MODULI OF MAP GERMS, THOM POLYNOMIALS AND THE GREEN-GRIFFITHS CONJECTURE

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1. Introduction

Let $J_k(n,m)$ denote the complex vector space of k-jets of map germs from $\mathbb{C}^n$ to $\mathbb{C}^m$ mapping the origin to the origin. The open dense subset $J_k^{\mathrm{reg}}(n,m)$ consists of jets with regular linear part. $J_k^{\mathrm{reg}}(1,1)$ is a group under composition of jets, and it acts via reparametrisation on $J_k(1,n)$.

The dimension of the complex vector space $J_k(1,1)$ is k, and with a natural choice of basis $J_k^{\mathrm{reg}}(1,1)$ can be identified with the following linear subgroup of $GL(k)$:

$$\mathbf{G}_k = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \dots & 2\alpha_1\alpha_{k-1} + \dots \\ 0 & 0 & \alpha_1^3 & \dots & 3\alpha_1^2\alpha_{k-2} + \dots \\ 0 & 0 & 0 & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \alpha_1^k \end{pmatrix} : \alpha_1 \in \mathbb{C}^*, \alpha_i \in \mathbb{C} \right\},$$

where the polynomial in the (i,j) entry is

$$p_{i,j}(\bar{\alpha}) = \sum_{a_1+a_2+\ldots+a_i=j} \alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_i}.$$

This paper is an exploration of this subgroup of GL_k and the non-reductive quotient $J^k(1,n)/J_k(1,1)$, which is roughly speaking the moduli of k-jets of curves in $\mathbb{C}^n$. Principles and ideas of classical reductive geometric invariant theory of Mumford do not apply in this situation, for more details about the background see [13, 5].

We illustrate the importance of this moduli space for two classical problems.

The first problem goes back to René Thom and his study of degreneracy loci of holomorphic maps between manifolds. Consider a holomorphic map $f : N \to M$ between two complex manifolds, of dimensions $n \le m$. For a singularity class $O \subset J_k(n,m)$ we can define the set

$$Z_O[f] = \{p \in N;\ f_p \in O\},$$

that is the set of points where the germ f_p belongs to O. Then, under some additional technical assumptions, $Z_O[f]$ is an analytic subvariety of N. The computation of the

Poincaré dual class $\alpha_O[f] \in H^*(N, \mathbb{Z})$ of this subvariety is one of the fundamental problems of global singularity theory. It turns out that these classes—the Thom polynomials of singularities—are certain equivariant intersection numbers on the moduli space $J_k(1,n)/J_k^{\mathrm{reg}}(1,1)$.

The second problem is an old conjecture of Green and Griffiths about holomorphic curves in smooth projective varieties. Their conjecture, from 1979, says that any projective variety X of general type contains a proper subvariety $Y \subsetneq X$ such that any entire holomorphic curve $f : \mathbb{C} \to X$ sits in Y, that is $f(\mathbb{C}) \subset Y$. The strategy of Green, Griffiths, Demailly and Siu, and the recent work of Diverio, Merker and Rousseau [12] leads us to prove the positivity of an intersection number on the Demailly bundle, whose fibers are canonically isomorphic to $J_k(1,n)/J_k^{\mathrm{reg}}(1,1)$.

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Most results presented here have already been published in the papers [3, 4, 5]. The only exception is the formula for the Euler characteristic of Demailly jet bundles in §8 Appendix and the relation to the curvilinear Hilbert scheme in the last section.

2. Equivariant cohomology

It is well-known that any group action on a topological space carries topological information about the space.

Let G be a topological group. A principal G-bundle is a map $E \to B$, which is locally a projection $U \times G \to U$. One of the main fundamental principles in topology is to find universal objects such that all objects in a given category can be "pulled-back" from this. Here a universal principal G-bundle is a bundle $\pi : EG \to BG$ such that every principal G-bundle $E \to B$ is a pull-back via a map $B \to BG$, which is unique up to homotopy. EG is contractible. In fact, if P is a contractible space with a free G-action then $P \to P/G$ is a universal principle G-bundle.

Theorem 1. *EG exists for all topological group G, and unique up to equivariant homotopy.*

Example 1. *$B\mathbb{C}^* = \mathbb{P}^\infty(\mathbb{C})$, and $\mathbb{C}^\infty \to \mathbb{P}^\infty(\mathbb{C})$ is a universal principle $\mathbb{C}^*$ bundle.*

Similarly,

$$BGL_n = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^\infty)/GL_n = \mathrm{Gr}(n, \infty),$$

and $EGL(n) = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^\infty) \to \mathrm{Gr}(n, \infty)$ is the universal principle $GL(n)$-bundle. From this we can construct

$$EGL_n \times_{GL_n} \mathbb{C}^n \to BGL_n,$$

which is a universal vector bundle, namely any vector bundle of rank n can be pulled back from this.

The next step is to define equivariant cohomology. Let X be a G-space, i.e. a topological space with a G-action. If the action is free, then G-equivariant cohomology is the ordinary cohomology of the quotient $H^*(X/G)$. For non-free actions the quotient X/G is not well-behaved and $H^*(X/G)$ does not carries enough information. We need to "resolve" the action by replacing X with $X \times EG$. This has a free (diagonal) G-action, and define

$$H^*_G(X) = H^*(EG \times_G X)$$

Example 2. *$H^*_G(pt) = H^*(BG) = \mathbb{C}[\mathfrak{h}]^W$, where $\mathfrak{h} = LieT$ is the Cartan algebra acted on by the Weil group W. For example $H^*_{GL_n}(pt) = S^W = \mathbb{C}[x_1, \dots, x_n]^{S_n}$, the algebra of symmetric polynomials.*

Properties of equivariant cohomology:

(1) $f : X \to Y$ G-map induces $H(f) : H_G(Y) \to H_G(X)$
(2) $h : G \to H$ homomorphism, then EH can serve as EG and we have a projection $EH \times_G X \to EH \times_H X$ which induces $H(h) : H_H(X) \to H_G(X)$
(3) $H^*_G(pt) = H^*(BG) = \mathbb{C}[\mathfrak{h}]^W$, and $H^*_G(X)$ is a $H^*_G(pt)$-module. For example $H^*_{GL_n}(pt) = S^W = \mathbb{C}[x_1, \dots, x_n]^{S_n}$.

Proposition 1. ***Induction, Restriction*** *Let X be a G-space.*

- *Restriction: If $H \subset G$ then X is naturally a H-space, and there is an induced map $H^*_G(pt) \to H^*_H(pt)$.*
$$H^*_H(X) = H^*_H(pt) \otimes_{H^*_G(pt)} H^*_G(X)$$
- *Induction: If $G \subset K$ then $K \times_G X$ is naturally a K-space, and there is an induced map $H^*_K(pt) \to H^*_G(pt)$.*
$$H_K(K \times_G X) = H_G(X)$$
but as a $H_K(pt)$-module.

Example 3. *Let $G = GL_n$. We have a left-right action of the upper Borel $B \subset GL_n$ on $G \times_B G$. We compute $H_{B\times B}(G \times_B G)$ in the following steps:*

$$H^*_{B\times B}(B) = H^*(B) = S,$$

so by induction

$$H^*_{G\times B}(G \times_B B) = S \in S^W\text{-}mod\text{-}S,$$

therefore by restriction

$$H^*_{B\times B}(G) = S \otimes_{S^W} S \in S\text{-}mod\text{-}S$$

and by induction again

$$H^*_{G\times B}(G \times_B G) = S \otimes_{S^W} S \in S^W\text{-}mod\text{-}S$$

and by restriction

$$H^*_{B\times B}(G \times_B G) = S \otimes_{S^W} S \otimes_{S^W} S \in S\text{-}mod\text{-}S.$$

2.1. **The equivariant DeRham model.** Let G be a Lie group with Lie algebra $\mathfrak{g}$. For a smooth G-manifold M we can define equivariant differential forms, for more details see [7]. The equivariant differential forms are differential form valued polynomial functions on $\mathfrak{g}$:

$$\Omega_G(M) = \{\alpha : \mathfrak{g} \to \Omega(M) : \alpha(gX) = g\alpha(X) \text{ for } g \in G, X \in \mathfrak{g}\} = (\mathbb{C}[\mathfrak{g}] \otimes \Omega(M))^G$$

where $(g \cdot \alpha)(X) = g \cdot (\alpha(g^{-1} \cdot X))$. Here $\mathbb{C}[\mathfrak{g}]$ denotes the algebra of complex values polynomial functions on $\mathfrak{g}$.

We define an equivariant exterior differential d_G on $\mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$ by the formula

$$(d_G\alpha)(X) = (d - \iota(X_M))\alpha(X),$$

where $\iota(X_M)$ denotes the contraction by the vector field X_M. This increases the degree of an equivariant form by one if the $\mathbb{Z}$-grading is given on $\mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$ by

$$\deg(P \otimes \alpha) = 2\deg(P) + \deg(\alpha)$$

for $P \in \mathbb{C}[\mathfrak{g}], \alpha \in \Omega(M)$. The homotopy formula $\iota(X)d + d\iota(X) = \mathcal{L}(X)$ implies that

$$d_G^2(\alpha)(X) = -\mathcal{L}(X)\alpha(X) = 0$$

for any $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$, and therefore $(d_G, \Omega_G(M))$ is a complex.

Definition 1. *The equivariant cohomology of the G-manifold M is the cohomology of the complex $(d_G, \Omega_G(M))$:*

$$H_G^*(M) = H_{d_G}^*$$

Note that $\alpha \in \Omega_G(M)$ is equivariantly closed if

$$\alpha(X) = \alpha(X)_0 + \ldots + \alpha(X)_n \text{ such that } \iota(X_M)\alpha(X)_i = d\alpha(X)_{i-2}.$$

Here $\alpha(X)_i \in \Omega_i(M)$ is the degree-i part of $\alpha(X) \in \Omega(M)$. In other words, $\alpha_i : \mathfrak{g} \to \Omega^i(m)$ is a polynomial function.

