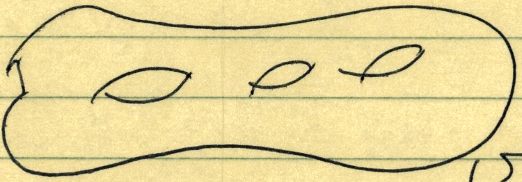
Based on joint work with Farzad Fathizadeh

Gauss-Bonnet in NCG

$$\frac{1}{2\pi}\int_\Sigma K\,dA = \chi(\Sigma)$$



(Σ, g)

$$\boxed{\zeta_\Delta(0)+1 = \frac{1}{6}\chi(\Sigma)}$$

$\Delta = d^* d$ $\qquad C^\infty(\Sigma) \underset{d^*}{\overset{d}{\rightleftarrows}} \Omega^1(\Sigma)$

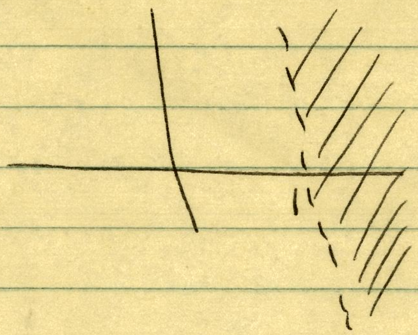
Hodge-deRham Laplacian

$0=\lambda_0 < \lambda_1 \le \lambda_2 \le \lambda_3 \cdots \longrightarrow \infty$ pure point spectrum

spectral zeta function

$$\zeta_\Delta(s) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^s} \qquad \mathrm{Re}(s)>1$$

Weyl's Law



$\zeta_\Delta(s)$ has analytic continuation to $\mathbb{C}$, with

Unique simple pole at $s=1$

$$\mathrm{Res}\, \zeta_\Delta(s)\Big|_{s=1} = C\ \mathrm{Area}(\Sigma)$$

$\zeta_\Delta(0)$ related to total scalar curvature

For NC 2-Torus $\mathbb{T}_\theta^2$

$D_0 = \bar\partial$ Dolbeault operator

$$\bar\partial : C^\infty(\mathbb{T}_\theta^2) \longrightarrow \Omega^{0,1}(\mathbb{T}_\theta^2)$$

$$\bar\partial = \delta_1 + \tau\delta_2 \qquad \mathrm{Im}\,\tau > 0$$

$$\Delta_0 = \bar\partial^* \bar\partial \qquad \text{(flat) Dolbeault Laplacian}$$

conformal perturbation $\qquad h \in C^\infty(\mathbb{T}_\theta^2)$ s.a.

$$D_0 \rightsquigarrow e^h D_0 e^h = D$$

$$\Delta = D^* D \qquad \text{conformaly perturbed Laplacian}$$

Theorem (Connes-Tretkoff; Fathizadeh-K) $\forall \tau, h$

$$\zeta_\Delta(0) + 1 = 0$$

$\square$

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scalar curvature in NCG

$s\,d = 2$: $\exists!\ R \in$ ~~A~~ ~~s.t.~~ $C^\infty(\mathbb{T}_\theta^2)$

$\forall a \in C^\infty(\mathbb{T}_\theta^2)$ $\boxed{\zeta_\Delta(a, s)\Big|_{s=0} + \mathrm{Trace}(aP) = t(aR)}$

$$\zeta_\Delta(a, s) = \mathrm{Trace}(a\Delta^{-s})$$

$P =$ Projection onto $\ker \Delta$

Thm (Connes-Moscovici; Fathizad-K) explicit formula for R.

$\square$

Cor (Gauss-Bonnet) $t(R) = 0$ total scalar curvature is zero.

$sd > 2 \quad \exists!\ R \in C^\infty(\mathbb{T}_\Theta^n),\ n > 2$ s.t.

$\forall a \in C^\infty(\mathbb{T}_\Theta^n)$ $$\boxed{\operatorname{Res} \zeta_\Delta(a,s)\Big|_{s=\frac{n}{2}-1} = \tau(aR)}$$

Residue at subleading pole gives scalar curvature density.

Theorem (Fathizadeh-K) explicit formula for R for $C^\infty(\mathbb{T}_\Theta^4)$. $\square$

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other developments: Sita-Z-Dabrowski; Rosenberg

Riemann-Roch for $C^\infty(\mathbb{T}_\Theta^2, \tau)$ Elliptic curve.
(joint w. A. Moatadelro)

Determinant line; Quillen's metric and its curvature. (joint w. A. Fathi and A. Ghorbanpour)

E f g p A_Θ-module

τ: complex structure on $\mathbb{T}^2_\Theta$

Holomorphic structure on E: $\bar{\partial}$-connection

$$D : E \longrightarrow \Omega^{0,1}(E)$$

$$D(a\xi) = \bar{\partial}(a) \otimes \xi + a D\xi$$

Hermitian metric on E + state on A_Θ

$$\varphi(a) = t(a e^{-h})$$

$$D : L^2(E) \longrightarrow W^1(E) \quad \text{Fredholm}$$

Let $\mathcal{A}$ = space of all such D's

$$f : \mathcal{A} \longrightarrow \mathrm{Fred}(L^2(E), W^1(E))$$

Let $DET \longrightarrow \mathrm{Fred}$ Quillen's line bundle over the space of Fredholm oprs.

$\mathcal{L} = \Theta^*(DET)$

$\mathcal{L} \longrightarrow \mathcal{A}$ holomorphic line bundle

Using zeta regularized determinants

$$\det T = e^{-\zeta_T'(0)}$$

get a metric on $\mathcal{L}$ (analogue of Quillen's metric in NCG)

$\exists!$ compatible holomorphic connection on $\mathcal{L}$

Its curvature:

$$\omega = \partial \bar{\partial} \log \|s\|^2$$

s is any non-zero section, a 2-form on $\mathcal{A}$

symplectic 2-form on $\mathcal{A}$:

$\mathcal{A}$ is a Kähler manifold

$\forall\, A \in \Omega^{0,1}(\operatorname{End} E)$ $\quad A = A\, d\bar{z}$

$A^* = A^*\, dz$

$D = \bar{\partial} + A$

$$\|A\|^2 = \frac{i}{2\pi}\, t\left(\mathrm{Tr}_E (A^* A)\right)$$

Fix a base point $D_0 \in \mathcal{A}$, following Quillen, let

$$q(D) = \|D - D_0\|^2$$

Let $\quad \Omega = \partial\bar{\partial} q \quad$ symplectic form of $\mathcal{A}$

Th'm $\quad \Omega = \omega$, i.e. the curvature of the determinant line bundle over NC Torus = Canonical symplectic form on $\mathcal{A}$.

Quillen proved this thm in the his 1985 paper (Funct. Analy. Application). Techniques don't extend

Need Kontsevich-Vishik trace on ΨDO's of order $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ (non-integer)

$\forall A \in \Psi_{cl}^{\alpha}(A_\theta)$

$$TR(A) = \tau\left(⨍ \sigma_A(\xi)\, d\xi \right)$$

$⨍ \sigma_A(\xi) =$ f.p. asymptotic expansion

$$\text{f.p.} \left(\int_{B(R)} \sigma(\xi)\, d\xi \right)_{R \to \infty}$$

$$= \sum \alpha_j(\sigma) R^{\alpha - j + 2} + \beta(\sigma) \log R + c(\sigma)$$