The influence of Oscar Zariski on algebraic geometry

Piotr Blass

Introduction

Oscar Zariski was born in 1899 in Kobryn in the Ukraine (today Kobryn lies in Belarus) in a Jewish family. His first name, as he once told me, was sometimes pronounced Aszer in his childhood.
He loved mathematics and original creative thinking about mathematics from a very early age. He recalled the exhilaration of doing mathematics as a boy.
His father died when Oscar was very young; his mother was a business lady, as he used to say. She would sell various things in the Jewish Nalewki district in Warsaw.

Oscar Zariski (1899-1986) (picture by George M. Bergman)

In 1921, Oscar Zariski went to study in Rome. He had previously studied in Kiev and recalled that he was strongly interested in algebra and also in number theory. The latter subject was by tradition strongly cultivated in Russia.
He went to Rome on a Polish passport. His stay in Rome must have been very exciting. It lasted from 1921 until 1927. He became a student at the University of Rome and he married a wonderful woman who was his unfailing companion and

This paper was originally published in the *Australian Mathematical Society Gazette* vol. 16, No. 6 (December 1989). We thank the Australian Mathematical Publishing Association Inc. for permission to reprint an updated version of this paper. The updated version of the paper has been prepared by Piotr Pragacz; the typing and pictures are due to Maria Donten-Bury.

soulmate for the next 65 years, Yole. They were inseparable and she was a tower of strength to him in good and bad times.
In fact, during the latter years of his life when his hearing became impaired, she took over a lot of his communication tasks with other people. He continued to be creative almost to the last year of his life.

1. Zariski in Rome

But let us return to Rome, 1921. There were three mathematicians at the University of Rome who became synonymous with the beauty, excitement and perhaps slightly cavalier approach to proofs in the Italian school of algebraic geometry. Guido Castelnuovo, Federigo Enriques and Francesco Severi. The first two were of Jewish origin and were related to each other. Yole Zariski came from the family that belonged to the same highly sophisticated social group as they did (in my impression at least). Zariski always spoke very warmly about Castelnuovo and Enriques. I don't feel that they liked Severi very much, although they respected his work.
This may be a good moment to trace Zariski's mathematical ancestry. Italian algebraic geometry started with Luigi Cremona who was a fighter in Garibaldi's army, became a senator, and was contemporary with the Romanic Revival in Italy and perhaps a part of it. (Yes, mathematicians do fit into general culture!) Cremona, who had apparently studied with Chasles, influenced Corrado Segre who, in turn, taught Castelnuovo. Castelnuovo influenced Enriques – this really was a partnership – and finally Guido Castelnuovo became Oscar Zariski's thesis adviser. I remember Zariski telling me about an important and dramatic conversation with Castelnuovo during his early days in Rome. Castelnuovo was so impressed with young Zariski that he helped to cut a lot of "red tape" to speed up Oscar's studies and became Zariski's dissertation adviser.
The Italians considered Zariski to be an "unpolished diamond". They sensed that his view of geometry would eventually be different from their own. Castelnuovo once told him "You are here with us but you are not one of us". This was not said in reproach but good-naturedly for Castelnuovo himself told Zariski time and time again that the methods of the Italian geometric school had done all they could do, had reached a dead end and were inadequate for further progress in the field of algebraic geometry. (This is reported by Zariski in the introduction to his collected papers.) Castelnuovo perhaps suspected that the way out of this predicament would lie in increasing use of algebra and topology in algebraic geometry and, well aware of Oscar's algebraic inclinations, he suggested to him a thesis problem that was closely related to Galois theory and to topology.

2. Zariski's Thesis Problem (over $\mathbb{C}$)

He proved the following based on the results of his thesis:
Given an algebraic equation $f(x, y) = 0$ of genus > 6 and generic (of general moduli) it is not possible to introduce a parameter t, a rational function of x, y, so that x and y can be expressed through t by radicals.
Another formulation is contained in the following theorem. Let X be a curve. We call a map of curves $X \to \mathbb{P}^1$ solvable if and only if the corresponding extension $k(x) \supset k(t)$ is a field extension solvable by radicals.