The functoriality properties of equivariant cohomology now come for free:

(1) If $H \to G$ is a homomorphism of Lie groups then the restriction map $\mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}]$ induces a homomorphism of differential graded algebras $\Omega_G(M) \to \Omega_H(M)$ and finally a homomorphism $H_G(M) \to H_H(M)$.
(2) If $\phi : N \to M$ is a map of G-manifolds which interwines the actions of G then pull-back by ϕ induces a homomorphism of differential graded algebras $\phi^* : \Omega_G(M) \to \Omega_G(N)$ and homomorphism $H_G(M) \to H_G(N)$.

3. Equivariant localization

3.1. **Integrating equivariant forms.** If G is a Lie group and M is a G-manifold, we can integrate equivariant forms obtaining a map

$$\int_M : \Omega_G(M) \to \mathbb{C}[\mathfrak{g}]^G$$

by the formula

$$\left(\int_M \alpha\right)(X) = \int_M \alpha(X) = \int_M \alpha_{[n]}(X)$$

That is, if α is an equivariant differential form, then we integrate the top degree part of it, and a result is a polynomial function on $\mathfrak{g}$. This is well-defined: if α is equivariantly exact, i.e. $\alpha = d_G\beta$ for some $\beta \in \Omega_G(M)$ then $\alpha(X)_n = d\beta(X)_n$, and therefore $\int_M \alpha(X) = 0$. Thus if α is equivariantly closed then $\int_M \alpha$ only depends on the equivariant cohomology class represented by α.

It can be shown (see Proposition 7.10 in [7]) that if G is a compact Lie group, and $M_0(X)$ is the zero locus of the vector field X_M, then the form $\alpha(X)_n$ is exact outside $M_0(X)$. This suggests that the integral $\int_M \alpha(X)$ only depends on the restriction $\alpha(X)|_{M_0(X)}$.

Here we state the localization thorem in the special case when X_M has isolated zeros.

Theorem 2 (Atiyah/Bott/Berline/Vergne)**.** *Let $G = T$ be a complex torus, M a T-manifold, $\alpha \in \Omega_T(M)$. Then*

$$\int_M \alpha = (2\pi)^l \sum_{p\in M^T} \frac{\alpha_0(p)}{\mathrm{Euler}^T(T_pM)}$$

In other words:

$$\int_M \alpha(X) = (2\pi)^l \sum_{p\in M^T} \frac{\alpha(X)_0(p)}{\prod_i \lambda_i}$$

where λ_i are the weights of the Lie action

$$X : \xi \in T_pM \to [X_M(p), \xi] \in T_pM.$$

Most often we apply localization to compute certain intersection numbers on M. My favorite example illustrating the strength of the localization method is the following.

3.2. **How many lines intersect 2 given lines and go through a point in $\mathbb{P}^3$?** We think points, lines and planes in $\mathbb{P}^3$ as $1, 2, 3$-dimensional subspaces in $\mathbb{C}^4$. For $R \in \mathrm{Grass}(3, \mathbb{C}^4)$, $L \in \mathrm{Grass}(1, \mathbb{C}^4)$ define

$$C_2(R) = \{V \in \mathrm{Grass}(2, 4) : V \subset R\},\ C_1(L) = \{V \in \mathrm{Grass}(2, 4) : L \subset V\}$$

Standard Schubert calculus says that $C_1(L)$ (resp $C_2(R)$) represents the cohomology class $c_1(\tau)$ (resp $c_2(\tau)$) where τ is the tautological rank 2 bundle over $\mathrm{Grass}(2, 4)$.

So the answer is

$$C_1(L_1) \cap C_1(L_2) \cap C_2(R) = \int_{\mathrm{Grass}(2,4)} c_1(\tau)^2 c_2(\tau).$$

Apply equivariant localization. The sufficient data are the following.

- The diagonal torus $T^4 \subset GL(4)$ acts on $\mathbb{C}^4$ with weights $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathfrak{t}^* \subset H^*_T(pt)$.

- The induced action on Grass(2, 4) has $\binom{4}{2}$ fixed points, the coordinate subspaces indexed by (i, j).
- The tangent space of Grass(2, 4) at (i, j) is $(\mathbb{C}^2)^*_{i,j} \otimes \mathbb{C}^2_{s,t}$, where $\{s, t\} = \{1, 2, 3, 4\} \setminus \{i, j\}$, and $\mathbb{C}^2_{i,j} \in \mathrm{Grass}(2, 4)$ is the subspace spanned by the i, j basis. Therefore, the weights on $T_{(i,j)}\mathrm{Grass}$ are $\mu_s - \mu_i, \mu_s - \mu_j$ with $s \neq i, j$.
- The weights of τ are identified with the Chern roots, so $c_i(\tau)$ is represented by the ith elementary symmetric polynomial in the weights of τ.

ABBV localization then gives

$$\int_{Gr(2,4)} c_1(\tau)^2 c_2(\tau) = \sum_{\sigma \in S_4/S_2} \sigma \cdot \frac{(\mu_1 + \mu_2)^2 \mu_1 \mu_2}{(\mu_3 - \mu_1)(\mu_4 - \mu_1)(\mu_3 - \mu_2)(\mu_4 - \mu_2)} = 2. \tag{1}$$

On the right hand side we sum over all $\binom{4}{2}$ fixed points by taking appropriate permutation of the indices.

It is not clear at first glance, why this rational expression is an integer. But it turns out that the sum is indepenent of μ_i's and it is 2.

3.3. **Iterated residues.** We saw in the previous example that the ABBV localization results a sum of rational expressions, but adding these together is not an obvious task. There is a short and elegant way to do this by identifying the summands as iterated residues of a certain meromorphic differential form on $\mathbb{C}^d$ for some d, and then by applying the Residue theorem saying that the sum of the residues at finite points is equal to minus the residue at infinity.

The set-up is the following.

- $z_1, \ldots, z_d$ – coordinates on $\mathbb{C}^d$.
- $\omega_1, \ldots, \omega_N$ – affine linear forms on $\mathbb{C}^d$; $\omega_i = a_i^0 + a_i^1 z_1 + \ldots + a_i^d z_d$.
- $h(\mathbf{z})$ a function $h(z_1 \ldots z_d)$, and $d\mathbf{z} = dz_1 \wedge \cdots \wedge dz_d$ holomorphic d-form.

Definition 2. *We define the iterated residue of $\frac{h(\mathbf{z})\, d\mathbf{z}}{\prod_{i=1}^N \omega_i}$ at infinity as follows*

$$\operatorname*{Res}_{z_1=\infty} \ldots \operatorname*{Res}_{z_d=\infty} \frac{h(\mathbf{z})\ d\mathbf{z}}{\prod_{i=1}^N \omega_i} \overset{\text{def}}{=} \left(\frac{1}{2\pi i}\right)^d \int_{|z_1|=R_1} \cdots \int_{|z_d|=R_d} \frac{h(\mathbf{z})\ d\mathbf{z}}{\prod_{i=1}^N \omega_i}, \tag{2}$$

where $1 \ll R_1 \ll \ldots \ll R_d$. The torus $\{|z_m| = R_m;\ m = 1 \ldots d\}$ is oriented in such a way that $\mathrm{Res}_{z_1=\infty} \ldots \mathrm{Res}_{z_d=\infty}\, d\mathbf{z}/(z_1 \cdots z_d) = (-1)^d$.

In practice, the iterated residue 2 may be computed using the following **algorithm**: for each i, use the expansion

$$\frac{1}{\omega_i} = \sum_{j=0}^{\infty} (-1)^j \frac{(a_i^0 + a_i^1 z_1 + \ldots + a_i^{q(i)-1} z_{q(i)-1})^j}{(a_i^{q(i)} z_{q(i)})^{j+1}}, \tag{3}$$

where $q(i)$ is the largest value of m for which $a_i^m \neq 0$, then multiply the product of these expressions with $(-1)^d h(z_1 \cdots z_d)$, and then take the coefficient of $z_1^{-1} \dots z_d^{-1}$ in the resulting Laurent series.

Example 4.

- $\frac{1}{z_1(z_1-z_2)}$ *has two different Laurent expansions, but on* $|z_1| \ll |z_2|$ *we use* $\frac{1}{z_1(z_1-z_2)} = \sum_{i=0}^{\infty}(-1)^i \frac{z_1^{i-1}}{z_2^{i+1}}$ *to get* $\mathrm{Res}_\infty \frac{1}{z_1-z_2} = 1$.
- $\mathrm{Res}_{\mathbf{z}=\infty} \frac{1}{(z_1-z_2)(2z_1-z_2)} = \mathrm{coeff}_{(z_1z_2)^{-1}} \frac{1}{z_2^2}(1 + \frac{z_1}{z_2} + \frac{z_1^2}{z_2^2} + \dots)(1 + \frac{2z_1}{z_2} + \frac{4z_1^2}{z_2^2} + \dots) = 3$

Let's turn back to our toy example presented in §3.2. Define the differential form

$$\omega = \frac{(z_2 - z_1)^2(z_1 + z_2)^2 z_1 z_2 \, d\mathbf{z}}{\prod_{i=1}^4(\mu_i - z_1) \prod_{i=1}^4(\mu_i - z_2)}$$

This is a meromorphic form in z_2 on $\mathbb{P}^1$ with poles at $z_2 = \mu_i$, $1 \leq i \leq 4$ and $z_2 = \infty$. The poles at μ_i are non-degenerate and therefore applying the Residue Theorem we get

$$\operatorname*{Res}_{z_2=\infty} \omega = \sum_{i=1}^{4} \underbrace{-\frac{(\mu_i - z_1)^2(\mu_i + z_1)^2 \mu_i z_1 dz_1}{\prod_{j=1}^4(\mu_j - z_1) \prod_{j\neq i}(\mu_j - \mu_i)}}_{z_2=\mu_i} = \sum_{i=1}^{4} -\frac{(\mu_i - z_1)(\mu_i + z_1)^2 \mu_i z_1 dz_1}{\prod_{j\neq i}(\mu_j - z_1) \prod_{j\neq i}(\mu_j - \mu_i)}$$

