A.Carey, P.M.Hajac, N.Higson, A.Rennie, W.Szymanski			

Objectives: Just as there is more to principal bundles than Cartesian products of groups with spaces, we argue that there is more to principal extensions of C*-algebras than crossed products. In the algebraic context of Hopf--Galois extensions, one might prove that a given extension is not a crossed product taking an advantage of the fact that algebraic crossed products have many nontrivial invertible elements, whereas a given polynomial algebra might lack such elements. This argument does not apply on the C*-level because it is easy to construct nontrivial invertible elements in a C*-algebra. It is difficult to decide whether a given C*-algebra is a crossed product with its fixed-point subalgebra because, contrary to the more restricted setting of von Neumann algebras, there appears to be no theory to manage such problems. Von Neumann algebras are at the opposite extreme to polynomial algebras. For them theory tells us when an extension of algebras has a crossed-product structure. (Think of measurable global sections of nontrivial topological bundles.) Therefore, we find the C*-context most interesting to analyse this crossed-product problem.

In algebraic topology, to prove that a principal bundle is nontrivial, one computes an invariant of an associated vector bundle. Following this pattern, in [14], we used the result of an index computation for finitely generated projective modules over the C*-algebra of a mirror quantum sphere to prove that the C*-algebras of noncommutative lens spaces cannot be Z-crossed products with the sphere C*-algebra. Along these lines, the next research task is:

Construct more general (higher dimensional) classes of examples of principal actions on unital C-algebras and apply noncommutative index theory to prove that they are not crossed products with the invariant C*-subalgebras.*

Work package number	**4**		
Work package title	From non-contractibility of compact quantum groups to non-triviality of principal extensions		
Partner short name	IMPAN, PENN STATE, SISSA, UMSNH, SU		
Researchers involved: P.Baum, L.Dabrowski, P.M.Hajac, N.Higson, W.Szymanski, E.Wagner			

Objectives: A simple topological construction called *join* allows one to produce non-trivial principal bundles for all non-contractible compact groups. (We have to exclude the trivial compact group.) For instance, this way one can obtain the edge of the Moebius strip over the circle, the Hopf fibration, and the instanton fibration. We carried out this construction for an arbitrary Hopf *-algebra and thus obtained, in explicit terms, a large family of principal extensions. In particular, this includes the quantum SU(2) instanton fibration that can be derived from [20] This brings us to the following obstruction-theory type problem:

Study the non-contractibility of compact quantum groups and apply it to prove the non-triviality of general classes of noncommutative join constructions.

Work package number	**5**		
Work package title	Principal actions of locally compact quantum groups		
Partner short name	IMPAN, UW, SU, UCB		
Researchers involved: P.M.Hajac, P.Kasprzak, M.Rieffel, P.Soltan, P.Stachura, W.Szymanski			

Objectives: Although the theory of compact quantum spaces (based on unital algebras) is the best understood part of noncommutative geometry, the building of non-compact theory (with non-unital algebras) is gaining a momentum now. In particular, serious steps towards formalising non-unital spectral triples have been already taken, and the theory of locally compact quantum groups has made great strides. Thus, it is timely and important to complement these advances with development of the theory of locally compact quantum principal bundles that would enjoy the existence of the Chern character analogous to the one invented in [3].

As a starting point, we take the multiplier Hopf algebra Galois extensions [23] and the C*-algebraic characterization of free and proper (principal) actions of quantum groups on (nonunital) C*-algebras due to Ellwood. In order to proceed further, it is necessary to analyse the noncommutative analogue of the module of continuous sections of a vector bundle associated to a principal bundle with a locally compact quantum group as its structure group. This is likely to require a modification of the cotensor product construction (which proved very helpful in understanding the compact case). The theory of connections on quantum locally compact bundles must come next.

Furthermore, the study of Dirac operators on quantum homogeneous spaces G/T, where G is a q-deformation of a Lie group and T is its compact subgroup, plays a role in connection with the Baum-Connes conjecture. More precisely, progress in this direction might provide test models of the noncommutative universal proper space for a locally compact quantum group G, and thus significantly contribute to the Meyer and Nest programme of exploring the Baum-Connes conjecture for locally compact quantum groups. Summarising, our aim is:

Develop general theory of locally compact quantum principal bundles hand-in-hand with detailed analysis of concrete examples.