Theorem 1. *A general curve of genus ≥ 7 admits no solvable map into $\mathbb{P}^1$.*

During his stay in Rome Zariski was constantly involved in and exposed to research on algebraic surfaces (over $\mathbb{C}$). This was the favourite topic of his teachers. He says, in the introduction to [**6**],

> *In my student days in Rome algebraic geometry was almost synonymous with the theory of algebraic surfaces. This was the topic about which my Italian teachers lectured most frequently and in which arguments and controversy were also most frequent. Old proofs were questioned, corrections were offered and these corrections were – rightly so – questioned in their turn. At any rate the theory of algebraic surfaces was very much on my mind...*

Nevertheless, most of his publications from this period still deal with algebraic curves and also certain foundational philosophical questions (e.g. Dedekind's theory of real numbers and Cantor's and Zermelo's recently created set theory). To that interest, he was influenced by Enriques, who was himself a philosopher and a historian of mathematics. Zariski must have considered the theory of algebraic curves to be a necessary training ground for an algebraic geometer.

Indeed, when I asked him to teach me algebraic geometry in Harvard in 1970 he made sure that I had some knowledge on algebraic curves. When I asked him later about the best introduction to this subject, he said: the book by Enriques-Chisini [**3**].

3. Illinois and Johns Hopkins

In 1927, Oscar and Yole left Italy for the United States. The reasons for this were, I think, the emergence of fascism in Italy and difficulty of finding a suitable academic position in Italy. Upon arrival in the US, the Zariskis spent some time in Illinois at a rather mediocre university but soon his brilliance was recognized and he was offered a position at Johns Hopkins University in Baltimore. This was the school that brought Sylvester to the USA before. Little did they realize that they had another Sylvester on their hands. Or did they?

Zariski spent considerable time preparing his monograph *Algebraic Surfaces*. In it he presented the current status of the theory of algebraic surfaces as of 1933. He examined every argument carefully and found a number of significant gaps in the classical proofs of the Italian school. As he says: "The geometric paradise was lost once and for all" – new tools, and a new framework and language were needed. This was a crisis. But to a mind like Zariski's in 1937 a crisis was just an exciting opportunity. He found his new tools in the commutative algebra and valuation theory that were being developed by Krull and van der Waerden who actually was trying his hand quite impressively in applying modern algebra to algebraic geometry even slightly before Zariski. But we are getting ahead of the story. During 1927–1937, Zariski made frequent trips to talk with Solomon Lefschetz in Princeton. Castelnuovo had the highest respect for Lefschetz and told Zariski about his work. Solomon Lefschetz, another Jewish immigrant from Russia, "stuck the harpoon of topology into the whale of algebraic geometry".

Lefschetz was another great genius with an unusual, even romantic, life story. Originally trained as an engineer he lost both hands in a terrible industrial accident. He had to give up his career and entered Clark University in Worcester, Massachusetts, to get his PhD in mathematics. I was on the Faculty of Clark for a year, and I have had the pleasure of examining Lefschetz's thesis written under the supervision

of Storey. It was very concrete, quite "Italian", geometry. Lefschetz spent many years in Nebraska and thirteen years in Lawrence, Kansas, in complete isolation, which he later considered a blessing. He read papers of Picard and Poincaré about integrals and their periods on algebraic varieties. He was deeply impressed by Picard's method of fibering an algebraic surface by a suitable pencil of curves (now called a Lefschetz pencil). Using such pencils and monodromy, Lefschetz obtained very deep and subtle results on the topology and algebraic geometry of algebraic varieties over $\mathbb{C}$ and later over fields of characteristics zero. Lefschetz's genius was recognized and he was called to be a professor in Princeton (miracles still happen in America – at least they did in 1924). While he was talking to Lefschetz, Zariski was also doing work on topology of algebraic varieties. Under Lefschetz's influence a young American, Walker, rigorously proved a difficult and important theorem: resolution of singularities of algebraic surfaces over algebraically closed fields of characteristic zero.
Zariski examined Walker's proof and declared it to be correct in his *Algebraic Surfaces* monograph mentioned above. Let me quote from Zariski's 1934 introduction to *Algebraic Surfaces*:

> *It is especially true in algebraic geometry that in this domain the methods employed are at least as important as the results. The author has therefore avoided as much as possible purely formal accounts of the theory... and then due to exigencies of simplicity and rigor the proofs given in the text differ to a greater or lesser extent from the proofs given in the original papers.*

Thus Zariski essentially proves anew and clarifies a great deal of material. This book was the work of a master geometer.
During the period 1935–50, Zariski continued his work on topology of algebraic varieties, their fundamental group, purity of branch locus, cyclic multiple planes, etc. For lack of space, I must refer the reader for the details to the very descriptive and insightful summaries of the editors of Zariski's collected papers [**7**].

4. Zariski Applies Modern Algebra to Resolution of Singularities

Between 1935–37, Zariski studied modern algebra, saying "I had to start somewhere". He took valuation theory and the notion of integral dependence from Krull's Idealtheorie [**5**] and applied them to algebraic varieties – more specifically – to two problems:

I. Local uniformisation, and
II. Reduction of singularities or resolution of singularities.

Much later, in 1958, he wrote about:

III. Purity of branch locus.

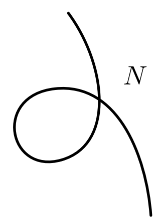
I will restrict myself to describing problem II, i.e. resolution of singularities and Zariski's contribution to it.

Definition 1. Let $V \subseteq \mathbb{P}^n_k$ be an irreducible projective algebraic variety (i.e. a set of common zeros of a set of homogeneous forms in the homogeneous variables $\{x_0, x_1, \ldots, x_n\}$). We call a projective variety $W \subseteq \mathbb{P}^m_k$ a *desingularisation* of V if there exists a regular algebraic morphism $\pi : W \to V$ such that

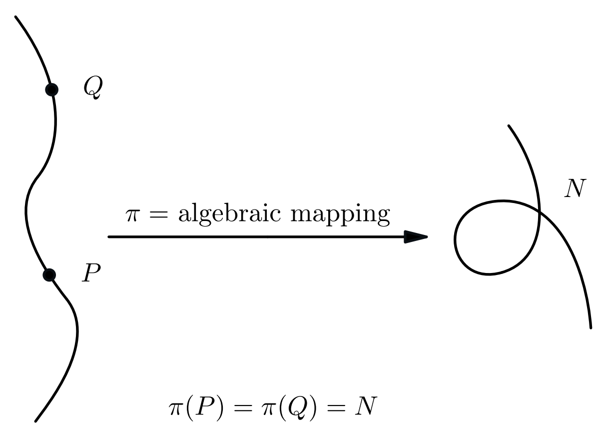
(1) W is a smooth variety

(2) π is proper, i.e. universally closed for any base change (which is automatically satisfied in our context),
(3) π induces an isomorphism of $\pi^{-1}(V - \operatorname{Sing} V)$ with $V - \operatorname{Sing} V$.

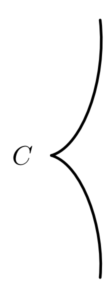
The simplest example is a plane curve with a node. For example, $V : x^3 + y^3 - xy = 0$. The essential features of this curve can be pictured as follows:



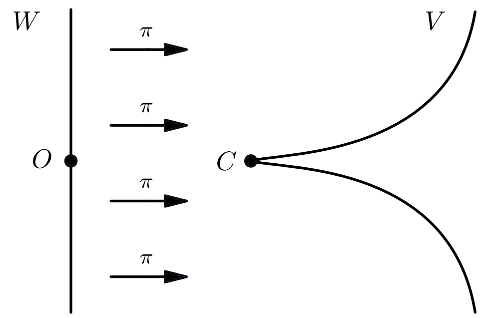
N, the origin, is a node, a very well known type of singularity. The desingularisation W is a smooth (space) curve where the two branches at N are pulled apart.



