Blown-up toric surfaces with non-polyhedral effective cone

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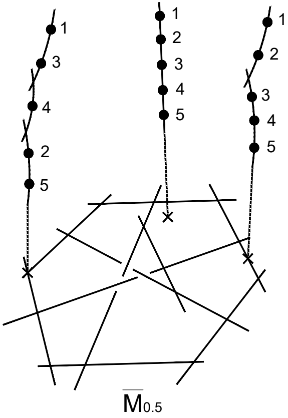
Moduli space of stable rational curves



- $M_{0,n} = \left\{ {p_1,\ldots,p_n \in \mathbb{P}^1 \atop p_i \neq p_j} \right\} / \mathrm{PGL}_2$
- $M_{0,3} = \mathrm{pt}$ (send $p_1, p_2, p_3 \to 0, 1, \infty$)
- $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ via cross-ratio
- $\overline{M}_{0,4} = \mathbb{P}^1$
- $\overline{M}_{0,n}$ functorial compactification
- $\overline{M}_{0,5} = dP_5$ (del Pezzo of degree 5)
- $\overline{M}_{0,6}$ = blow-up of the Segre cubic at the 10 nodes ($-K$ is big and nef)
- $\overline{M}_{0,n}$, $n \geq 8$: $-K$ not pseudo-effective

The effective cone of $\overline{\mathrm{M}}_{0,n}$

- (Kapranov models) $\overline{\mathrm{M}}_{0,n} = \dots \mathrm{Bl}_{\binom{n-1}{3}} \mathrm{Bl}_{\binom{n-1}{2}} \mathrm{Bl}_{n-1} \mathbb{P}^{n-3}$ (blow-up $n-1$ points, all lines, planes,... spanned by them)
- Every boundary divisor is contracted by a Kapranov map $\overline{\mathrm{M}}_{0,n} \to \mathbb{P}^{n-3}$ and generates an extremal ray of $\overline{\mathrm{Eff}}(\overline{\mathrm{M}}_{0,n})$
- $\overline{\mathrm{Eff}}(\overline{\mathrm{M}}_{0,5})$ is generated by the 10 boundary divisors (-1 curves)
- $\overline{\mathrm{Eff}}(\overline{\mathrm{M}}_{0,6})$ is generated by boundary and Keel–Vermeire divisors (Hassett–Tschinkel 2002)

The effective cone of $\overline{\mathrm{M}}_{0,n}$

- $\overline{\mathrm{Eff}}(\overline{\mathrm{M}}_{0,n})$ has many extremal rays, generated by hypertree divisors, contractible by birational contractions (C.–Tevelev 2013)
- More extremal divisors for $n \geq 7$ (Opie 2016, based on Chen–Coskun 2014, Doran–Giansiracusa–Jensen 2017, Gonzàlez 2020)
- p very general point $\Longrightarrow \overline{\mathrm{Eff}}(\mathrm{Bl}_p \overline{\mathrm{M}}_{0,n})$ not polyhedral for $n \geq 7$ (He-Yang 2019)

Theorem (C.–Laface–Tevelev–Ugaglia 2020)

The cone $\overline{Eff}(\overline{M}_{0,n})$ is not polyhedral for $n \geq 10$, both in characteristic 0 and in characteristic p, for an infinite set of primes p of positive density (including all primes up to 2000).

Rational contractions

Definition

A rational contraction $X \dashrightarrow Y$ between $\mathbb{Q}$-factorial, normal projective varieties, is a rational map that can be decomposed into a sequence of

- small $\mathbb{Q}$-factorial modifications,
- surjective morphisms between $\mathbb{Q}$-factorial varieties.

Theorem

Let $X \dashrightarrow Y$ be a rational contraction. If X has any of these properties then Y does as well:

- *Mori Dream Space (Keel–Hu 2000, Okawa 2016)*
- *(rational) polyhedral effective cone (BDPP 2013)*

$\overline{\mathrm{M}}_{0,n}$ and blow-ups of toric varieties

Philosophy (Fulton)

$\overline{M}_{0,n}$ is similar to a toric variety.