Work package number	**6**		
Work package title	K-theory and homology theory for principal extensions:		
Partner short name	IMPAN, UOG, UCB		
Researchers involved: U.Krähmer, T.Maszczyk, M.Wodzicki			

Objectives: The idea of extending the notion of K_0 groups from algebras to monoidal categories is based on the following observation. Both the generators of $K_0(A)$ (idempotents in matrix algebras with entries in A) and the equivalence relation between them can be defined in the framework of monoidal categories without any reference to finitely generated projective A-modules usually used for defining the K_0-group. This allows us to work exclusively in the framework of monoidal functors, which can be interpreted as geometric morphisms between noncommutative spaces, according to [19]. In this formalism, K_0-group of the algebra A becomes a noncommutative counterpart of the classical notion of the 0-th cohomology group, counting the number of connected components of a classical topological space X. This statement (attributed to Alain Connes) can be made rigorous in this formalism. Then the following problem arises:

Define higher noncommutative Eilenberg-MacLane spaces and construct the Chern character to the representable cohomology.

On the other hand, noncommutative Galois extensions can be given a geometric form also in the framework of monoidal categories [19]. From this perspective, they are nothing but (noncommutative) covers in flat topology. Taking this into account and aiming to achieve a conceptual understanding of relations between K-theory and the Galois condition, one is tempted to raise the following problem:

Show that there exists a spectral sequence computing the K-groups for a noncommutative Galois extension such that its second term combines the K-theory of the base with some coefficients related to the structure of the extension (generalized Galois group data). Prove that this sequence is compatible, in an appropriate sense, with an analogous spectral sequence for representable cohomology via the aforementioned Chern character.

Work package number	**7**		1
Work package title	The Chern-Galois character and Hopf-cyclic homology		
Partner short name	IMPAN, JU, PENN STATE, SWANSEA, UNB		
Researchers involved: E.Beggs, T.Brzezinski, P.M.Hajac, H.Moscovici, B.Rangipour, A.Sitarz, B.Zielinski			

Objectives: The Hopf-algebraic symmetry discovered by Connes and Moscovici while studying the transverse geometry of foliations [5] led to the invention of a new type of cyclic theory, where morphisms defining the differential complex are determined by Hopf-algebraic structures. This new theory was picked up in [10,11,17] where a general type of coefficients (called stable anti-Yetter-Drinfeld modules) was introduced.

There has been a lot of progress both in the theory and applications of Hopf-cyclic cohomology with coefficients (e.g., the proof of the cup-product conjecture). Despite this, the geometric understanding of this theory remains unsatisfactory. Firstly, in classical differential geometry, a de Rham complex often appears “twisted” by tensoring with the module of sections of a (non-trivial) flat vector bundle as coefficients. Therefore, the much desired introduction of coefficients into cyclic theory prompts the question whether they can play the role of flat vector bundles in noncommutative geometry. Here the main difficulty appears to be that stable anti-Yetter-Drinfeld modules are modules comodules over a Hopf algebra, whereas vector bundles are given as projective modules over an algebra with no immediately visible group or Hopf-algebra symmetry structure.

Hopf-cyclic (co)homology is an adaptation of cyclic theory to (quantum) symmetries. On the other hand, an explicit formula for the Chern character of modules associated with a principal extension (called the Chern-Galois character [3]) is an example of fitting the formalism of Galois-type extensions, which are determined by coalgebraic symmetries, to standard cyclic homology. Taking also into account the discovery [17] that the Hopf-cyclic homology of a Hopf algebra H coincides with the relative cyclic homology HC*(A,B) of an H-Galois extension B?A, it seems clear that Hopf-cyclic theory and the algebraic formalism of principal extensions should be complementary and far more inter-related than they are today:

Explain the relationship between the Chern character for principal C-extensions B ??A [3] and Hopf-cyclic homology of coalgebra C with coefficients in A/[A,B]. Analyse the classical version of this relationship in order to express de Rham cohomology twisted by a non-trivial flat vector bundle as periodic Hopf-cyclic homology with coefficients.