An equally famous and simple example is the cusp $V : y^2 - x^3$. This can be pictured as follows:

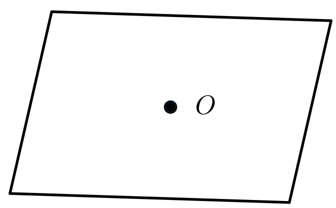


The origin C is a cusp, the desingularisation W is a straight line S with a mapping $\pi : W \to V$ that can be pictured as follows:

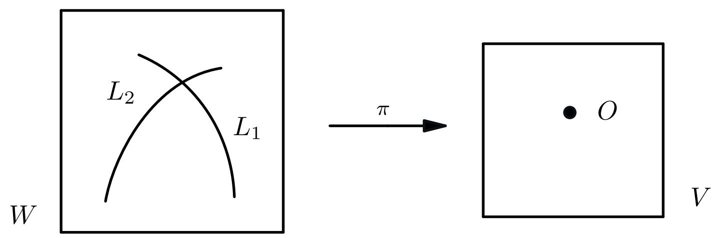


Here is an example of a surface singularity:

$$V:\ z^3 = xy$$

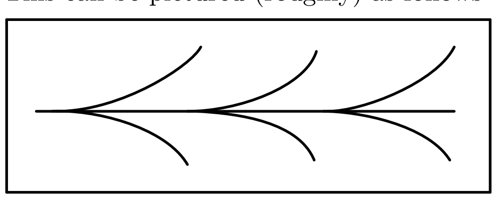


O is a biplanar double point of V. The desingularisation $\pi : W \to V$ can be pictured as follows:



$$\pi(L_1) = \{O\} = \pi(L_2)$$

Here L_1 and L_2 are (projective) lines that are contracted by π to the point O. The map π restricted to $W - (L_1 \cup L_2)$ establishes an isomorphism with $V - \{O\}$.
In other words, after blowing up or quadratic transformation the singular point is replaced by a pair of projective lines intersecting at one point. (The intersection matrix is $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.)
Singularities do not have to be isolated. For example, $x^3 = y^2 z$ has a whole curve of cusps $x = z = 0$. This can be pictured (roughly) as follows



(This singularity gets "worse" when $y = 0$.)
Resolution can be a very complicated process and general existence theorems tend to be very hard in higher dimensions. Zariski, first of all, showed that the algebraic notion of integral closure, which he called normalisation, gives resolution of singularities for curves. Thus:

$\dim X = 1$, any characteristic, resolution is possible,
$\dim X = 2$, the case of algebraic surfaces; he proved that resolution could be obtained by alternating use of normalisation and quadratic transformations with point centers:
Normalize $\to$ Blow up $\to$ Normalize $\to \ldots$

His proof worked in characteristic zero and, i.e., he proved the same type of result as the one by Walker, mentioned before. However, he masterfully wrote his paper in such a way that made it very transparent what still had to be done in char $p > 0$. This was then completed by some brilliant work of Abhyankar in the early 1950s. Abhyankar was Zariski's first student at Harvard. So resolution for surfaces is now well known in all characteristics. A beautiful and conceptual proof was supplied by another student of Zariski, Joseph Lipman, around 1980.
Zariski then turned to the very difficult case of a 3-dimensional variety in characteristic zero. Again he succeeded but the proof was now very long (70 pages in the *Annals of Math.*; reprinted in [**7**], volume I, where all of Zariski's resolution papers may be found). He says in the introduction (about the n-dimensional problem):

> *How much more difficult is the general problem is of course impossible to say with certainty and precision at the present moment. We are inclined to conjecture that the difficulties in the general case and in the three-dimensional case are of comparable order of magnitude... The 3-dimensional case offers an excellent testing ground...*

