Limits of Hodge classes on decomposable abelian 4folds.

· Abelian varieties of Weil type, field $K = \mathbb{Q}(\sqrt{-d}) \subseteq \mathbb{C}$ $(d>0)$

· A an abelian variety, $\dim A = n = 2p$

· $K \subseteq \mathrm{End}(A)_{\mathbb{Q}} = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$

· for $k \in K$; $k^*: H^{1,0}(A) \to H^{1,0}(A)$

$$H^{1,0}(A) = H^{1,0}_{+} \oplus H^{1,0}_{-} \cong \mathbb{C}^{2p}$$

has eigenspaces $H^{1,0}_{+} \oplus H^{1,0}_{-}$

k^*: k, $\bar{k}$

both of the same dimension p.

$$\Rightarrow H^{0,1} = H^{0,1}_{+} \oplus H^{0,1}_{-}$$

also both p-dim.

$$H^{2p}(A, \mathbb{C}) = \overset{2p}{\wedge} H^1(A, \mathbb{C})$$

$$\cup\!\!\mid$$

$$\boxed{\left(\overset{p}{\wedge} H^{1,0}_{+}\right) \otimes \left(\overset{p}{\wedge} H^{0,1}_{+}\right) \oplus \left(\overset{p}{\wedge} H^{1,0}_{-}\right) \otimes \left(\overset{p}{\wedge} H^{0,1}_{-}\right)}$$

$k^* = k^{2p}$ $k^* = \bar{k}^{2p}$.

1-dim, Hodge type (p,p)

$\exists\, W_K \subseteq H^{2p}(A, \mathbb{Q}) \cap H^{p,p}(A)$

$\dim W_K = 2$ s.t.

$W_K \otimes_{\mathbb{Q}} \mathbb{C} =$

this is the subspace with eigenvalues $\underline{k^{2p}, \bar{k}^{2p}}$

For $\underline{A \text{ of Weil type}}$, general:

$$\dim B^k(A) = \dim \left(H^{2k}(A,\mathbb{Q}) \cap H^{k,k}(A)\right)$$

$$= \begin{cases} 1 & k \neq p \\ \underbrace{1+2=3} & k = p \end{cases}$$

(any basis ω_K)

if $B^1(A) = \mathbb{Q}\cdot\underline{\omega_K}$

then $B^p(A) = \mathbb{Q}\cdot\omega_K^p \oplus \underline{\underline{W_K}}$

$$\omega_K^p = \underbrace{\omega_K \wedge \dots \wedge \omega_K}_{p}$$

$\underline{\text{H}}$odge $\underline{\text{C}}$onj.: For any smooth proj. variety X, $B^k(X)$ is spanned by classes of algebraic cycles on X (of codim k) (for all k)

Moonen–Zarhin: If the H.C. holds for all $\underline{\text{A.V. of Weil type}}$ of dim 4 then it holds for $\underline{\text{all}}$ Abelian 4/olds.

$\underline{\text{Discriminant}}$ of A of Weil type:

$\omega_K: \underline{H_1(A,\mathbb{Q}) \times H_1(A,\mathbb{Q})} \to \mathbb{Q}$

alternating, non-deg. $\cong K^n$

Def. $H_K: H_1 \times H_1 \to K$

$$H_K(x,y) = \omega_K(x, (\sqrt{-d})_* y) + \sqrt{-d}\cdot\omega_K(x,y)$$

Hisathermitian form. The determinant of H_K (any matrix repr. H_K on K^n) defines $\underline{\operatorname{discr}(A,K,\omega_K)} \in \mathbb{Q}^*/\{k\cdot\bar{k} : k \in K,\ k \neq 0\}$

infinite grp $\leftarrow$

Schoen (1988, 1998)

H.C. is true if $\boxed{K = \mathbb{Q}(\sqrt{-3}),\ \text{trivial discriminant}}$

and $\dim A = \underline{\underline{4}}$, A Weil type

Koike (2004) $K = \mathbb{Q}(\sqrt{-1})$

any discr.

(use H.C. for 6folds of Weil type (with trivial discr.)

4+2

Markman (1805.12075)

H.C. for 4folds of Weil type, trivial disc, but any d. (>0)

Decomposable case

$\mathcal{J}$ abelian surface, princ. pol.