Doing the same again with z_1 we get

$$\operatorname*{Res}_{z_1=\infty} \operatorname*{Res}_{z_2=\infty} \omega = \sum_{i=1}^{4} \sum_{j\neq i} -\frac{(\mu_i - \mu_j)(\mu_i + \mu_j)^2 \mu_i \mu_j}{\prod_{k\neq i,j}(\mu_k - \mu_j) \prod_{j\neq i}(\mu_j - \mu_i)} =$$

$$= \sum_{i=1}^{4} \sum_{j\neq i} \frac{(\mu_i + \mu_j)^2 \mu_i \mu_j}{\prod_{k\neq i,j}(\mu_k - \mu_j) \prod_{k\neq i,j}(\mu_k - \mu_i)} = \int_{Gr(2,4)} c_1(\tau)^2 c_2(\tau).$$

On the other hand, using the above algorithm by expanding the rational form ω we get

$$\operatorname*{Res}_{z_1=\infty} \operatorname*{Res}_{z_2=\infty} \omega = 2.$$

We give an other example, the so called Giembelli-Thom-Porteous formula in section 4.2.

3.4. **Localization on partial flag manifolds.** The following Proposition is a far-reaching generalization of the idea presented in the previous section, and it provides a meromorphic differential form whose residue at infinity gives back the localization formula for a large class of forms.

Let

$$\mathrm{Flag}_d(n) = \{V_1 \subset \dots \subset V_d \subset \mathbb{C}^n : \dim(V_i) = i\}$$

denote the full flags of d-dimensional subspaces of $\mathbb{C}^n$. The maximal torus $T \subset GL(n)$ acts on $\mathrm{Flag}_d(n)$, and the fixed points are parametrized by coordinate flags corresponding to certain permutations $\sigma \in (\mathrm{Flag}_d(n))^T$. The Chern classes of the tautological rank-d

bundle over $\mathrm{Flag}_d(n)$ are elementary symmetric polynomials in the weight of T on $\mathbb{C}^n$, and the intersection numbers of τ can be computed as iterated residues according to

Proposition 2 ([3]). *Let $Q(\mathbf{z}) = Q(z_1, \ldots, z_d)$ be a polynomial on $\mathbb{C}^d$ of degree* $\dim(\mathrm{Flag}_d(n))$. *Then*

$$\sum_{\sigma \in (\mathrm{Flag}_d(n))^T} \frac{Q(\lambda_{\sigma \cdot 1} \cdots \lambda_{\sigma \cdot d})}{\prod_{1 \le m \le d} \prod_{i=m+1}^{n} (\lambda_{\sigma \cdot i} - \lambda_{\sigma \cdot m})} = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{\prod_{1 \le m < l \le d} (z_m - z_l)\, Q(\mathbf{z})\, d\mathbf{z}}{\prod_{l=1}^{d} \prod_{i=1}^{n} (\lambda_i - z_l)} \tag{4}$$

where the permutation $\sigma = (\sigma(1), \ldots, \sigma(n)) \in \mathrm{Flag}_d(n))^T = S_n/S_{n-d}$ represents the torus fixed flag $\mathbb{C}e_{\sigma(1)} \subset \ldots \subset \mathbb{C}e_{\sigma(1)} \oplus \ldots \oplus \mathbb{C}e_{\sigma(d)} \subset \mathbb{C}^n$.

4. Singularities of maps

The first problem we address goes back to the 1950's and the work of René Thom. For more details of the history and background of the problem see [1, 3].

The usual set-up for studying singularities of map germs is the following.

Set up: We fix integers $k \le n \le m$.

- Let A be a nilpotent algebra, $\dim A/\mathbb{C} = k$. We will take $A_k = z\mathbb{C}[z]/z^{k+1}$.
- Define $J_k(n,m) = \{p = (p_1, \ldots, p_m) \in \mathrm{Poly}(\mathbb{C}^n, \mathbb{C}^m) : \deg p_i \le k, p_i(0) = 0\}$. This is the vector space of k-jets of map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$.
- Let $\Sigma_A = \{p \in J_k(n,m) : \mathbb{C}[x_1, \ldots, x_n]/\langle p_1, \ldots, p_m \rangle = A\}$ be the set of map-germs with local algebra isomorphic to A.
- The germs $J_k^{\mathrm{reg}}(n,n)$ with non-degenerate linear part form a group, and $J_k^{\mathrm{reg}}(n,n) \times J_k^{\mathrm{reg}}(m,m)$ naturally acts on $J_k(n,m)$ with
$$(A,B)p = BpA^{-1}$$
These are the polynomial reparametrizations of map germs.

The central problem of global singularity theory is the computing the (co)hohomology classes of singularity loci of holomorphic maps between complex manifolds. Given a holomorphic map $f : N^n \to M^m$ define

$$Z(f) = \left\{ p \in N | \widehat{f_p} \in \Sigma_A \right\},$$

where $\widehat{f_p}$ is the germ of f at $p \in N$.

It was already known by Thom, which is now called the Thom principle, that for generic map f, $Z(f)$ represents a cohomology cycle and there is a well-defined polynomial

$$MD_A^{n \to m} \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]^{S_n \times S_m}$$

such that

$$[Z(f)] = MD_A(TN, f^*(TM)) \in H^*(N, \mathbb{C}).$$

Here MD_A stands for multidegree, for explanation see the next section.

Furthermore, in [18] Haefliger and Kosinski proves that if

$$c(q) = c_0 + c_1 q + c_2 q^2 + \ldots = \frac{c(f^*(TM))}{c(TN)} = \frac{\prod_{m=1}^{k}(1+\theta_m q)}{\prod_{i=1}^{n}(1+\lambda_i q)}$$

is the Chern classes of the difference bundle then

$$MD_A(TN, f^*(TM)) = \mathrm{Tp}_A^{k\to n}(c_1, c_2, \ldots)$$

That is, MD_A is a polynomial in these difference Chern classes, and Tp_A is called the Thom polynomial of the algebra A.

4.1. **Multidegrees.** The polynomial MD_A stands for multidegree, which is also called equivariant Hilbert polynomial or equivariant Poincaré dual in the literature. This is defined for any G-invariant subvarieties of a complex G-vector spaces (i.e. G-representations, where G is a Lie algebra) as follows.

Set up:

(1) $V = \mathbb{C}^N$ complex vector space, with a G-action.
(2) $\Sigma \subset V$ is a G-invariant closed subvariety.
(3) $H_G^*(V) = H_G^*(pt)$ is the G-equivariant cohomology ring of V. Recall that $H^*_{GL(d)}(pt) = \mathbb{C}[x_1, \ldots, x_d]^{S_d}$.

We give two definitions of a polynomial

$$\mathrm{mdeg}[\Sigma, V] \in H_G^{\mathrm{codim}(\Sigma\in V)}(pt),$$

called the multidegree of Σ: one topological and one algebraic definition.

Vergne's integral definition-topology

If $\Sigma \subset V$ is a subvariety then $EG \times_G \Sigma \subset EG \times_G V$ represents a homology cycle, and the multidegree is the ordinary Poincaré dual of the Borel construction $EG \times_G \Sigma$:

$$\mathrm{mdeg}[\Sigma, V] = PD(EG \times_G \Sigma \subset EG \times_G V).$$

By definition $\mathrm{mdeg}[\Sigma, V] \in H^*(EG \times_G V) = H_G^*(pt)$ is a polynomial.

Theorem 3 ([38]). *There is an equivariant Thom class:*

$$\mathrm{Thom}_G(V) \in H_G^{\dim V}(V)$$

such that for any $\Sigma \subset V$ G-invariant subvariety

$$\mathrm{mdeg}[\Sigma, V] = \int_\Sigma \mathrm{Thom}_G(V).$$

We give an other, more algebraic definition of the multidegree, which also provides an algorith to compute these polynomials.

Sturmfels' axiomatic definition

Theorem 4 ([27]). *Let $\Sigma \subset V$ be a G-invariant subset of the G-representation V. Then* $\mathrm{mdeg}[\Sigma, V]$ *is characterized by the following axioms:*

additivity: *If* $\Sigma \doteq \cup\Sigma_i$ *is the set of maximal irreducible components of* Σ*, then*

$$\mathrm{mdeg}[\Sigma, V] = \sum_{i=1}^{c} \mathrm{mult}(\Sigma_i) \cdot \mathrm{mdeg}[\Sigma_i, W].$$

degeneration: *The multidegree is constant under flat deformation of* Σ.

normalization: *For* T*-invariant linear subspaces of* V*,* $\mathrm{mdeg}[\Sigma, V]$ *is defined to be equal to the product of weights in the normal direction.*

The recipe to compute the multidegree (although this recipe often ends up with difficult commutative algebra computations) is to choose a proper flat deformation of Σ into the union of coordinate spaces, that is, to deform its ideal into a monomial ideal. For example, choosing a monomial order on the coordinate ring, the initial ideal is monomial, and then normalization and additive properties of the multidegree give you the result.