Again the paper was written in such a way that Abhyankar was able to extend this proof brilliantly to the fields of characteristics $p > 5$. (This has been only recently extended to all positive characteristics by V. Cossart and O. Piltant [**2**].)
Heisuke Hironaka, advised and prompted by Zariski, then solved the general case and proved that resolution of singularities exists in all dimensions in characteristic zero. This was in 1964 – the *Annals* paper [**4**] is one of the best ever written (it is 217 pages long). As Abyankar points out (Kyoto 2008), Hironaka proved this result at first in dimension 4. In Abyankar's opinion, people working on resolution of singularities in positive characteristic, should follow this strategy. Hironaka received a Field's Medal for this achievement. I remember talking with Zariski about that achievement of Hironaka. I could still feel in 1971 the excitement over this result; somehow Zariski made it clear to me that he was very helpful in Hironaka's work without taking away any of the credit, rightly earned by his brilliant former student.
Zariski not only proved general theorems about resolution, he also knew how to resolve singularities in practice when they are given by explicit equations. He taught that to his students and to me. He also knew how to use resolution to study differentials and numerical invariants of varieties. He could thus make precise a great deal of Italian geometry. In fact, being able to resolve singularities is a trademark of the Zariski school. Abhyankar once said of Zariski: "Without his blessing, who can resolve singularities?" In the near future – I feel – the knowledge of resolution of singularities and its possible computer implementations should become useful to engineers and other scientists who work with systems of algebraic equations.
In connectionn with the resolution problem, I mentioned the Indian Mathematician Abhyankar several times. He is a professor in Purdue and in Poona, India, he has had a large number of students who consider Zariski *paramguru* (guru of your guru). Thus Zariski's influence is being felt in the new generation of geometers in Japan

(Hironaka's influence) and in India (Abhyankar's). In the US, David Mumford, Michael Artin, Joseph Lipman and Steve Kleiman have given great impetus to algebraic geometry and have had numerous students. Daniel Gorenstein was also an early Zariski student. His thesis was about curves – hence Gorenstein rings. He moved out of algebraic geometry but we will forgive him since he led the magnificent effort to classify finite simple groups.

5. Linear Systems, Simple Points, Zariski's Main Theorem

During the period 1937–45, Zariski, in addition to his work in resolution and local uniformisation, took up in a rigorous way such topics as linear systems, simple points, Bertini theorems, applying modern algebra to all of these topics that were studied less rigorously by the Italian geometers. Around 1945–46, he started to develop his theory of holomorphic functions and continuation in abstract, algebraic geometry. His teaching load at Johns Hopkins was 18 hours a week; it was wartime. He was however invited to spend at least one year in São Paulo in January 1945. There he developed his theory of holomorphic functions in relative peace and quiet. He had a superlative audience consisting of one person, André Weil, with whom he took frequent walks and talks together. An important paper appeared in 1946 in Brazil and was completed in a 1951 AMS Memoir. There are several noteworthy effects that came out of Zariski's theory of holomorphic functions:

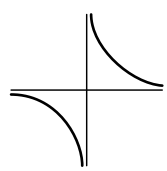
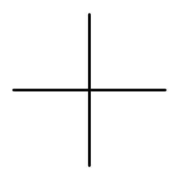
(i) Zariski's Main Theorem, and
(ii) the connectedness principle.

Also,

(iii) it inspired Grothendieck's theory of formal schemes and several deep theorems in cohomology of schemes and thus is now the mainstream and lifeblood of modern algebraic geometry.

The easiest one to explain is the connectedness principle. Enriques stated it as follows:

> If an irreducible variety V varies in a continuous system and degenerates into a reducible variety V_0, then V_0 is connected.

Example: $xy = c$  degenerates to $xy = 0$ 

(Over $\mathbb{C}$ it is obvious because V_0 is a continuous image of V but over an abstract field it is much harder.)
As for (iii), first of all Grothendieck reformulated and generalized the connectedness theorem as follows:

> If $f : V^1 \to V$ is a proper morphism and $f_*\mathcal{O}_{V^1} = \mathcal{O}_V$, then the geometric fibers of f are connected.

See Artin's introduction to Zariski's collected papers, ([**7**]) volume II.