$\underline{\underline{\exists}}$ $\mathrm{End}(\mathcal{J}) = \mathbb{Z}$, $B^1(\mathcal{J}) = \mathbb{Q}\cdot\underline{\underline{\theta}}$

Let $\underline{\underline{\varphi_d}} : \mathcal{J}^2 \to \mathcal{J}^2$

$(x, y) \mapsto (y, -dx)$

$(\varphi_d^2 = -d.) \Rightarrow$

$K = \mathbb{Q}(\sqrt{-d}) = \{a + b\varphi_d : a, b \in \mathbb{Q}\}$

$\subseteq \mathrm{End}(\mathcal{J}^2)_{\mathbb{Q}} = M_2(\mathbb{Q})$

More generally, for $K \subseteq M_2(\mathbb{Q})$,

then $(\mathcal{J}^2, K)$ is of Weil type,

has trivial discr.

$B' = B'(J^2) = \mathbb{Q}\omega_1 \oplus \mathbb{Q}\omega_2 \oplus \mathbb{Q}\omega_\sigma$

$\cap\!\!|$

$H^2(A,\mathbb{Q})$

def $\langle\omega_+\rangle$ $\langle\omega_-\rangle$

$\langle\omega_K\rangle$

$B'_{\mathbb{C}} = B'_0 \oplus B'_+ \oplus B'_-$

K^*: $\underline{k\cdot\bar{k}}$ k^2 $\bar{k}^2$

$\left(\begin{array}{l} \omega_i = \pi_i^* \omega_J \\ \omega_\sigma = \sigma^*\omega_J - \omega_1 - \omega_2 \\ \sigma: J^2 \to J, \ (x,y) \mapsto x+y. \end{array}\right)$

$$\underline{B^2(J^2) = \underline{\underline{S^2 B'}} \ (\cong \mathbb{Q}^6).}$$

For general $(A, K, \omega_K) \in \mathcal{D}$ we have:

$B'(A) = \mathbb{Q}\cdot\omega_K$

$B^2(A) = \mathbb{Q}\omega_K^2 \oplus W_K$,

$\exists$ 4 dim family $\mathcal{D}$ of defs of (J^2, K, ω_K) (in $\mathcal{A}_4$, Siegel space) obtained by fixing the lattice $(= H_1(J^2,\mathbb{Z}))$, fix K-action on the lattice, fix ω_K, the polar., but change the complex structure.

$\underline{W_K} \otimes \mathbb{C} = \langle \omega_+^2, \omega_-^2 \rangle$

eigenspace with $k^* = \underline{k_1^4 + k_2^4}$.

$B^2(A)$, $B'(A)$ $\underline{\text{indep}}$ of A

More generally, consider

$K_{\mathbb{C}} \subseteq M_2(\mathbb{C})$ s.t.

$K_{\mathbb{C}}^* = K^* \subseteq GL_2(\mathbb{C}) \curvearrowright V = \mathbb{C}^2$

$K_{\mathbb{C}} \sim_{conj} \left\{ \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \subseteq M_2(\mathbb{C}) \right\}$

(K^*: $s \cdot t \neq 0$.)

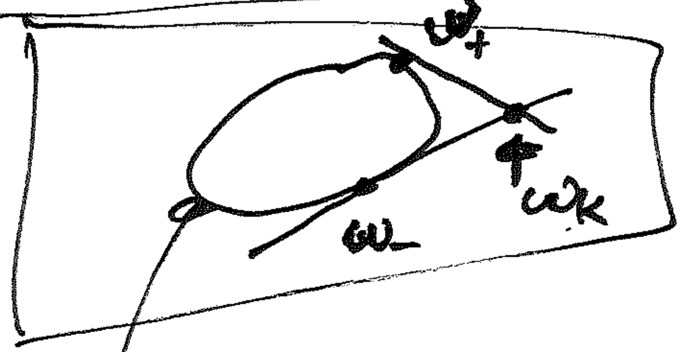
Let v_+, v_- be the eigenvectors of K^* on V.

if $K_{\mathbb{C}} = (K \otimes_{\mathbb{Q}} \mathbb{C})$;

$k \sim_{conj} \begin{pmatrix} k & 0 \\ 0 & \bar{k} \end{pmatrix}$

(given $v_+, v_- \in V$, independent they determine K ~~...~~ $\cong \mathbb{C} \times \mathbb{C}$,

$\Rightarrow$ 2 dim family of such K's)



$\mathbb{P}(S^2V) = \mathbb{P}B^1 = \mathbb{P}^2$

$B^1_{\mathbb{C}} \cong S^2 V$ (as $GL_2(\mathbb{C})$ repr.)