Example 5. $(\mathbb{C}^*)^3$ *acts on* $\mathbb{C}^4$ *with weights* $\eta_1, \ldots, \eta_4$*. Let* $\eta_1 + \eta_2 = \eta_3 + \eta_4$*, and*

$$\Sigma = \mathrm{Spec}(\mathbb{C}[y_1, y_2, y_3, y_4]/(y_1y_2 - y_3y_4)).$$

Define the flat deformation

$$\Sigma_t = \mathrm{Spec}(\mathbb{C}[y_1, y_2, y_3, y_4]/(y_1y_2 - ty_3y_4)),$$

For $t = 0$*,* $\Sigma_0 = \{y_1y_2 = 0\}$*, so normalization and additivity says*

$$\mathrm{mdeg}[\Sigma, \mathbb{C}^4] = \eta_1 + \eta_2 = \eta_3 + \eta_4.$$

Now we can state Thom's principle more precisely:

Theorem 5 (Thom). *Let* $\Sigma_A \subset J_k(n,m)$ *denote the set of germs with local algebra isomorphic to* A*. This is a* $\mathrm{GL}_n \times \mathrm{GL}_m$*-invariant subvariety of* $J_k(n,m)$*, and*

$$MD_A^{n\to m} = \mathrm{mdeg}^{\mathrm{GL}_n\times\mathrm{GL}_m}[\Sigma_A, J_k(n,m)].$$

4.2. **Degeneraci loci of sections via localization.** Here is an other illustrating example for transformation of localization formulas into iterated residues. Given a a rank-n vector bundle on a manifold M, and n generic sections $\sigma_1, \ldots, \sigma_n$, it is an old question in topology to determine the cohomology class dual to the locus where the sections are linearly dependent. This class is the Thom polynomial Tp_A with $A = t\mathbb{C}[t]/t^2$, and we have

$$\Sigma_A = \Sigma_1 = \{A \in \mathrm{Hom}(n,m);\ \dim\ker A = 1\} = \{A \in \mathrm{Hom}(n,m)\exists![v] \in \mathbb{P}^{n-1} : Av = 0\}.$$

The goal is to compute $\mathrm{mdeg}[\Sigma_1, \mathrm{Hom}(n,m)]$.

We have the fibration $\pi : \Sigma_1 \to \mathbb{P}^{n-1}$ sending a linear map to its kernel. This is equivariant with respect to the $GL_n \times GL_m$ action, and GL_m acts fiberwise whereas GL_n acts on $\mathbb{P}^{n-1}$. According to Vergne's definition, we want to integrate the equivariant Thom class over Σ_1. The idea is to integrate first over the base $\mathbb{P}^{n-1}$ and then along the fibers, and to apply ABBV localization on $\mathbb{P}^{n-1}$.

We have n fixed points on $\mathbb{P}^{n-1}$ corresponding to the coordinate axes. Let $\lambda_1,\dots,\lambda_n$ denote the weights of $T^n\subset GL_n$ on $\mathbb{C}^n$. The weights of $T_{p_i}\mathbb{P}^{n-1}$ are $\{\lambda_s-\lambda_i;\ s\neq i\}$, and the fiber at p_i is the set of matrices A with all entries in the ith column vanishing. The normalization axiom says that the multidegree of the fiber at p_i is $\prod_{j=1}^m(\theta_j-\lambda_i)$, so:

$$\mathrm{mdeg}[\Sigma_1,\mathrm{Hom}(n,k)]=\int_{\Sigma_1}\mathrm{Thom}_{(\mathbb{C}^*)^{n+m}}(\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^m))=$$

$$=\int_{\mathbb{P}^{n-1}}\int_{fiber}\mathrm{Thom}_{(\mathbb{C}^*)^{n+m}}=\sum_{i=1}^n\frac{\prod_{j=1}^m(\theta_j-\lambda_i)}{\prod_{s\neq i}(\lambda_s-\lambda_i)}$$

Consider the rational differential form

$$-\frac{\prod_{j=1}^m(\theta_j-z)}{\prod_{i=1}^n(\lambda_i-z)}\,dz.$$

The residues of this form at finite poles: $\{z=\lambda_i;\ i=1\dots n\}$ exactly recover the terms of the above sum. Applying the residue theorem, and change of variables $z=-1/q$, we get

$$\mathrm{mdeg}[\Sigma_1,\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^m)]=\mathrm{res}_{q=0}\frac{\prod_{j=1}^m(1+q\theta_j)}{\prod_{i=1}^n(1+q\lambda_i)}\,\frac{dq}{q^{m-n+2}}=c_{m-n+1},$$

where c_{m-n+1} is the $m-n+1$th Chern class of the difference bundle $f^*(TM)-TN$. This gives us the Thom polynomial $Tp^{n\to m}_{t\mathbb{C}[t]/t^2}=c_{m-n+1}$. Note that it depends only on $m-n$.

5. Computing multidegrees of singularities

Recall the following notations from the previous section.

- $J_k(n,m)=\{(p_1,\dots,p_m)\in\mathrm{Poly}(\mathbb{C}^n,\mathbb{C}^m):\deg p_i\leq k, p_i(0)=0\}$ is the set of k-jets of map germs.
- $\Sigma_k=\left\{p\in J_k(n,m):\mathbb{C}[x_1,\dots,x_n]/\langle p_1,\dots,p_m\rangle\cong z\mathbb{C}[z]/z^{k+1}\right\}$ the set of germs with A_k-singularity.
- $\mathcal{D}=J_k^{\mathrm{reg}}(n,n)\times J_k^{\mathrm{reg}}(m,m)$ naturally acts on $J_k(n,m)$ with $(A,B)p=BpA^{-1}$. Note that $\mathrm{GL}_n\times\mathrm{GL}_m\subset\mathcal{D}$

The goal now is to compute

$$\mathrm{Tp}_k^{n\to m}=\mathrm{mdeg}^{\mathrm{GL}_n\times\mathrm{GL}_m}[\Sigma_k,J_k(n,m)],$$

the Thom polynomial of Morin singularities.

The following theorem has first appeared in the work of Porteous and Gaffney, see [16].

Theorem 6 (The test curve model of Morin singularities).

$$\Sigma_k(n,m)\doteq\left\{\Psi\in J_k(n,m)|\exists\gamma\in J_k^{\mathrm{reg}}(1,n)\ such\ that\ \Psi\circ\gamma=0\ in\ J_k(1,m)\right\}.$$

Here $\doteq$ denotes birational equality, that is there Zariski closure are equal.

$$(\mathbb{C},0) \xrightarrow{\gamma} (\mathbb{C}^n,0) \xrightarrow{\Psi} (\mathbb{C}^m,0) \tag{5}$$

Note that if $\varphi \in J_k^{\mathrm{reg}}(1,1) = \mathbf{G}_k$, then

$$\Psi \circ \gamma = 0 \quad \Rightarrow \quad \Psi \circ (\gamma \circ \varphi) = 0$$

$$(\mathbb{C},0) \xrightarrow{\varphi} (\mathbb{C},0) \xrightarrow{\gamma} (\mathbb{C}^n,0) \xrightarrow{\Psi} (\mathbb{C}^m,0) \tag{6}$$

It can be shown that for $\Psi \in J_k(n,m)$ whose linear part has corank 1

$$\Psi \circ \gamma_1 = \Psi \circ \gamma_2 = 0 \Leftrightarrow \exists \alpha \in J_k^{\mathrm{reg}}(1,1) \text{ s.t } \gamma_1 = \gamma_2 \circ \alpha.$$

Therefore:

Proposition 3. *The Zariski open subset* $\Sigma_k^0 = \{\Psi \in \Sigma_k : \dim\ker\Psi = 1\} \doteq \Sigma_k$ *fibers with linear fibres over* $J_k^{reg}(1,n)/\mathbf{G}_k$.

What are these fibers, and why are they linear? If $\gamma = v_1 t + v_2 t^2 + \ldots + v_d t^d \in J_k^{\mathrm{reg}}(1,n)$ with $v_i \in \mathbb{C}^n$ and $v_1 \neq 0$ and $\Psi(v) = Av + Bv^2 + \ldots$ with $A \in \mathrm{Hom}(\mathbb{C}^n, C^k)$, $B \in \mathrm{Hom}(\mathrm{Sym}^2(\mathbb{C}^n), \mathbb{C}^k)$, *etc*, then $\Psi \circ \gamma = 0$ is equivalent with the following k equations:

$$\begin{gathered} A(v_1) = 0, \\ A(v_2) + B(v_1,v_1) = 0, \\ A(v_3) + 2B(v_1,v_2) + C(v_1,v_1,v_1) = 0, \\ \ldots \end{gathered} \tag{7}$$

For fixed $\gamma = (\gamma_1, \ldots, \gamma_k)$ these are linear equations determining the fiber. According to Proposition 3

$$\Sigma_k(n,m) \doteq \bigcup \left\{ \mathrm{Sol}_\gamma \mid \gamma \in J_k^{reg}(1,n) \right\},$$

where

$$\mathrm{Sol}_\gamma = \mathrm{Ann}(\gamma) \otimes \mathbb{C}^m \subset J_k(n,m)$$

is the annihilator tensored by $\mathbb{C}^k$.

To linearize the action of $\mathbf{G}_k$ let's make the following identifications

- Identify $J_k(1,n)$ with $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ by putting the coordinates $\gamma = (v_1, \ldots, v_k)$ into the columns of a matrix;
- Identify $J_k(n,1)$ with $\mathrm{Sym}^{\leq k}\mathbb{C}^n = \oplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n$, and then $J_k(n,m) = \mathrm{Sym}^{\leq k}\mathbb{C}^n \otimes \mathbb{C}^m$.

Then $\mathbf{G}_k$ acts on $J_k(1,n)$ by multiplication on the right by the following matrix group:

$$\left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{k-1} + \ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{k-2} + \ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^k \end{pmatrix} : \alpha_1 \in \mathbb{C}^*, \alpha_i \in \mathbb{C} \right\}; \tag{8}$$

where the polynomial in the (i, j) entry is

$$p_{i,j}(\bar{\alpha}) = \sum_{a_1+a_2+\ldots+a_i=j} \alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_i}.$$

This group is the central object of our study in this paper. It is a non-reductive linear subgroup of GL_k, and therefore Mumford's geometric invariant theory does not help us in handling the quotient $J_k^{\mathrm{reg}}(1,n)/\mathbf{G}_k$. The following construction, which was the starting point of a general construction in [6] first appeared in [3].