6. Back to Surfaces (with Vengeance). Main Course

After much foundational work, Zariski returned to his old Italian love: algebraic surfaces. Now he had powerful algebraic tools, resolution of singularities, Bertini

Theorems, and adequate notions of smoothness. He could proceed much more confidently. Also, in many cases he was able to deal with varieties of any dimension. Mumford remarked that this must have seemed like a dessert after the foundational work. Zariski good-naturedly corrected him and said to him this was the main course. (Mumford was perhaps Oscar's favorite and most trusted student.)

Zariski moved to Harvard in 1949. He was at the pinnacle of his career and became world famous. (It is remarkable that his greatest achievements started when he was almost 40 years old, thus dispelling once and for all the myth that mathematics is a young person's game. There is hope for all of us!) During 1946–1955, he reigned supreme in algebraic geometry. Then came Serre and Grothendieck, but we will come to this later.

Zariski published several important papers about linear systems, algebraic surfaces and algebraic varieties of higher dimension, studying a number of global questions this time. In the period 1948–1962 roughly, he dealt with such topics as invariance of arithmetic genus under birational transformations, the so-called lemma of Enriques-Severi-Zariski, Riemann-Roch for surfaces, and minimal models of surfaces. In a paper written in 1958, he generalized a famous theorem of his teacher Castelnuovo to surfaces in characteristic $p > 0$.

Let us try to explain Castelnuovo's theorem (and criterion). A surface S is called rational if it can be parametrized "almost everywhere" by two independent parameters. In purely algebraic terms,

$$k(S) = \text{the function field of } S \cong k(T_1, T_2),$$

with T_1, T_2 algebraically independent over k. A surface S is called unirational if there is an extension

$$k(S) \to k(T_1, T_2)\,.$$

When $k = \mathbb{C}$, Castelnuovo proved that

$$S \text{ unirational} \Rightarrow S \text{ rational} \tag{1}$$

(about 1895). This work was a jewel of Italian geometry, Castelnuovo's argument being long and subtle. Castelnuovo used his criterion

$$\left.\begin{array}{c} \text{arithmetic genus of } S = 0 \\ \text{bigenus of } S = 0 \end{array}\right\} \Rightarrow S \text{ rational.} \tag{2}$$

In modern terms (a simply connected surface with no regular 2-forms of weight two is rational),

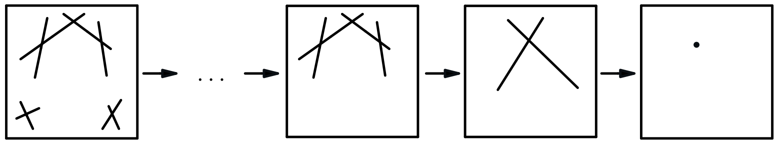
$$\left.\begin{array}{c} h^2(S) - h^1(S) = 0 \\ h^0(2K) = 0 \end{array}\right\} \Rightarrow S \text{ rational.}$$

Zariski generalized (2) to all fields of char $p > 0$, $p = 2$ being the hardest case in his approach. However (1) is false in char $p > 0$; there exist unirational surfaces that are not rational! (Unless you assume $k(S) \to k(T_1, T_2)$ is a separable extension, in which case (1) is true, as Zariski showed in 1958.)

Zariski wrote down an explicit example:

$p \geq 3$ prime
F: $z^p = x^{p+1} + y^{p+1} - \frac{x^2}{2} - \frac{y^2}{2}$
$\tilde{F}$:= closure of F in projective space, $\pi : \overline{F} \to \tilde{F}$ desingularisation

The desingularisation is:

($\frac{1}{2}(p-1)$ steps) 

Clearly, $k(\tilde{F}) = k(x, y, z) \subseteq$ (inseparable ext)$k(x^{1/p}, y^{1/p})$. Thus $\tilde{F}$ is unirational but Zariski checked that, for example, $dxdz/(y^p - y)$ defines a regular differential 2-form on $\tilde{F}$. Thus $\tilde{F}$ cannot be rational.
This example led to my own thesis topic, suggested to me by Hironaka in 1970. Zariski's example (half a page) has blossomed into a substantial theory of Zariski surfaces (a 450 page monograph [**1**] that uses all the tools of modern algebraic geometry and is tied up with computer science and, it seems, coding theory. Thus in this case, as in many others, Zariski's idea has blossomed into a large theory. How typical!