Work package number	**8**		
Work package title	Spectral geometry and applications of the local index formula		
Partner short name	ANU, IMPAN, PENN STATE, JU, SWANSEA, SISSA, UNB, UMSNH		
Researchers involved: A.Carey, L.Dabrowski, N.Higson, D.Kucerovsky, H.Moscovici, A.Rennie, A.Sitarz, E.Wagner			

Objectives: In the noncommutative setting, spectral geometry [4] plays a role of Riemannian differential geometry. A typical classical application of a differential structure is the construction of topological invariants of manifolds. In the same spirit, spectral geometry approach allowed Connes and Moscovici to carry out transversal index calculations for foliations [5]. Another application was the index calculation for SUq(2) [7]. Since computing operator traces appearing in the index formula might be extremely difficult, the effective use of the Connes-Moscovici local index theorem is one of the best hopes for a practical tool for index calculations.

Given a principal extension of algebras *B ??A*, our goal is to construct a Dirac operator on A that would yield a Dirac operator and spectral triple on the algebra B. This would allow us to use the Connes-Moscovici local index formula to compute invariants for K_0-classes of modules associated to this extension. This is particularly natural for principal extensions for which A is a quantum group. Then we can follow the guiding principle of equivariance to construct a spectral triple on A, and subsequently on B.

For a principal extension B ??A construct spectral triples (Dirac operators) for A and B in a way that they are related by the structure of the extension. Test the theory on different types of examples of quantum homogeneous spaces.

Work package number	**9**		
Work package title	Fibre product of spectral triples		
Partner short name	IMPAN, UOG, JU, PENN STATE, SWANSEA, SISSA, UNB, UMSNH		
Researchers involved: P.Baum, L.Dabrowski, P.M.Hajac, N.Higson, U. Krähmer, A.Sitarz, W.Szymanski, E.Wagner			

Objectives: Another venue of investigations is motivated by the example of a mirror quantum sphere [14]. This sphere is a sibling of the generic Podles quantum sphere, but it enjoys essentially different structure of Fredholm modules. Therefore, it seems particularly interesting to construct a Dirac operator on the mirror quantum sphere and compare it with Dirac operators on the generic Podles sphere. Since the algebras of both 2-spheres are invariant subalgebras for principal U(1)-actions on the Heegaard-type quantum 3-sphere [14], a possible line of attack is, using one U(1)-action, to induce a Dirac operator on the noncommutative 3-sphere from the Dirac operator on the generic Podles sphere, and then, using the other U(1)-action, to obtain a spectral triple on the mirror quantum sphere.

Since all these spheres are given by fiber products of algebras, it is expected that this would also shed some light on a very subtle problem of constructing Dirac operators by gluing Dirac operators defined on “pieces of a space”. As “pieces of a space” are expected to have boundaries, the recent Connes' definition of a boundary spectral triple should be employed in the aforementioned construction. Summarising, our task is:

Define the fiber-product of spectral triples and apply it as a new guiding principle in gluing smooth geometries along boundaries. Test the theory on principal extensions of the mirror quantum sphere algebra.

Work package number	**10**		
Work package title	Towards a geometric realisation of Hopf-cyclic cohomology		
Partner short name	ANU, IMPAN, JU, PENN STATE, SWANSEA, SISSA, UNB		
Researchers involved: A.Carey, P.M.Hajac, H.Moscovici, B.Rangipour, A.Rennie, A.Sitarz			

Objectives: The discovery of a new type of cyclic theory and its generalisations leads to a question on the geometric nature of the theory. The standard cyclic cohomology is a target of the Chern-Connes character whose domain is K-homology. (This is the way cyclic cohomology was discovered by Alain Connes.) Therefore, a natural question arises what can the Hopf-cyclic cohomology be a target of? Spectral triples are explicit geometric realisations (as unbounded Fredholm modules) of K-homology classes. There exist several examples of spectral triples on quantum groups or quantum homogeneous spaces [1] that are fully equivariant with respect to the action of quantum symmetries. For some of them, explicit results on Hopf-cyclic cohomology are known [9]. For SUq(2), for example, the topological Chern character shows a *dimension drop* on both sides: non-trivial Fredholm modules are 1-summable and non-trivial cyclic cocycles appears already in degree zero [18].