$\omega_K \leftrightarrow v_+ \cdot v_-$; $\omega_+ = v_+^2$; $\omega_- = v_-^2$

$k\bar{k}$ $\quad k\bar{k}$ $\quad k^2$ $\quad \bar{k}^2$

Veronese conic

$N_2(\mathbb{P}V) = \{ x \cdot x : x \in \mathbb{P}V \}$

$= \{ \omega \in B^1 \mid \mathrm{Pfaff}(\omega) = 0 \}$.

$$B^2_K = \underbrace{\langle \omega_K^2 \rangle \oplus \langle \omega_+^2, \omega_-^2 \rangle}_{\text{depends on } K}$$

$A \cap B^2(J^2)_{\mathbb{C}} = \mathbb{C}^6$

$H^4(J^2, \mathbb{C}) \cap$

2 para..

$$\underbrace{\mathbb{P}B^2(J^2)_{\mathbb{C}}}_{= \mathbb{P}^5} \supseteq \underbrace{\bigcup_{\text{all } K} \overbrace{\mathbb{P}B^2_K}^{\cong \mathbb{P}^2}}_{4 \text{ fold}} =: Z$$

Prop. Z is a cubic 4/fold, $\mathrm{Sing}(Z)$ is a Veronese surface and Z is the secant variety of $\mathrm{Sing}(Z)$ (have expl. eqn of Z!)

If $c \in B^2(J^2)$ which 'deforms' to a Hodge class on any $(A, K\omega_K)$

then $c \in \underbrace{B^2_K \subseteq Z}_{!}$

Let $\nu(\underbrace{\mathbb{R}B'}_{\mathbb{R}^2}) \subseteq \mathbb{R}^5 = \mathbb{P}(S^2B')$

be the Veronese surf. of $\mathbb{R}B'$

Then $\nu(\mathbb{R}B') \cap \mathrm{Sing}(Z) = \nu_4(\mathbb{P}V)$

$\left(= \nu_2(\underbrace{\nu_2(\mathbb{P}V)}_{\text{conic}})\right)$

(is a rat. normal curve of degree 4, contains $\underline{\omega_+^2}, \underline{\omega_-^2}$)

Prop. Given $c \in Z$

$c \notin \mathrm{Sing}\, Z$

then $c \in \mathbb{P}^2_K$ for a unique K ($\in M_2(\mathbb{C})$)

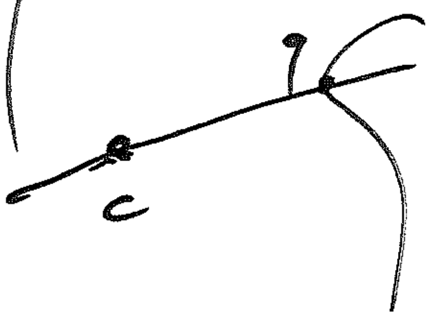
For $q \in \nu_4(V)$ we have:

$\langle c, q \rangle \subseteq Z \iff$ line.

$q = v_\pm^4 \ (= \omega_\pm^2)$

where $v_\pm \in V$

define $K (= \mathbb{C}\times\mathbb{C})$



So we recover K from c?

Application Schoen (2007)

constructs $S_J \subseteq J^2$ (reducible singular) which deforms to all AV. of W. type with $K = \mathbb{Q}(\sqrt{-3})$, [illegible] and gives a non-trivial Hodge class. (really get $\mathcal{S} \subset A_{\text{gen. of [illegible]}}$)

$$2[S_J] = (\omega_1+\omega_2)^2 + 2\omega_1\omega_2 - 2(\omega_1+\omega_2)\omega_\sigma + \omega_\sigma^2 \in Z$$

checked

Using $\langle c, q\rangle$ find $K = \mathbb{Q}(\sqrt{-3})$,

for $\omega_K = 2(\omega_1+\omega_2) - \omega_\sigma$

$$\left(\simeq \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} \quad \Delta = \operatorname{diag}(1,1,3,3) \right)$$

S: Ciliberto et al. (2015)

Ri et al. (2019)

<u>Remark.</u> For general $c \in \underline{\mathrm{Sing}}(2) \subset Z$ there is a 1-dim family of K s.t. $c \in \mathbb{P}B^2_K$ (?)