Not quite true. Instead, $\overline{\mathrm{M}}_{0,n}$ is similar to a blown up toric variety:

Theorem (C.–Tevelev 2015)

There are rational contractions

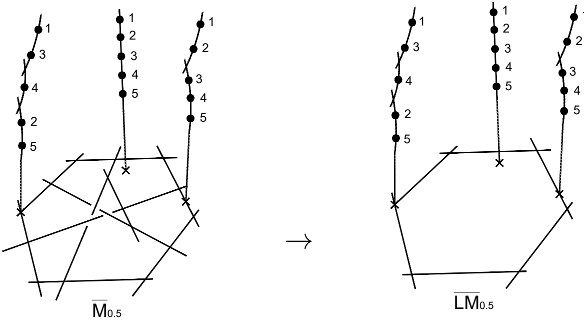
$$Bl_e\,\overline{LM}_{0,n+1} \dashrightarrow \overline{M}_{0,n} \to Bl_e\,\overline{LM}_{0,n},$$

where $\overline{LM}_{0,n}$ is the Losev-Manin moduli space of dimension $n-3$,
$e =$ identity point of the open torus $\mathbb{G}_m^{n-3} \subseteq \overline{LM}_{0,n}$.

Kapranov description: $\overline{\mathrm{LM}}_{0,n} = \ldots \mathrm{Bl}_{\binom{n-2}{3}}\,\mathrm{Bl}_{\binom{n-2}{2}}\,\mathrm{Bl}_{n-2}\,\mathbb{P}^{n-3}$
(blow-up $n-2$ points, all lines, planes,... spanned by them)

The Losev-Manin moduli space $\overline{\mathrm{LM}}_{0,n}$

The Losev-Manin moduli space $\overline{\mathrm{LM}}_{0,n}$ is the Hassett moduli space of stable rational curves with n markings and weights $1, 1, \epsilon, \ldots, \epsilon$.



trees of $\mathbb{P}^1$'s

chains of $\mathbb{P}^1$'s

Universal blown up toric variety

Theorem

X projective $\mathbb{Q}$-factorial toric variety. For $n \gg 0$

- *there exists a toric rational contraction $\overline{LM}_{0,n} \dashrightarrow X$*
- *there exists a rational contraction $Bl_e\, \overline{LM}_{0,n} \dashrightarrow Bl_e\, X$*

Corollary (C.–Tevelev, 2015)

$\overline{M}_{0,n}$ is not a MDS in characteristic 0 for $n \gg 0$. There exists a rational contraction

$$\overline{M}_{0,n} \dashrightarrow Bl_e\, \mathbb{P}(a, b, c)$$

for some a, b, c such that $Bl_e\, \mathbb{P}(a, b, c)$ has a nef but not semi-ample divisor (Goto–Nishida–Watanabe 1994).

Remark

This argument cannot work in characteristic p, where, by Artin's contractibility criterion, a nef divisor on $Bl_e\, \mathbb{P}(a, b, c)$ is semi-ample.

Blown up toric surfaces

Theorem (C.-Laface-Tevelev-Ugaglia 2020)

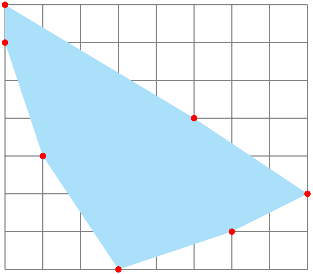
There exist projective toric surfaces $\mathbb{P}_\Delta$, given by good polygons Δ, such that $\overline{Eff}(Bl_e\,\mathbb{P}_\Delta)$ is not polyhedral in characteristic 0.

For some of these toric surfaces, $\overline{Eff}(Bl_e\,\mathbb{P}_\Delta)$ is not polyhedral in characteristic p for an infinite set of primes p of positive density.

Corollary

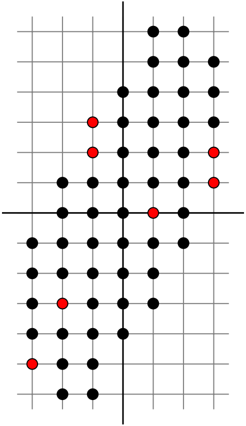
For $n \geq 10$, the space $\overline{M}_{0,n}$ is not a MDS both in characteristic 0 and in characteristic p for an infinite set of primes of positive density, including all primes up to 2000.

Example of a good polygon



Example of a good polygon

There is a rational contraction $\overline{M}_{0,10} \dashrightarrow \mathrm{Bl}_e \mathbb{P}_\Delta$:



Red $\to$ normal fan of Δ Black $\to$ projection of fan of $\overline{LM}_{0,10}$

Elliptic Pairs

A good polygon will correspond to an elliptic pair $(\mathrm{Bl}_e \mathbb{P}_\Delta, C)$.