Define the map

$$\rho : \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \to \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^{\leq k}\mathbb{C}^n) \tag{9}$$

$$\rho(v_1, \ldots, v_k) = (v_1, v_2 + v_1^2, \ldots, \sum_{a_1+a_2+\ldots+a_i=k} v_{a_1}v_{a_2}\ldots v_{a_i}),$$

where in the jth coordinate we sum over all ordered partitions of j into positive integers. Note that these correspond to the monomials in jth column of the matrix $\mathbf{G}_k$. For more details see [6].

Theorem 7 ([3]). *Let* $\mathrm{Hom}^0(\mathbb{C}^k, \mathbb{C}^n) = \{(v_1, \ldots, v_k \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) : v_1 \neq 0\} = J_k^{reg}(1,n)$. *Then ρ (defined in* (9)*) descends to an injective map on the orbits*

$$\rho^{\mathrm{Grass}} : \mathrm{Hom}^0(\mathbb{C}^k, \mathbb{C}^n)/\mathbf{G}_k \hookrightarrow \mathrm{Grass}(k, \mathrm{Sym}^{\leq k}\mathbb{C}^n).$$

and therefore descends also to

$$\rho^{\mathrm{Flag}} : \mathrm{Hom}^0(\mathbb{C}^k, \mathbb{C}^n)/\mathbf{G}_k \hookrightarrow \mathrm{Flag}_k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$$

Composing with the Plucker embedding we get

$$\rho^{\mathrm{Proj}} = Pluck \circ \rho^{\mathrm{Grass}} : \mathrm{Hom}^0(\mathbb{C}^k, \mathbb{C}^n)/\mathbf{G}_k \hookrightarrow \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$$

Note that ρ is GL_n-equivariant with respect to the multiplication on the left on $\mathrm{Hom}^0(\mathbb{C}^k, \mathbb{C}^n)/\mathbf{G}_k$ and the induced action on $\mathrm{Grass}(n, \mathrm{Sym}^{\leq k}\mathbb{C}^n)$ coming from the standard action on $\mathbb{C}^n$.

This embedding allows us to give a geometric description of some generators in the invariant ring

$$\mathbb{C}[J_k^{reg}(1,n)]^{U_k} \subset \mathbb{C}[f', \ldots, f^{(k)}],$$

where $U_k \subset \mathbf{G}_k$ is the maximal unipotent subgroup. Namely, the coordinate ring of the image is a subring of the invariant ring:

$$\mathbb{C}[\mathrm{im}(\rho)] \subset \mathbb{C}[J_k^{reg}(1,n)]^{U_k}$$

$\mathbb{C}[J_k^{reg}(1,n)]^{U_k}$ has been long studied in relation with the Green-Griffiths conjecture. In his seminal paper [10], Demailly suggested a strategy to the Green-Griffiths conjecture through the investigation of the invariant jet differentials. These are sections of a bundle, whose fibers are canonically isomorphic to the invariant ring $\mathbb{C}[J_k^{reg}(1,n)]^{U_k}$. It is a major unsolved problem to prove the finite generation of this ring, and to find the generators. The main obstacle is that $\mathbf{G}_k$ is a non-reductive group, and therefore classical

Geometric Invariant Theory ([28]) does not apply. For an introduction on non-reductive group actions see [13].

Following Demailly's notation, let $(f', f'', \ldots, f^{(k)}) \in J_k^{\mathrm{reg}}(1,n)$ denote the k-jet of a germ f and $f_i^{(j)}$ denote the ith coordinate of the jth derivative, $1 \le i \le n, 1 \le j \le k$. This is a simple rescaling, namely $v_i = f^{(i)}/i!$.

Theorem 8 ([5])**.** *Let*

$$I = (\Delta_{\mathbf{i}_1,\ldots,\mathbf{i}_k}(f) : (\mathbf{i}_1, \ldots, \mathbf{i}_k) \in \binom{\dim(\mathrm{Sym}^{\le k}(n))}{k}) \lhd \mathbb{C}[f', \ldots, f^{(k)}]$$

be the ideal generated by the $k \times k$ minors of $\rho(f' \ldots, f^{(k)}) \in \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^{\le k}\mathbb{C}^n)$. Then

$$I \subset \mathbb{C}[J_k^{reg}(1,n)]^{U_k}$$

Example 6. *$n = 2, k = 4$. In this case*

$$J_4^{\mathrm{reg}}(1,2) = \{(f_1', f_2', f_1'', f_2'', f_1''', f_2''', f_1'''', f_2'''') \in (\mathbb{C}^2)^4; (f_1', f_2') \neq (0,0)\},$$

and fixing a basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ and

$$\{e_1, e_2, e_1^2, e_1e_2, e_2^2, e_1^3, \ldots, e_1e_2^4, e_2^4\}$$

of $\mathrm{Sym}^{\le 4}\mathbb{C}^2$ the map $\rho : J_4(1,2) \to \mathrm{Hom}(\mathbb{C}^4, \mathrm{Sym}^{\le 4}\mathbb{C}^2)$ sends

$$(f_1', f_2', f_1'', f_2'', f_1''', f_2''', f_1'''', f_2''''$$

to a 4×15 matrix, whose first 5 columns (corresponding to $\mathrm{Sym}^{\le 2}\mathbb{C}^2$) are

$$\begin{pmatrix} f_1' & f_2' & 0 & 0 & 0 \\ \frac{1}{2!}f_1'' & \frac{1}{2!}f_2'' & (f_1')^2 & f_1'f_2' & (f_2')^2 \\ \frac{1}{3!}f_1''' & \frac{1}{3!}f_2''' & f_1'f_1'' & (f_1'f_2'' + f_1''f_2') & f_2'f_2'' \\ \frac{1}{4!}f_1'''' & \frac{1}{4!}f_2'''' & \frac{2}{3!}f_1'f_1''' + \frac{1}{2!2!}(f_1'')^2 & \frac{2}{3!}(f_1'f_2''' + f_1'''f_2') + \frac{1}{2!}f_1''f_2'' & \frac{2}{3!}f_2'f_2''' + \frac{1}{2!2!}(f_2'')^2 \end{pmatrix},$$

and next four columns (corresponding to $\mathrm{Sym}^3\mathbb{C}^2$) are

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (f_1')^3 & (f_1')^2f_2' & f_1'(f_2')^2 & (f_2')^3 \\ \frac{3}{2!}((f_1')^2f_1'') & \frac{3}{2!}((f_1')^2f_2'' + 2f_1'f_2'f_1'') & \frac{3}{2!}((f_2')^2f_1'' + 2f_2'f_1'f_2'') & \frac{3}{2!}((f_2')^2f_2'') \end{pmatrix},$$

and the remaining five columns (corresponding to $\mathrm{Sym}^3\mathbb{C}^3$) are

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (f_1')^4 & (f_1')^3f_2' & (f_1')^2(f_2')^2 & f_1'(f_2')^3 & (f_2')^4 \end{pmatrix}.$$

Then the weight $1 + 2 + 3 + 4 = 10$ piece $\mathbb{C}[J_4(1,2)]_{10}^{U_4}$ of the invariant algebra $\mathbb{C}[J_4(1,2)]^{U_4}$ is generated by the 4×4 minors of this 4×15 matrix.

5.1. **The computation: double localization+vanishing theorem.** According to Proposition 3 and Theorem 7 we have the following picture, also called the snowman-model after the figure in Section §6 in [3]:

$$\begin{array}{cc}
\Sigma_k(n,m) & \subset J_k(n,m) \\
\downarrow & \\
J_k^{\mathrm{reg}}(1,n)/\mathbf{G}_k & \subset \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n) \\
\downarrow & \\
J_k^{\mathrm{reg}}(1,n)/B_k = \mathrm{Flag}_k(\mathbb{C}^n) &
\end{array} \tag{10}$$

Here $B_k \subset GL_k$ is the upper Borel subgroup. Now we apply ABBV localization to compute $\mathrm{mdeg}[\Sigma_k(n,m), J_k(n,m)]$. According to Vergne, we have to compute $\int_{\Sigma_k(n,m)} \mathrm{Thom}(J_k(n,m),$ and we do this in two steps: first we localize on $\mathrm{Flag}_k(\mathbb{C}^n)$ and use Proposition 2 to turn the localization formula into an iterated residue. Then we integrate along the fibers. The fibers are canonically isomorphic to $B_k/\mathbf{G}_k$ and in the second step we apply ABBV localization on the image $\overline{\rho(fiber)} \subset \mathrm{Grass}(k, \mathrm{Sym}^{\le k}\mathbb{C}^n)$. Surprisingly—for some unclear geometric reason—in this second localization all fixed points but a distinguished one contributes 0 to the sum, and a lenghty computation leads us in [3] to

Theorem 9 ([3])**.**

$$\mathrm{Tp}_k^{m-n} = \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j)\, Q_k(z_1 \dots z_k)}{\prod_{i+j\le l\le k}(z_i + z_j - z_l)} \cdot \prod_{l=1}^{k} c\left(\frac{1}{z_l}\right) z_l^{m-n}\, dz_l, \tag{11}$$

where

- *We integrate on the cycle* $|z_1| > |z_2| > \dots |z_k|$, *which determines the Laurent expansion.*
- $c(q) = 1 + c_1 q + c_2 q^2 + \dots$
- $Q_k(z_1, \dots, z_k)$ *is the multidegree of a Borel-orbit in* $(\mathbb{C}^k)^* \otimes \mathrm{Sym}^2(\mathbb{C}^k)$, *for details see* [3], *and*

$$Q_1 = Q_2 = Q_3 = 1, Q_4 = 2z_1 + z_2 - z_4$$

The polynomial Q_k is known up to $k \le 6$, but with enough computer capacity—in principle—it can be computed for any k. But no general formula is known at the moment.

