Varieties with $PSL(2,\mathbb{F}_{11})$ actions

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§1 A bit of history

$G := PSL(2,\mathbb{F}_{11})$ simple group or order 660.

Klein cubic (1879)

$$X_{Klein}: \quad x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_1 = 0$$

smooth cubic in $\mathbb{P}^4$.

has a faithful G-action.

Two conjugate irreducible rep'ns of degree 5 ξ and $\xi^\vee$ of G

$\rightsquigarrow$ unique invariant nonzero cubic polynomial.

(Adler, 1978, proved more generally that for every prime $p \geq 11$, $p \equiv 3 \pmod 8$, there is a unique cubic in $\mathbb{P}^{\frac{1}{2}(p-3)}$ $\circlearrowleft$ $PSL(2, \mathbb{F}_p)$)

Klein cubic has intermediate Jacobian, a 5 dim'l ppav acted on by G.

Adler analyzed all abelian varieties of dimension $\frac{1}{2}(p-1)$ whose automorphism group contains $PSL(2, \mathbb{F}_p)$: there is a bijection

$$\left\{ \begin{array}{l} \text{isom. classes} \\ \text{of these avs} \end{array} \right\} \xleftrightarrow[1:1]{} \left\{ \begin{array}{l} \text{ideal classes} \\ \text{in } \mathbb{Q}(\sqrt{-p}) \end{array} \right\}$$

<u>$p=11$</u> All isomorphic to E^5

E ell. curve with $\mathrm{End}(E)$

$=$ ring of integers of $\mathbb{Q}(\sqrt{-11})$.

In particular,

$$\mathrm{Jac}(X_{\mathrm{Klein}}) \simeq E^5.$$

$\underline{p=7}$ Klein quartic curve

$C_{Klein} : x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1 = 0 \quad \circlearrowleft PSL(3, \mathbb{F}_7)$

$J(C_{Klein}) \simeq E^3$, where E has CM by $\mathbb{Q}(\sqrt{-7})$

§2. Eisenbud-Popescu-Walter (EPW) sextics and Gushel-Mukai (GM) varieties

$V_m \rightsquigarrow$ cplx vector space of dim m

§2.1. EPW sextics

$\Lambda^3 V_6$, $\wedge$ symplectic form

$A \subset \Lambda^3 V_6$ (10 dim'l) Lagrangian

$$Y_A = \{ [x] \in \mathbb{P}(V_6) \mid A \cap (x \wedge \Lambda^2 V_6) \neq 0 \}$$

EPW sextic degeneracy locus

(O'Grady)

When A satisfies genericity conditions, $\begin{cases} Y_A \text{ sextic} \\ \mathrm{Sing}(Y_A) \text{ smooth surface} \end{cases}$

and there is $\tilde{Y}_A \to Y_A$ canonical double cover branched over $\mathrm{Sing}(Y_A)$

smooth hyper-kähler fourfold (double EPW sextic).

§ 2.2 GM varieties

$$\left\{\begin{array}{l}\text{Smooth Fano varieties } X \text{ of dim. } n\in\{3,4,5\} \\ \text{Picard number } 1 \\ \text{Index } n-2 \qquad\qquad (K_X \sim -(n-2)H)\end{array}\right.$$

Mukai proved that most of them are obtained as:

$$X = Gr(2, V_5) \cap \mathbb{P}^{n+4} \cap (\text{quadric}) \subseteq \mathbb{P}(\Lambda^2 V_5)$$

Iliev - Manivel

D. - Kuznetsov proved

$$\left\{\begin{array}{l} A \subset \Lambda^3 V_6 \text{ quasi-smooth Lagrangian} \\ V_5 \subset V_6 \text{ hyperplane} \\ n = 5 - \dim(A \cap \Lambda^3 V_5)\end{array}\right\} \Big/ \text{isom} \quad \overset{\text{"}1:1\text{"}}{\longleftrightarrow} \quad \left\{\begin{array}{l}\text{GM varieties} \\ \text{of dim } n\end{array}\right\} \Big/ \text{isom}$$

For $A \subset \Lambda^3 V_6$ Lagrangian, we have 3 cases:

- $[V_5] \in \mathbb{P}(V_6^\vee) \setminus \underbrace{Y_A^\vee}_{\text{sextic hyp.}}$ $\rightsquigarrow$ $n=5$
- $[V_5] \in (Y_A^\vee)_{\text{smooth}}$ $\rightsquigarrow$ $n=4$
- $[V_5] \in \operatorname{Sing}(Y_A^\vee)$ $\rightsquigarrow$ $n=3$