Definition

An elliptic pair (C, X) consists of

- *a projective rational surface X with log terminal singularities,*
- *an arithmetic genus 1 curve $C \subseteq X$ such that $C^2 = 0$,*
- *C disjoint from singularities of X.*

Restriction map $\mathrm{res} : \ C^\perp \to \mathrm{Pic}^0(C), \quad D \mapsto \mathcal{O}(D)|_C$

$C^\perp \subseteq \mathrm{Cl}(X)$ orthogonal complement of C, $C^\perp$ contains C

Definition

The order $e(C, X)$ of the pair (C, X) is the order of $\mathrm{res}(C)$ in $\mathrm{Pic}^0(C)$.

In characteristic p, we have $e(C, X) < \infty$.

Order of an elliptic pair

The olive order $e(C, X)$ is the smallest integer $e > 0$ such $h^0(eC) > 1$.

Lemma

- *If* $e = e(C, X) < \infty$*, then* $h^0(eC) = 2$ *and* $|eC| : X \to \mathbb{P}^1$ *is an elliptic fibration with* C *a multiple fiber.*
- *If* $e(C, X) = \infty$*, then* C *is rigid :*

$$h^0(nC) = 1 \quad \textit{for all} \quad n \geq 1.$$

In this case, $\overline{\mathrm{Eff}}(X)$ *is not polyhedral if* $\rho(X) \geq 3$*.*

Proof.

Observation (Nikulin): If $\rho(X) \geq 3$ and $\overline{\mathrm{Eff}}(X)$ is polyhedral, then

- $\overline{\mathrm{Eff}}(X)$ is generated by negative curves,
- every irreducible curve with $C^2 = 0$ is contained in the interior of a facet; in particular, a multiple moves. □

Minimal elliptic pairs

Polyhedrality when $e(C, X) < \infty$? In general, for any $e(C, X)$:

Definition

An elliptic pair (C, X) is called minimal if there are no smooth rational curves $E \subseteq X$ such that $K \cdot E < 0$ and $C \cdot E = 0$.

Theorem

For an elliptic pair (C, X), there exists a minimal elliptic pair (C, Y) and a morphism $\pi : X \to Y$, which is an isomorphism in a neighborhood of C.

In particular, $e(C, X) = e(C, Y)$.

Proof.

$\mathcal{O}(K + C)|_C \simeq \mathcal{O}_C \Rightarrow K \cdot C = 0$

(C, X) is minimal $\Leftrightarrow K + C$ is nef $\Leftrightarrow K + C \sim \alpha C$, $\alpha \in \mathbb{Q}$

Run $(K + C)$-MMP: contract all curves $E \subseteq X$ with $K \cdot E < 0$, $C \cdot E = 0$.

□

Minimal + Du Val singularities

Definition

Since $K \cdot C = 0$*, define on* $\mathrm{Cl}_0(X) = C^\perp/\langle K\rangle$ *the reduced restriction map*

$$\overline{\mathrm{res}}: \ \mathrm{Cl}_0(X) \to \mathrm{Pic}^0(C)/\langle \mathrm{res}(K)\rangle$$

Theorem

Let (C, Y) *be an elliptic pair such that* Y *has Du Val singularities. Let* Z *be the minimal resolution of* Y*. Then*

$$(C, Y) \quad minimal \quad \Leftrightarrow \quad (C, Z) \quad minimal \quad \Leftrightarrow \quad \rho(Z) = 10.$$

In this case $\mathrm{Cl}_0(Z) \simeq \mathbb{E}_8$*.*

Assume (C, Y) *minimal elliptic pair with* $\rho(Y) \geq 3$ *and* $e(C, Y) < \infty$*:*

$$\overline{Eff}(Y) \quad polyhedral \quad \Leftrightarrow \quad \overline{Eff}(Z) \quad polyhedral \quad \Leftrightarrow$$

$$\mathrm{Ker}(\overline{\mathrm{res}}) \quad contains\ 8\ linearly\ independent\ roots\ of \quad \mathbb{E}_8.$$

Upshot

$(C, Y) =$ minimal model of elliptic pair (C, X)

- $e(C,X) = \infty \Rightarrow \overline{\mathrm{Eff}}(X)$, $\overline{\mathrm{Eff}}(Y)$ not polyhedral (if $\rho \geq 3$)

  In this case, Y *is* Du Val: $\mathcal{O}(C)|_C$ not torsion implies $-K_Y \sim C$
- $e(C,X) < \infty$ *and* Y is Du Val $\Rightarrow$ polyhedrality criterion for $\overline{\mathrm{Eff}}(Y)$