We give a concise–and not complete–summary of the history of Thom polynomial computations. For a more detailed overview see [3, 20].

- Multidegrees of singularities have been studied for nearly 60 years now. We call these Thom polynomials in the honour of René Thom and his pioneering work in the 1950's. He proved the existence of these polynomials ([37]). He studied real manifolds and singularities of differentiable maps between them. Later Damon in [9] studied complex contact singularities.
- The case $k = 1$ is the classical formula of Porteous: $\mathrm{Tp}_1^{n\to m} = c_{m-n+1}$. The $k = 2$ case was computed by Ronga in [32]. An explicit formula for $\mathrm{Tp}_3^{n\to k}$ was proposed in [2] and P. Pragacz has given a proof [29]. He also studied Thom polynomials in [30, 22], the latter written with A. Lascoux. Finally, using his method of restriction equations, Rimányi [31] was able to treat the $n = k$ case, and computed $\mathrm{Tp}_k^{n\to n}$ for $k \le 8$ (cf. [16] for the case $d = 4$).
- More recently, Kazarian ([21]) has worked out a framework for computing Thom polynomials of contact singularities in general. He suggests studying certain non-commutative associative algebras to get a polynomial Q_A and an iterated residue formula similar to (11) for any local algebra A. Unfortunately, the explicit computation of Q_A is difficult, his description does not give more information for Morin singularities, where Q_k is unknown for $k > 6$. The structure of Thom polynomials of contact singularities was also studied in [14, 15].

Finally, let us mention a conjecture of R. Rimányi about the positivity of these Thom polynomials.

Conjecture 1 (**Rimányi, 1998**).

$$\mathrm{Tp}_k^{m-n} \in \mathbb{N}[c_1, \dots, c_{k(m-n+1)}],$$

i.e. the coefficients of the Thom polynomials are nonnegative. This would follow from the more general conjecture, that

$$\frac{\prod_{i<j}(z_i - z_j)\, Q_k(z_1 \dots z_k)}{\prod_{i+j\le l\le k}(z_i + z_j - z_l)} > 0,$$

the coefficients of the Thom series are nonnegative.

6. The Green-Griffiths conjecture

First we list some results related to hyperbolic varieties and the Green-Griffiths conjecture. This is a selection of classical results and it is far from being complete.

6.1. **Hyperbolic varieties.** Let X be a complex manifold, $n = \dim_{\mathbb{C}}(X)$. X is said to be hyperbolic

- in the sense of Brody, if there are no non-constant entire holomorphic curves $f : \mathbb{C} \to X$.

- in the sense of Kobayashi, if the Kobayashi-Royden pseudo-metric on T_X is nondegenerate. This pseudo-metric is defined as follows. The infinitesimal Kobayashi-Royden metric is

  $$k_X(\xi) = \inf\{\lambda > 0 : \exists f : \Delta \to X, f(0) = x, \lambda f'(0) = \xi\} \text{ for } x \in X, \xi \in T_{X,x}.$$

  The Kobayashi pseudo-distance $d(x, y)$ is the geodesic pseudo-distance obtained by integrating the Kobayashi-Royden infinitesimal metric. X is hyperbolic in the sense of Kobayashi if $d(x, y) > 0$ for $x \neq y$.

The following theorem of Kobayashi tells that positivity of the cotangent bundle implies hyperbolicity.

Theorem 10 (**Kobayashi, '75**). *X-smooth projective variety with ample cotangent bundle. Then X is hyperbolic.*

Conversely,

Conjecture 2. *If a compact manifold X is hyperbolic, then it should be of general type, i.e. $K_X = \wedge^n T^*X$ should be big. (That is, X has maximal Kodaira dimension, i.e. $\dim \oplus_{i=0}^{\infty} H^0(X, K^i) = \dim X$.)*

Conjecture 3 (**Green-Griffiths, '79**). *Let X be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.*

Diophantine properties

Theorem 11 (**Faltings, '83**). *A curve of genus greater than 1 has only finitely many rational points.*

Theorem 12 (**Moriwaki, '95**). *Let K be a number field (finitely generated over $\mathbb{Q}$), and X a smooth projective variety. If T^*X is ample and globally generated then $X(K)$ is finite.*

Conjecture 4 (**S. Lang**).
1. *If a projective variety X is hyperbolic, then it is mordellic, i.e. $X(K)$ is finite for any K finitely generated over $\mathbb{Q}$.*
2. *Let $Exc(X) = \cup\overline{\{f(\mathbb{C}) : f : \mathbb{C} \to X\}}$. Then $X \setminus Exc(X)$ is mordellic.*

Highlights in the history of the Green-Griffiths conjecture

Here is a short (incomplete) list of results related to the Green-Griffiths conjecture, which first appeared in [17].

- In [24] McQuillen gave a positive answer to the conjecture for general surfaces if the second Segre class $c_1^2 - c_2 > 0$ is positive.
- In the seminal paper [10] Demailly—using ideas of Green, Griffiths and Bloch—works out a strategy for projective hypersurfaces.
- In [33, 34] Siu gives positive answer for hypersurfaces of high degree, without effective lower bound for the degree.

- In [12] Diverio, Merker and Rousseau give effective lower bound, proving that for a generic projective hypersurface of dimension n and degree $> 2^{n^5}$ the Green-Griffiths conjecture holds.
- Recently, Merker ([26]) has proved the existence of global jet differentials of high order for projective hyperpersurfaces in the optimal degree. Demailly in [11] has proved the existence of global jet differentials (of possibly high order) for compact manifolds in general.

6.2. **A promising strategy.** (**Green, Griffiths, Demailly, Siu, Diverio, Merker, Rousseau**)

The main idea of this strategy is to find differential equations which must be satisfied by (the jet of) any entire holomorphic curve in X, and then to find enough independent equations such that their solution set is a proper subvariety of X. For more details on the history of this approach see [12, 10].

Let

$$f:\mathbb{C}\to X,\ \ t\to f(t)=(f_1(t),f_2(t),\ldots,f_n(t))$$

be a curve written in some local holomorphic coordinates $(z_1,\ldots,z_n)$ on X. Let J_kX be the k-jet bundle over X of holomorphic curves, whose fiber at $x\in X$ is

$$(J_kX)_x=\{\hat{f}_{[k]}:f:(\mathbb{C},0)\to(X,x)\}\to X$$

sending $f_{[k]}$ to $f(0)$. This fibre is canonically isomorphic to $J_k(1,n)$.

The group of reparametrizations $\mathbf{G}_k=J_k^{\mathrm{reg}}(1,1)$ acts fiberwise on J_kX. The fibres of J_kX can be identified with $J_k(1,n)$, and the action is linearised as in (8) before. Note that $\mathbf{G}_k=\mathbb{C}^*\ltimes U_d$ is a $\mathbb{C}^*$-extension of its maximal unipotent subgroup, and for $\lambda\in\mathbb{C}^*$

$$(\lambda\cdot f)(t)=f(\lambda\cdot t),\ \text{so}\ \lambda\cdot(f',f'',\ldots,f^{(k)})=(\lambda f',\lambda^2f'',\ldots,\lambda^kf^{(k)}).$$

Algebraic differential operators correspond to polynomial functions on J_kX, and we call these polynomial functions jet differentials, they have the form

$$Q(f',f'',\ldots,f^{(k)})=\sum_{\alpha_i\in\mathbb{N}^n}a_{\alpha_1,\alpha_2,\ldots\alpha_k}(f(t))(f'(t)^{\alpha_1}f''(t)^{\alpha_2}\cdots f^{(k)}(t)^{\alpha_k}),$$

where $a_{\alpha_1,\alpha_2,\ldots\alpha_k}(z)$ are holomorphic coefficients on X and $t\to z=f(t)$ is a curve.

Q is homogeneous of weighted degree m under the $\mathbb{C}^*$ action if and only if

$$Q(\lambda f',\lambda^2f'',\ldots,\lambda^kf^{(k)})=\lambda^mQ(f',f'',\ldots,f^{(k)}).$$

Definition 3.
- *(**Green-Griffiths '78**) Let $E_{k,m}^{GG}$ denote the sheaf of algebraic differential operators of order k and weighted degree m.*
- *(**Demailly, '95**) The bundle of invariant jet differentials of order k and weighted degree m is the subbundle $E_{k,m}\subset E_{k,m}^{GG}$, whose elements are invariant under arbitrary changes of parametrization, i.e. for $\phi\in\mathbf{G}_k$*

$$Q((f\circ\phi)',(f\circ\phi)'',\ldots,(f\circ\phi)^{(k)})=\phi'(0)^mQ(f',f'',\ldots,f^{(k)}).$$

We want to apply the general pinciple that for a G-space X the ring of invariant functions on X can be identified with polynomial functions on the quotient X/G. Roughly speaking we want

$$\oplus_m (E_{k,m})_x = \oplus_m (E^{GG}_{k,m})^{\mathbb{U}}_x = \mathcal{O}((J_k X)_x)^{\mathbf{G}_k} = \mathcal{O}(J_k(1,n)/\mathbf{G}_k)$$

Applying Theorem 7 fibrewise we get

Proposition 4. (1) *The quotient $J_k X/\mathbf{G}_k$ has the structure of a locally trivial bundle over X, and there is a holomorphic embedding*

$$\phi^{\mathbb{P}} : J_k X/\mathbf{G}_k \hookrightarrow \mathbb{P}(\wedge^k (T_X^* \oplus \mathrm{Sym}^2(T_X^*) \oplus \ldots \oplus \mathrm{Sym}^k(T_X^*)).$$

The fibrewise closure of the image $\mathcal{X}_k = \overline{\mathrm{im}\phi^{\mathbb{P}}}$ is a relative compactification of $J_k(T_X^*)/\mathbf{G}_k$ over X.