7. Zariski at Harvard (1949–1986)

Zariski officially retired at Harvard around 1970. Upon Hassler Whitney's suggestion (and that a lot of other people, or so they say now), Zariski was called to Harvard in approximately 1944. At a recent algebraic geometry meeting, someone (perhaps Abhyankar) asked for Zariski's students and students of his students to raise their hands. Almost everybody in the room did.
Zariski can be considered the father of the American algebraic geometry school. He did not stand in the path of progress, on the contrary he welcomed it. While at Harvard from 1949 Zariski quickly established himself as the undisputed leader in algebra and algebraic geometry. In fact, Garrett Birkhoff stopped teaching algebra after a couple of years and moved to computer science. Harvard had great graduate students. Zariski's students included Abhyankar, Gorenstein, Mike Artin (Emil Artin's son), David Mumford, Steven Kleiman, Joseph Lipman, Heisuke Hironaka and Alberto Azevedo (from Brazil). Mumford and Hironaka went on to receive Fields Medals.
In 1955 Serre and in 1958 Grothendieck suddenly revolutionized algebraic geometry by introducing the notion of sheaves, schemes and cohomology. They were inspired by Zariski but in some ways their theories could go much further.
Zariski, who was approaching 60 at that time, organized a summer school in algebraic sheaf theory. He wrote an account of Serre's work.
Grothendieck was welcomed at Harvard and taught a remarkable class with Mumford, Artin, Hironaka, Tate (who was on the faculty), Shatz and others in the audience. Zariski's students became Grothendieck followers, but they never forgot what they learned from Zariski. Thus the Zariski school adopted the scheme theoretic and cohomological techniques of Grothendieck. Grothendieck dedicated his EGA *Elements of Algebraic Geometry* treatise to Oscar Zariski and André Weil.
Zariski spent the last fifteen or so years of his life working on the question of equisingularity. Once again he created an important and impressive theory which, roughly speaking, attempts to compare singularities at different points on varieties and decide when they are in some sense the same (or similar). It was touching and inspiring to see him work in his 80s. He agonized when he felt that his mathematical powers could be leaving him. I overheard him once talking about it to Mumford.

"Maybe I should quit," said the 85 year old Oscar. "Take a little vacation," said Mumford. And so he did.

His wife, Yole, was incredibly important to him during this time, as always. His hearing became disturbed with constant ringing. It was hard to converse with him. You had to write everything. He was depressed because results were coming slowly.

8. Conclusion – Personal Memories

Zariski was serious and professional about mathematics and that was picked up by all of his students. He claimed to be "slow", which forced people to really explain things. His approach was to do something almost every day. A lemma a day... (he used to say).

Let me add a personal note. I met Oscar at Harvard when I was about 21 years old. He was retired but I worked with him as much as I could. He was very helpful with my thesis problem; I would meet with him once a week to talk about geometry and my progress, although Hironaka was my official advisor. Sometimes we would all go to lunch together at the Harvard Faculty Club.

Zariski told me quite a lot about his youth; I came from a nearby part of the world. (He visited his country of birth, Russia, in 1937.) I visited him at Purdue where he would spend his summer (to get away from Harvard "that was like a madhouse sometimes"). I named my own son Oscar.

I guess we all loved him like a father.

A lot of us came to his memorial meeting in September 1986. His was a wonderful life. He is survived by his wife, children, several grandchildren and most importantly by numerous geometers all over the world. As a teacher he was quite strict and made you want to learn and do all that you are capable of doing just to keep up with him. A word of praise from him was something you really treasured. But somehow he made you feel like part of the family.

In 1973, I was on leave from the Israeli army and from the war. I decided that I had to see Oscar. He was warm, his house enlivened by a couple of grandchildren. His presence and Yole's and their conversation made me stronger to face the hardships and come back to mathematics. Probably the greatest praise I ever heard from him was that he called my thesis interesting and wrote me a nice letter about it. I saved all of his letters and mathematical notes.