Problem

- *Suppose* $C, X, \mathrm{Cl}(X)$ *are defined over* $\mathbb{Q}$, $e(C,X) = \infty$
- $X \to Y$ *extends to the morphism of integral models* $\mathcal{X} \to \mathcal{Y}$ *over* $\operatorname{Spec}\mathbb{Z}$ *(outside of finitely many primes of bad reduction)*
- Y *is Du Val* $\Rightarrow$ Y_p *is Du Val*
- $e(C_p, X_p) < \infty$. *Study distribution of "polyhedral" primes*

Blown up toric surfaces

Lattice polygon $\Delta \subseteq \mathbb{R}^2 \Longrightarrow (\mathbb{P}_\Delta, \mathcal{L}_\Delta)$ associated polarized toric surface

Set $X = \mathrm{Bl}_e \mathbb{P}_\Delta$ and let $m > 0$ integer. Then $X, \mathrm{Cl}(X)$ are defined over $\mathbb{Q}$.

Definition

A lattice polygon Δ with at least 4 vertices is good if there exists

$$C \in |\mathcal{L}_\Delta - mE|$$

irreducible such that (C, X) is an elliptic pair with $e(C, X) = \infty$:

(I) *The Newton polygon of C coincides with Δ ($\Leftrightarrow C \subseteq X^{smooth}$),*

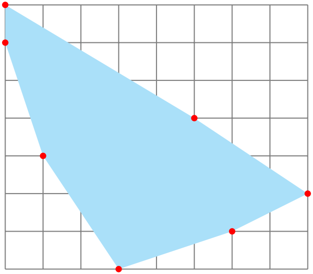
(II) *$Vol(\Delta) = m^2$ and $|\partial\Delta \cap \mathbb{Z}^2| = m$ ($\Leftrightarrow C^2 = 0$, $p_a(C) = 1$),*

(III) *The restriction $\mathrm{res}(C) = \mathcal{O}_X(C)|_C$ is not torsion in $\mathrm{Pic}^0(C)$ over $\mathbb{Q}$.*

Theorem

If Δ is a good polygon, then $\overline{Eff}(X)$ is not polyhedral in characteristic 0.

Example of a good polygon

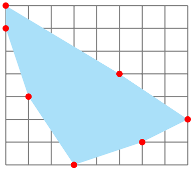


$$\mathrm{Vol}(\Delta) = 49, \quad |\partial\Delta \cap \mathbb{Z}^2| = 7$$

The linear system $|\mathcal{L}_\Delta - 7E|$ contains a unique curve C with equation

$$-u^8v^2 + 4u^7v^2 + 8u^6v^3 - 5u^6v^2 - 3u^6v - 5u^5v^4 - 50u^5v^3+$$
$$+21u^5v^2 + 6u^5v + 40u^4v^4 + 85u^4v^3 - 55u^4v^2 - 6u^3v^5 - 85u^3v^4-$$
$$-40u^3v^3 + 56u^3v^2 - 10u^3v + u^3 + 15u^2v^5 + 80u^2v^4 - 40u^2v^3+$$
$$+u^2v^2 + 3uv^6 - 30uv^5 + 5uv^4 + 2uv^3 - v^7 + 4v^6 = 0.$$

Example of a good polygon



The curve C is a smooth elliptic curve labelled 446.a1 in the LMFDB database. It has the minimal equation

$$y^2 + xy = x^3 - x^2 - 4x + 4$$

The Mordell-Weil group is $\mathbb{Z} \times \mathbb{Z}$, with generators

$$P = (0, 2), \quad Q = (-1, 3)$$

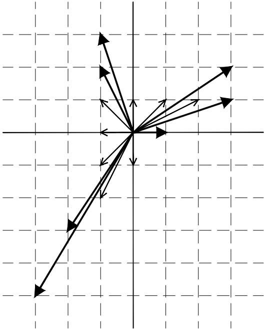
(identify $\mathrm{Pic}^0(C)$ with C)

Computation : $\mathrm{res}(C) = -Q = (-1, -2)$

$\mathrm{res}(C)$ not torsion in characteristic $0 \Longrightarrow \Delta$ is good

Example - Minimal resolution

Fan of the minimal resolution $\tilde{\mathbb{P}}_\Delta$ of $\mathbb{P}_\Delta$:



The proper transforms C_1, C_2 of 1-parameter subgroups $\{u = 1\}$, $\{u = v\}$

- have self-intersection -1 on $\mathrm{Bl}_e\,\tilde{\mathbb{P}}_\Delta$, and also on $X = \mathrm{Bl}_e\,\mathbb{P}_\Delta$
- have $C \cdot C_1 = C \cdot C_2 = 0$