(2) *We have*

$$(\pi_k)_* \mathcal{O}_{\mathcal{X}_k}(m) = \mathcal{O}(E_{k,m\binom{k+1}{2}})$$

where $\pi_k : \mathbb{P}(\wedge^k (T_X^* \oplus \mathrm{Sym}^2(T_X^*) \oplus \ldots \oplus \mathrm{Sym}^k(T_X^*))) \to X$ is the projection.

The strategy to solve the Green-Griffiths conjecture is based on the following

Theorem 13 (**Fundamental vanishing theorem ,Green-Griffiths '78, Demailly '95, Siu '96**)**.** *Let $P \in H^0(X, E_{k,m} \otimes \mathcal{O}(-A))$ be a global algebraic differential operator whose coefficients vanish on some ample divisor A. Then for any $f : \mathbb{C} \to X$, $P(f_{[k]}(\mathbb{C})) \equiv 0$. (Note that $f_{[k]}(\mathbb{C}) \subset J_k X$.)*

Corollary 1. (1) *Let σ be a nonzero element of*

$$H^0(\mathcal{X}_k, \mathcal{O}_{\mathcal{X}_k}(m) \otimes \pi^* \mathcal{O}(-A)) \simeq H^0(X, E_{k,m\binom{k+1}{2}} \otimes \mathcal{O}(-A)).$$

Then $f_{[k]}(\mathbb{C}) \subset Z_\sigma$, where $Z_\sigma \subset \mathcal{X}_d$ is the zero divisor of σ.

(2) *If $\{\sigma_j\}$ is a basis of global sections then the image $f(\mathbb{C})$ lies in $Y = \pi_k(\bigcap Z_{P_j})$, hence the Green-Griffiths conjecture holds if there are enough independent differential equations so that $Y = \pi_k(\bigcap(Z_{P_j})) \subsetneq X$.*

It is crucial to control in a more precise way the order of vanishing of these differential operators along the ample divisor. Thus, we need here a slightly different theorem.

Theorem 14 ([12])**.** *Assume that $n = k$, and there exist a $\delta = \delta(n) > 0$ and $D = D(n, \delta)$ such that*

$$H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}(m) \otimes \pi^* K_X^{-\delta m}) \simeq H^0(X, E_{n,m\binom{n+1}{2}} T_X^* \otimes K_X^{-\delta m}) \neq 0$$

whenever $\deg(X) > D(n, \delta)$ *provided that $m > m_{D,\delta,n}$ is large enough. Then the Green-Griffiths conjecture holds for*

$$\deg(X) \geq \max(D(n, \delta), \frac{n^2 + 2n}{\delta} + n + 2).$$

The goal is therefore to find a global section of $\mathcal{O}_{\mathcal{X}_n}(m) \otimes \pi^* K_X^{-\delta m}$ keeping $D(n, \delta)$ small. Following [12], we use the algebraic Morse inequalities of Demailly/Trapani.

Theorem 15. [36] *Let $L \to X$ be a holomorphic line bundle given as*

$$L = F \otimes G^{-1} \text{ with } F, G \text{ nef bundles.}$$

Then for any nonnegative integer q we have

$$\sum_{j=0}^{q}(-1)^{q-j}h^j(X, L^{\otimes m} \otimes E) \leq r\frac{m^n}{n!}\sum_{j=0}^{q}(-1)^{q-j}\binom{n}{j}F^{n-j}\cdot G^j + o(m^n).$$

Applying this with $q = 1$ we get

$$F^n - nF^{n-1}G > 0 \Rightarrow H^0(L^{\otimes m}) \neq 0 \text{ for } m \gg 0. \tag{12}$$

In [4] we prove that F and G are nef bundles in the following equality.

$$\underbrace{\mathcal{O}_{\mathcal{X}_n}(1) \otimes \pi^* K_X^{-\delta\binom{n+1}{2}}}_{L} = \underbrace{(\mathcal{O}_{\mathcal{X}_n}(1) \otimes \pi^*\mathcal{O}_X(2n^2))}_{F} \otimes \underbrace{(\pi^*\mathcal{O}_X(2n^2) \otimes \pi^* K_X^{\delta\binom{n+1}{2}})^{-1}}_{G}.$$

Introduce the following notations:

$$h = c_1(\mathcal{O}_X(1));\ c_1(K_X) = -c_1(X) = (d-n-2)h;\ \mathcal{O}_{\mathcal{X}_n}(1) = \det\tau$$

where $\tau \to \mathcal{X}_n$ is the tautological n-bundle. Now $\dim(\mathcal{X}_n) = n^2$, and according to (12) we want to prove the positivity of the following intersection number on $\mathcal{X}_n$.

$$\int_{\mathcal{X}_n}(c_1(\det\tau) + 2n^2\pi^*h)^{n^2} - n^2(c_1(\det\tau) + 2n^2\pi^*h)^{n^2-1}(2n^2\pi^*h + \delta\binom{n+1}{2}(d-n-2)h).$$

We apply localization using the double fibration model (10) on the fibers of $\mathcal{X}_n$. We need a stronger version of the vanishing property of the iterated residue to guarantee that only one fixed point's contribution is non-zero. After going through these technical difficulties in [4] we arrive at

Residue formula for the Demailly intersection number

$$I = \int_X \operatorname{Res}_{\mathbf{z}=\infty}\frac{\prod_{i<j}(z_i - z_j)\,Q_d(z_1\dots z_n)R(\mathbf{z}, h, d, \delta)}{\prod_{1\leq i+j\leq l\leq n}(z_i + z_j - z_l)(z_1\dots z_n)^n}\cdot$$
$$\cdot\prod_{l=1}^{n}\left(1 + \frac{dh}{z_l}\right)\prod_{l=1}^{n}\left(1 - \frac{h}{z_l} + \frac{h^2}{z_l^2} - \dots\right)^{n+2}$$

where

$$R(\mathbf{z}, h, d, \delta) = (-z_1 - \dots - z_n + 2n^2h)^{n^2} -$$
$$- n^2(-z_1 - \dots - z_n + 2n^2h)^{n^2-1}(2n^2h + \delta\binom{n+1}{2}(d-n-2)h).$$

Analysis of the formula

- The iterated residue is the coefficient of $\frac{1}{z_1\dots z_n}$, and has the form $h^n p(d, n, \delta)$.
- Integration on X is the substitution $h^n = d$, so the result is $dp(d, n, \delta)$.

- $p(n, d, \delta) = a_n(n, \delta)d^n + \ldots + a_0(n, \delta)$ is a degree-n polynomial in $d = \deg(X)$, with polynomial coefficients in n, δ.
- The leading coefficient is

$$a_n(n, \delta) = \left(1 - n^2\binom{n+1}{2}\delta\right)\Theta(n),$$

where

$$\Theta(n) = \text{constant term of } \frac{Q(\mathbf{z})\prod_{i<j}(z_i - z_j)(z_1 + \ldots + z_n)^{n^2}}{\prod_{i+j\le l\le n}(z_i + z_j - z_l)(z_1 \ldots z_n)^n}$$

Note that

$$\Theta(n) = \int_{X_n} c_1(\tau)^{n^2} > 0$$

is positive, as τ is an ample bundle. Therefore

Corollary 2. *For $\delta < \frac{2}{n^3(n+1)}$ the leading coefficient of the Demailly intersection number is positive.*

The backgroud and experimental evidences of the following conjecture is explained in [4]. It says that quotients of "neighbouring" coefficients of the Thom polynomial is polynomial.

Conjecture 5. *Define*

$$\mathrm{Tp}_k(z_1, \ldots, z_k) = \frac{\prod_{m<l}(z_m - z_l)\, Q_k(z_1 \ldots z_k)}{\prod_{m+r\le l\le k}(z_m + z_r - z_l)}$$

Then

$$\frac{\mathrm{coeff}_{z_1^{i_1}\ldots z_k^{i_k}} Tp_k}{\mathrm{coeff}_{z_1^{i_1}\ldots z_l^{i_l+1}\ldots z_m^{i_m-1}\ldots z_k^{i_k}} Tp_k} < k^2$$

Some further computations in [4] leads us to

Theorem 16 ([4]). *Conjecture 5 and Conjecture 1 for Thom polynomials of A_n singularities implies the Green-Griffiths conjecture for $d = \deg(X) > n^6$.*

7. Appendix

The given iterated residue formula is suitable to compute other intersection numbers as well. The Euler characteristic of the Demailly bundle is defined as

$$\chi(X, E_{k,m}T_X^*) = \sum_{i=0}^{n}(-1)^i \dim H^i(X, E_{k,m}T_X^*).$$

It is well-known (see [19]) that

$$\chi(X, E_{k,m}) = \int_X [\mathrm{Ch}(E_{k,m}) \cdot \mathrm{Td}(T_X)]_n$$

where $\mathrm{Ch}(\mathcal{O}_{X_n}(1))$ is the Chern character and $\mathrm{Td}(T_X) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \ldots$ is the Todd-class.