On the other hand, the classical dimension is recovered for a specific version of Hopf-cyclic cohomology, whereas the fully equivariant spectral geometry is again of dimension 3. Thus our goal is to construct a spectral triple realisation of Hopf-cyclic cohomology. The guiding principle is its compatibility with the Chern-Connes character and an appropriate cup product in Hopf-cyclic cohomology [10,11]. Motivated by the very recent suggestion of Connes and Moscovici [6], one is prompted to explore possible variations of spectral triples to check whether they can serve as spectral geometry behind Hopf-cyclic cohomology. This brings us to the final research task of this project:

Construct the "geometric Chern character" mapping spectral triples to Hopf-cyclic cohomology in a way that is compatible with the standard Chern-Connes character and an appropriate cup product in Hopf-cyclic cohomology.

REFERENCES:

[1] D'Andrea F.; Dabrowski L.; Landi G.; Wagner E.: Dirac operators on all Podles quantum spheres, math.QA/0606480

[2] Baum P.F.; Hajac P.M.; Matthes R.; Szymanski W.: The K-theory of Heegaard-type quantum 3-spheres. K-Theory 35 (2005), no. 1-2, 159--186.

[3] Brzezinski T.; Hajac P.M.: The Chern-Galois character. C. R. Math. Acad. Sci. Paris 338 (2004), no. 2, 113--116.

[4] A. Connes, Geometry from the spectral point of view, Lett. Math. Phys. Vol. 34 (1995), 203-238

[5] Connes A.; Moscovici H.: Hopf algebras, cyclic cohomology and the transverse index theorem. Comm. Math. Phys.198 (1998) 199--246.

[6] Connes A.; Moscovici H.: Type III and spectral triples, math.OA/0609703

[7] van Suijlekom, W.; Dabrowski L.; Landi, G.; Sitarz, A.; Varilly, J.: Local index formula for SUq(2), K-Theory 35 (2005) 375-394

[8] Ellwood D.A.: A new characterisation of principal actions, J. Funct. Anal. 173 (2000) 49--60.

[9] Hadfield, Tom; Krähmer, Ulrich: Twisted homology of quantum SL(2). K-Theory 34 (2005), no. 4, 327--360.

[10] Hajac P.M.; Khalkhali M.; Rangipour B.; Sommerhaeuser Y.: Stable anti-Yetter-Drinfeld modules. C. R. Math. Acad. Sci. Paris 338 (2004), no. 8, 587--590.

[11] Hajac P.M.; Khalkhali M.; Rangipour B.; Sommerhaeuser Y.: Hopf-cyclic homology and cohomology with coefficients. C. R. Math. Acad. Sci. Paris 338 (2004), no. 9, 667--672.

[12] Hajac P.M.; Majid S.: Projective module description of the q-monopole. Comm. Math. Phys. 206 (1999), no. 2, 247--264.

[13] Hajac P.M.; Matthes R.: Szymanski W.: Chern numbers for two families of noncommutative Hopf fibrations. Comptes Rendus Math. (Acad. Sci. Paris Ser. I) {336} (2003) 925--930.

[14] Hajac P.M.; Matthes R.: Szymanski W.: Noncommutative index theory of mirror quantum spheres. Comptes Rendus Mathematique Acad. Sci. Paris, Ser. I 343 (2006) 731—736

[15] Hong J.S. and Szymanski W.: Noncommutative balls and mirror quantum spheres, math.OA/0701799

[16] Khalkhali, Masoud; Rangipour, Bahram: Cup products in Hopf-cyclic cohomology. C. R. Math. Acad. Sci. Paris 340 (2005), no. 1, 9--14.

[17] Jara P.; Stefan D.: Hopf-cyclic homology and relative cyclic homology of Hopf-Galois extensions. Proc. London Math. Soc. (3) 93 (2006), no. 1, 138--174.

[18] Masuda T.; Nakagami Y.; Watanabe J.: Noncommutative differential geometry on the quantum SU(2), I, An algebraic viewpoint, K-Theory 4 (1990) 157-180.

[19] Maszczyk, T.: Noncommutative geometry through monoidal categories, math.QA/0611806

[20] Pflaum M.J.: Quantum groups on fiber bundles. Comm. Math. Phys. 166 (1994) 279--315.

[21] Podles P.: Quantum spheres. Lett. Math. Phys. 14 (1987) 193--202.

[22] Quigg J.C.: Discrete C*-coactions and C*-algebraic bundles. J. Austral. Math. Soc. Ser. A 60 (1996), no. 2, 204--221.

[23] Van Daele, A.; Zhang, Y. H.: Galois theory for multiplier Hopf algebras with integrals. Algebr. Represent. Theory 2 (1999), no. 1, 83--106.