Example - Minimal elliptic pair

(C, X) elliptic pair, $X = \mathrm{Bl}_e \mathbb{P}_\Delta$

Zariski decomposition $K_X + C = 2C_1 + C_2 + C_3$

$C_3 =$ curve whose image in $\mathbb{P}_\Delta$ has multiplicity 3 at e

To get minimal elliptic pair (C, Y), contract C_1, C_2, C_3.

$$\begin{array}{ccc} \mathrm{Bl}_e \tilde{\mathbb{P}}_\Delta & \longrightarrow & Z \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

$Z \to Y$ minimal resolution of Y

$$\rho(X) = 6, \quad \rho(Y) = 3, \quad \rho(Z) = 10$$

Example - Minimal resolution

$Z \to Y$ minimal resolution of Y, $\mathrm{Cl}(Z) = \mathrm{Cl}(Y) \oplus T$

T = sublattice spanned by classes of (-2) curves above singularities of Y

Computation : $T = \mathbb{A}^6 \oplus \mathbb{A}^1$

$\mathrm{Cl}_0(Y) = \mathrm{Cl}_0(Z)/T = \mathbb{E}_8/\mathbb{A}^6 \oplus \mathbb{A}^1 \cong \mathbb{Z}$

Reduced restriction map $\overline{\mathrm{res}} : \mathrm{Cl}_0(Y) \to \mathrm{Pic}^0(C)/\langle Q \rangle$, $Q = (-1, 3)$

$\overline{\mathrm{Eff}}(Y)$ is not polyhedral in characteristic $p \Leftrightarrow$

$\Leftrightarrow \overline{\mathrm{res}}(\beta) \neq 0$ for all β = image in $\mathrm{Cl}_0(Y)$ of a root in $\mathbb{E}_8 \setminus T$

If $\alpha \in \mathrm{Cl}_0(Y)$ generator $\Longrightarrow$ Images of roots of $\mathbb{E}_8$ are $\pm k\alpha$, for $0 \leq k \leq 4$

Computation : $\mathrm{res}(\tilde{\alpha}) = P = (0, 2)$ for some lift $\tilde{\alpha} \in C^{\perp} \subseteq \mathrm{Cl}(Y)$ of α

$\overline{\mathrm{Eff}}(Y)$ not polyhedral in characteristic $p \Leftrightarrow k\overline{P} \notin \langle \overline{Q} \rangle$ for $k = 1, 2, 3, 4$

Example - Non-polyhedral primes

Prove that the set of primes p such that

$$\overline{P}, 2\overline{P}, 3\overline{P}, 4\overline{P} \notin \langle \overline{Q} \rangle \subseteq C(\mathbb{F}_p)$$

has positive density.

Fix q prime. It suffices to prove that the set of primes p such that

- q divides the index of $\langle \overline{Q} \rangle \subseteq C(\mathbb{F}_p)$
- q does not divide the index of $\langle 12\overline{P} \rangle \subseteq C(\mathbb{F}_p)$

has positive density.

Chebotarev Density theorem + Lang-Trotter (1976) $\Longrightarrow$

C elliptic curve defined over $\mathbb{Q}$, without complex multiplication over $\overline{\mathbb{Q}}$.

If $x, y \in C(\mathbb{Q})$ are points of infinite order, with $\langle x \rangle \cap \langle y \rangle = 0$, then for a set of primes p of positive density, q divides the index of $\langle \overline{x} \rangle \subseteq C(\mathbb{F}_p)$, but not the index of $\langle \overline{y} \rangle \subseteq C(\mathbb{F}_p)$.

Non-polyhedral primes

The set of non-polyhedral primes $p < 2000$ for our example of a good polygon:

$$47, 71, 103, 197, 233, 239, 277, 313, 367, 379,$$
$$409, 503, 563, 599, 647, 677, 683, 691, 719, 727,$$
$$761, 829, 911, 997, 1103, 1123, 1151, 1171, 1187, 1231,$$
$$1283, 1327, 1481, 1493, 1709, 1723, 1861, 1907, 1997$$

This gives $39/303 = 12\%$ of the primes under 2000.

There are:

- 135 toric surfaces corresponding to good polygons with volume ≤ 49;
- Infinite sequences of good pentagons with all primes polyhedral;
- Infinite sequences of good heptagons. For all but finitely many, the set of non-polyhedral primes has positive density.