There was no conflict between research and teaching for Zariski. Teaching extended his research and increased its impact a hundredfold. He was a truly wise and happy man. There are not many like him. Oscar Zariski died in 1986, so in 2011 we celebrate the 25-th anniversary of his death.

In Summary

Zariski transformed algebraic geometry from its semi-art, semi-science status into both art and science. He made it mathematically precise without sacrificing any of its beauty. The most fundamental topology on an algebraic variety or a scheme is called the Zariski topology. Thus he is remembered whenever modern algebraic geometry is done. Also the terms "Zariski tangent space", "Zariski decomposition" for surfaces, are very common in algebraic geometry.

He was flexible enough to welcome and encourage the most modern trends as long as they contributed to solving hard classical problems. The total impact of the Zariski school may well be historically comparable to Riemann's or Hilbert's, especially

when combined with its logical allies and successors, Grothendieck's school and Šafarevič's in Russia.
Great triumphs of mathematics are the solution of the Mordell conjecture by Faltings and the proof of the Fermat's Last Theorem by Wiles, which are built on Grothendieck's machinery. Zariski provided a powerful bridge spanning several decades between nineteenth century mathematics and early twentieth century mathematics to the most modern developments.

Quote from Zariski

The Italian geometers have erected, on somewhat shaky foundations, a stupendous edifices: the theory of algebraic surfaces. It is the main object of modern algebraic geometers to strengthen, preserve and further embellish this edifice, while at the same time building up also the theory of varieties of higher dimensions. The bitter complaint that Poincaré had directed, in his time, against the modern theory of functions of real variable cannot be deservedly directed against modern algebraic geometry. We are not intent on proving that our fathers were wrong. On the contrary our whole purpose is to prove that our fathers were right.

The arithmetic trend in algebraic geometry is in itself a redical departure from the past. This trend goes back to Dedekind and Weber who have developed, in their classical memoir, an arithmetic theory of fields of algebraic functions of one variable. Abstract algebraic geometry is a direct continuation of the work of Dedekind and Weber except that our chief object is the study of fields of algebraic functions of more then one variable. The work of Dedekind and Weber has been greatly facilitated by previous developments of classical ideal theory. Similarly, modern algebraic geometry has become a reality partly because of the previous development of the great theory of ideals. But here the similarity ends. Classical ideal theory strikes at the very core of the theory of functions of one variable and there is in fact a striking parallelism between this theory and the theory of algebraic numbers. On the other hand the general theory of ideals strikes almost at the foundations of algebraic geometry and falls short of the deeper problems which we face at the post foundational stage. Furthermore there is nothing in modern commutative algebra that can be regarded even remotely as a development parallel to the theory of algebraic function fields of more than one variable. This theory is after all itself a chapter of algebra, but it is a chapter about which modern algebraists know very little. All our knowledge here comes from geometry. For all these reasons it is undeniably true that the arithmetisation of algebraic geometry represents a substantial advance in algebra itself. In helping geometry, modern algebra is helping itself above all. We maintain that abstract algebraic geometry is one of the best things that has happened to commutative algebra in a long time. [**7**]

Remark. For more detailed account of Oscar Zariski's life and work, see the book by Carol Parikh, *The unreal life of Oscar Zariski*, Academic Press (1991).

References

[1] Piotr Blass and Jeffrey Lang, *Zariski surfaces and differential equations in characteristic* $p > 0$, Marcel Dekker Monographs 106 (1987).

[2] V. Cossart and O. Piltant, Resolution of singularities of threefolds in positive characteristic II, J. Algebra 321 (2009), 1836–1976.

[3] Federigo Enriques and Oscar Chisini, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, in three volumes, Bologna (1915–24).

[4] Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Annals of Math., 79 (1964), 109–326.

[5] Wolfgang Krull, *Idealtheorie*, Springer, Ergebnisse Series (1968).

[6] Oscar Zariski, *Algebraic Surfaces*, Chelsea (1948).

[7] Oscar Zariski, *Collected Papers*, in four volumes, MIT Press (1972).