Theorem 17 (**Iterated residue formula for the Euler-characteristics**).

$$\chi(X, \pi_* \mathcal{O}_{X_n}(m)) = \int_X \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j)\, Q_n(z_1 \ldots z_n) \mathrm{Ch}(\mathcal{O}_{X_n}(m)) \mathrm{Td}(T_X)}{\prod_{1 \le i+j \le l \le n}(z_i + z_j - z_l)(z_1 \ldots z_n)^n} \cdot \cdot \prod_{l=1}^{n} \left(1 + \frac{dh}{z_l}\right) \prod_{l=1}^{n} \left(1 - \frac{h}{z_l} + \frac{h^2}{z_l^2} - \ldots\right)^{n+2}$$

where

$$\mathrm{Ch}(\mathcal{O}_{X_n}(1)) = e^{m(z_1+\ldots+z_n)}, \ \mathrm{Td}(T_X) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \ldots$$

8. Curviliear Hilbert schemes

The goal of this section is to give a general framework for our localization arguments. If $G_k = J_k^{\mathrm{reg}}(1,1)$ denotes the group of k-jets of reparametrization germs of $\mathbb{C}$ and $J_k^{\mathrm{reg}}(1,n)$ the k-jets of germs of curves $f : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$, then the quotient $J_k^{\mathrm{reg}}(1,n)/J_k^{\mathrm{reg}}(1,1)$ plays an important role in our applications, namely:

(1) Σ_k fibers over $J_k^{\mathrm{reg}}(1,n)/J_k^{\mathrm{reg}}(1,1)$ with linear fibers. The Thom polynomials of Morin singularities are certain equivariant intersection numbers on Σ_k.
(2) $J_k^{\mathrm{reg}}(1,n)/J_k^{\mathrm{reg}}(1,1)$ is isomorphic to the fibers of the Demailly jet bundle E_k over a smooth manifold of dimension n. The positivity of the Demailly intersection number implies the Green-Griffiths conjecture.

In both applications we compute certain (equivariant) intersection numbers on the quotient $J_k^{\mathrm{reg}}(1,n)/J_k^{\mathrm{reg}}(1,1)$, using equivariant localization on $\phi^{\mathrm{Grass}}(J_k^{\mathrm{reg}}(1,n)/J_k^{\mathrm{reg}}(1,1))$.

Let $\mathcal{H}_0(k,n))$ be the punctual Hilbert scheme of k points on $\mathbb{C}^n$, that is, the set of zero dimensional subschemes of $\mathbb{C}^n$ of length k supported at the origin. There is an important subset of $\mathcal{H}_0(k,n))$, namely the punctual curvilinear Hilbert scheme, defined as follows

Definition 4. *The punctual curvilinear Hilbert scheme is the closure of the set of ideals*

$$\mathcal{C}(k,n) = \{I \subset \mathbb{C}[x_1, \ldots, x_n] : \mathbb{C}[x_1, \ldots, x_n]/I \simeq t\mathbb{C}[t]/t^{k+1}\},$$

that is

$$\mathcal{CH}(k,n) = \overline{\mathcal{C}(k,n)}.$$

If $\mathfrak{m} = (x_1, \ldots, x_n) \subset \mathcal{O}_{\mathbb{C}^n,0}$ denotes the maximal ideal at the origin, then

$$\mathrm{Sym}^{\leq k}\mathbb{C}^n := \mathfrak{m}/\mathfrak{m}^{k+1} = \oplus_{i=1}^{k}\mathrm{Sym}^i\mathbb{C}^n$$

is the set of function-germs of degree $\leq n$, and the punctual Hilbert scheme naturally sits in the Grassmannian

$$\mathcal{H}_0(k,n) \subset \mathrm{Grass}(k, \mathrm{Sym}^{\leq k}\mathbb{C}^n).$$

Looking at our embedding ϕ^{Grass} it is not hard to check (see [6]) that

Proposition 5. *We have*

$$C\mathcal{H}(k,n) = \overline{\phi^{\mathrm{Grass}}(J_k^{\mathrm{reg}}(1,n)/J_k^{\mathrm{reg}}(1,1))}.$$

This roughly means that $C\mathcal{H}(k,n)$ can be described as certain compactification of a non-reductive quotient.

When $n = 2$ we furthermore know that the punctual curvilinear component $C\mathcal{H}(k,n)$ is dense in $\mathcal{H}_0(k,n)$, and therefore

Corollary 3. *We have*

$$\mathcal{H}_0(k,2) = \overline{\phi^{\mathrm{Grass}}(J_k^{\mathrm{reg}}(1,2)/J_k^{\mathrm{reg}}(1,1))}.$$

We have developed localization methods to compute intersection numbers on the punctual curvilinear Hilbert scheme $C\mathcal{H}(k,n)$ for $k \leq n$ A more detailed study of non-reductive quotients allows us to improve this technique, the details with more applications will be published later.

References

[1] V. I. Arnold, V. V. Goryunov, O. V. Lyashko, V. A. Vasilliev, Singularity theory I. *Dynamical systems VI, Encyclopaedia Math. Sci., Springer-Verlag, Berlin, 1998.*

[2] G. Bérczi, L. M. Fehér, R. Rimányi, Expressions for resultants coming from the global theory of singularities, *Topics in algebraic and noncommutative geometry, Contemp. Math., 324, (2003) 63-69.*

[3] G. Bérczi, A. Szenes, Thom polynomials of Morin singularities, *arxiv:math/0608285.*

[4] G. Bérczi, Thom polynomials and the Green-Griffiths conjecture, *ArXiv:1011.4710.*

[5] G. Bérczi, F. C. Kirwan, A geometric construction for invariant jet differentials, *arXiv:1012.1797.*

[6] G. Bérczi, F. Kirwan, A Grassmannian model for non-reductive quotients *in preparation.*

[7] N. Berline, E. Getzler, M. Vergne, Heat kernels and Dirac operators, *Springer, 1992.*

[8] N. Berline, M. Vergne, Zéros d'un champ de vecteurs et classes characteristiques équivariantes, *Duke Math. J. 50 no.2 (1973), 539-549.*

[9] J. Damon, Thom polynomials for contact class singularities, *Ph.D. Thesis, Harvard University, 1972.*

[10] J.-P. Demailly, Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials, *Proc. Sympos. Pure Math. 62 (1982), Amer. Math. Soc., Providence, RI, 1997, 285-360.*

[11] J.-P. Demailly, Holomorphic Morse inequalities and the Green-Griffiths-Lang conjecture, *arXiv:1011.3636*

[12] S. Diverio, J. Merker, E. Rousseau, Effective algebraic degeneracy, Invent. Math. 180(2010) 161-223.

[13] B. Doran, F. Kirwan, Towards non-reductive geometric invariant theory, *Pure Appl. Math. Q. 3(2007), 61-105.*

[14] L. M. Fehér, R. Rimányi, Thom olynomial computing strategies. A survey, *Adv. Studies in Pure Math. 43, Singularity Theory and Its Applications, Math. Soc. Japan, (2006), 45-53.*

[15] L. M. Fehér, R. Rimányi, On the Structure of Thom polynomials of Singularities, *Bull. London Math. Soc. 39, (2007) 541-549.*

[16] T. Gaffney, The Thom polynomial of P^{1111}, *Singularities, Part 1, Proc. Sympos. Pure Math., 40, (1983), 399-408.*

[17] M. Green, P. Griffiths, Two applications of algebraic geometry to entire holomorphic mappings, *The Chern Symposium 1979. (Proc. Intern. Sympos., Berkeley, California, 1979) 41-74, Springer, New York, 1980.*

[18] A. Haefliger, A. Kosinski, Un thèorème de Thom sur les singularitès des applications diffèrentiables, *Sèminaire Henri Cartan, 9 Exposè 8, (1956-57).*

[19] Hirzebruch, Topological methods in algebraic geometry, *Grundlehren der Mathematische Wissenschaften 131, Springer.*

[20] M. Kazarian, Thom polynomials, *Lecture notes of talks given at the Singularity Theory Conference, Sapporo, 2003.*

[21] M. Kazarian, not published

[22] L. Lascoux, P. Pragacz, Thom polynomials and Schur functions: the singularities A_3, *Publ. RIMS Kyoto Univ. 46(2010) 183-200.*

[23] M. McQuillan, Diophantine approximation and foliations, *Inst. Hautes Études Sci. Publ. Math. 87 (1998), 121-174.*

[24] M. McQuillan, Holomorphic curves on hyperplane sections of 3-folds, *Geom. Funct. Anal. 9 (1999), 370-392.*

[25] J. Merker, Applications of computational invariant theory to Kobayashi hyperbolicity and to Green-Griffiths algebraic degeneracy, *Journal of Symbolic Computations, 13(3), 255-299.*

[26] J. Merker, Complex projective hypersurfaces of general type: toward a conjecture of Green and Griffiths, *arXiv:1005.0405*

[27] E. Miller, B. Sturmfels, Combinatorial Commutative Algebra, *Springer Verlag, 2004.*

[28] D. Mumford, J. Fogarty, F. Kirwan, Geometric Invariant Theory, *3rd edition, Springer Verlag, 1994.*

[29] P. Pragacz, Thom polynomials and Schur-functions I. *math.AG/0509234, 2005*

[30] P. Pragacz, Thom polynomials and Schur-functions: towards the singularities A_i, *Contemporary Mathematics, 459(2008), 165-178*

[31] R. Rimányi, Thom polynomials, symmetries and incidences of singularities, *Invent. Math. 143 (2001), no. 3, 499-521.*

[32] F. Ronga, Le calcul des classes duales aux singularités de Boardman d'ordre 2, *C. R. Acad. Sci. Paris Sér. A-B 270, (1970) A582-A584*

[33] Y.-T. Siu, Some recent transcendental techniques in algebraic and complex geometry. *Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), 439-448, Higher Ed. Press, Beijing, 2002.*

[34] Y.-T. Siu, Hyperbolicity in complex geometry, *The legacy of Niels Henrik Abel, Springer, Berlin, 2004, 543-566.*

[35] Y.-T. Siu, S.-K. Yeung, Hyperbolicity of the complement of a generic smooth curve of high degree in the complex projective plane, *Invent. Math. 124., (1996), 573-618.*

[36] S. Trapani, Numerical criteria for the positivity of the difference of ample divisors. *Math. Z. 219. (1995), no. 3, 387-401.*

[37] R. Thom, Les singularités des applications diff´erentiables, *Ann. Inst. Fourier 6, (1955-56) 43-87.*

[38] M. Vergne, Polynomes de Joseph et representation de Springer, *Ann de l'ENS, 23 1990.*