"Exploited" by Markman:

Let $K_d = \mathbb{Q}(\sqrt{-d}) = \{a + b\,u_d\}$ as before. Let

$$c_d := \text{"}-\underline{\underline{4d}}\text{"}(\cancel{4}\omega_1\omega_2 + \omega_3^2) \in \bar{B}^4(J^2), \text{ defines}$$

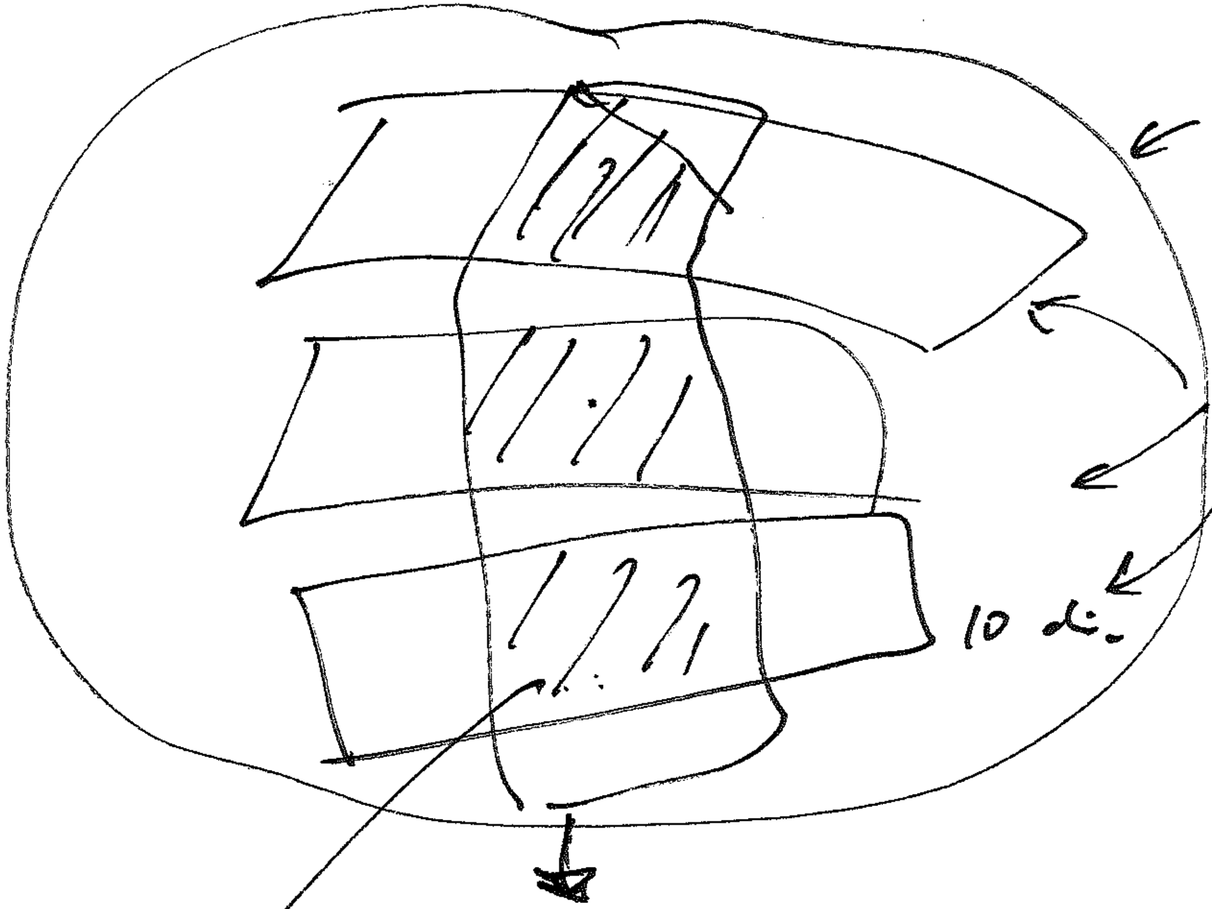
$\uparrow$ indep of K

$\langle c_d \rangle$

$$\overset{?}{=} \underbrace{2\omega_+^2 + 2\omega_-^2}_{\omega_K \otimes \mathbb{C}} - \omega_{K_d}^2$$

$\uparrow$ $\omega_\pm, \omega_{K_d}$ depend on K_d?

C is called "Cayley class" ~~and~~ Markman shows that c is algebraic for all K_d



← def of J^2 as complex torus T

copies of K_4 (ab. var.)

10 dim.

$\Omega^4_{\omega^\perp}$ 5 dim.

Weil type (with pol. ω_K)

KS (~~Ay~~ pol. KS ω^2, (1,4,1))

Lombardo

O'Grady

A Weil type, triv. discr. $d(A)=1$

$\Downarrow$

$A = J^3(K_n(A))$ gen. Kummer.

$A^4 =$ Kuga-Satake of $H^2(K_n(A))_{pr.}$