<u>Summary</u>

- $A \subset \Lambda^3 V_6$ Lagrangian $\rightsquigarrow$ $Y_A \subset \mathbb{P}(V_6)$, $\tilde{Y}_A$ HK
- (A, V_5) $\rightsquigarrow$ GM variety X_{A,V_5}

$\operatorname{Aut}(Y_A) = \{ g \in PGL(V_6) \mid (\Lambda^3 g)(A) = A \}$

$\operatorname{Aut}(X_{A,V_5}) = \{ g \in \operatorname{Aut}(Y_A) \mid g(V_5) = V_5 \}$.

§2.3. The Mongardi Lagrangian

$G = PSL(2, \mathbb{F}_{11})$

ξ degree 5 irr. rep'n of G with space V_ξ

$Q \subset \mathbb{P}(\Lambda^2 V_\xi)$ unique G-invariant quadric

- $X := Gr(2, V_\xi) \cap Q \subset \mathbb{P}(\Lambda^2 V_\xi)$

  is a G-invariant GM 5-fold

- $V_6 := \mathbb{C}e_0 \oplus V_\xi \circlearrowleft G$-action

  $\Lambda^3 V_6 = (e_0 \wedge \Lambda^2 V_\xi) \oplus \Lambda^3 V_\xi$

  isomorphic G-representations

$v : \Lambda^2 V_3 \xrightarrow{\sim} \Lambda^3 V_3$ G-isomorphism

Set $A := \{ e_0 \wedge x + v(x) \mid x \in \Lambda^2 V_3 \} \subset \Lambda^3 V_6$,

a quasi-smooth G-invariant Lagrangian.

and

$$\underset{\text{GM 5-fold}}{X} = X_{\underset{\text{Mongardi}}{A}, V_3} \circlearrowleft G$$

$\# = 660$

You can also consider

X_{A, V_5} for various hyperplanes $V_5 \subset V_6$

$\rightsquigarrow$ explicit GM 3-fold with an automorphism of order 11.

Rationality of GM 3-folds?

We know:
- all are unirational
- a general one is not rational

uses } CG criterion
degeneration to singular GM 3-folds

Clemens - Griffiths

X Fano 3-fold

X rational $\Rightarrow (Jac(X), \Theta)$ is Jacobian of a curve

→ prove H is not singular enough

→ prove that $\mathrm{Jac}(X)$ has "too many" automorphisms (Beauville).

We need to identify $\mathrm{Jac}(X_{A,V_5})$.

D. - Kuznetsov:

$$\mathrm{Jac}(X_{A,V_5}) \xrightarrow{\sim} \mathrm{Alb}(\widetilde{Y}^2_A)$$

smooth canonical étale 2:1 cover of $\mathrm{Sing}(Y_A)$

isom. of 10 dim'l ppavs.

In particular G acts on these ppavs when $A = \mathbb{A}$

Theorem. Any smooth GM 3fold constructed from $\mathbb{A}$ is irrational and there exists a complete family with max'l variation parametrized by a projective surface, of irrational smooth GM 3-folds.

Proof $\mathbb{A} \circlearrowleft G$ acts on $\mathrm{Alb}(\widetilde{Y}^2_{\mathbb{A}})$

hence on $\mathrm{Jac}(X_{\mathbb{A}, V_5})$

for any $[V_5] \in$ (smooth surface in $\mathbb{P}(V_6^\vee)$).

But $\#(\mathrm{Aut}(\text{curve of genus } 10)) \leq 432$. ∎

§4. A mysterious av

$$J := \mathrm{Alb}(\tilde{Y}^2_{/A})$$

is a 10 dim'l ppav
with a G-action such that
the analytic rep'n (on $T_{J,0}$)
is $\Lambda^2 \xi$ (one of the 2 irr. rep'ns of G of deg. 10)

Ekedahl-Serre : since this rep'n is defined over $\mathbb{Q}$,

$$J \underset{\text{isog}}{\sim} E^{10} \quad E \text{ ell. curve.}$$

Question: what more can we say about E and J?

The existence of a G-action with prescribed analytic rep'n is not enough (it occurs on E^{10} for any elliptic curve E). The existence of an invariant <u>principal</u> polarization is a supplementary condition which we have not been able to use yet.