"Gale duality, Blow-ups and Moduli spaces"

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§ Blow-ups and Mori dream Spaces.

$/\mathbb{C}$

$p_1, \ldots, p_k :=$ pts in general position in $\mathbb{P}^n$

$X^n_k := \mathrm{Bl}_{p_1, \ldots, p_k} \mathbb{P}^n$

Goal: Study the birational geometry of X^n_k.

§ Introduction

Remark A lot of important properties of projective algebraic varieties are encoded into particular convex cones.

X proj variety (think smooth).

- D and D' Cartier divisors are numerically equivalent if $D \cdot C = D' \cdot C$ for every curves C on X

$N^1(X)_{\mathbb{Z}}$:= finitely generated torsion free abelian group of Cartier divisors modulo numerical equivalence.

$\rightsquigarrow$ finite rank : $\rho(X)$ Picard rank.

$N^1(X)_{\mathbb{R}} := N^1(X)_{\mathbb{Z}} \otimes \mathbb{R}$

finite dim. vector space.

Def (Effective cone of X)

The Effective cone of X, $\mathrm{Eff}(X)$, is convex cone in $N^1(X)_{\mathbb{R}}$ generated by classes of effective divisors.

Example: Let X be a smooth projective surface.
Effective divisors on a surface are curves:

$$\mathrm{Eff}(X) = NE(X) = \left\{ \sum a_i [C_i] \;\middle|\; \begin{array}{l} C_i \text{ irreduc. curves} \\ a_i \geq 0 \end{array} \right\}$$

- Cone of curves
- Mori cone

Notation

X smooth projective variety

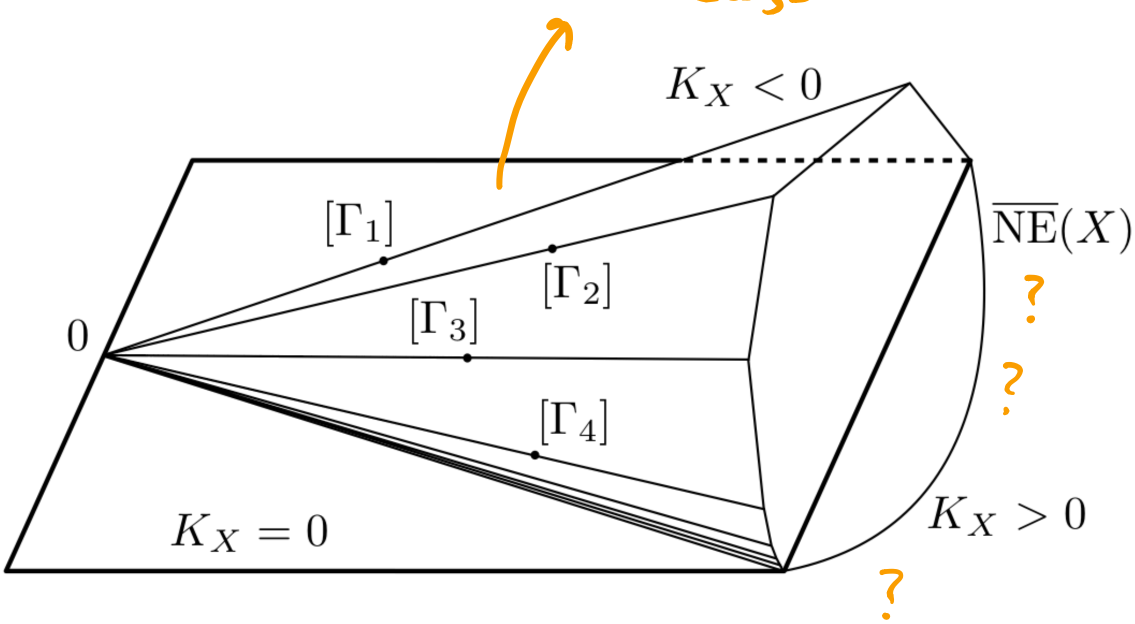
$\omega_X = \Lambda^{\dim X} \Omega_X^1$ canonical sheaf

$K_X \rightsquigarrow$ canonical divisor $\to$ any Cartier divisor st $\omega_X \simeq \mathcal{O}_X(K_X)$.

Cone theorem (Mori)

Let X be a smooth proj variety.

extremal rays $\rightsquigarrow$ birational contract.



The closed cone of curves

in $N_1(X)_{\mathbb{R}}$

Statement of the Cone thm

There are at most (countably) many rational curves $C_i \subseteq X$ such that
$$0 < -(C_i \cdot K_X) \leq \dim X + 1$$
(produced by Bend & Break)

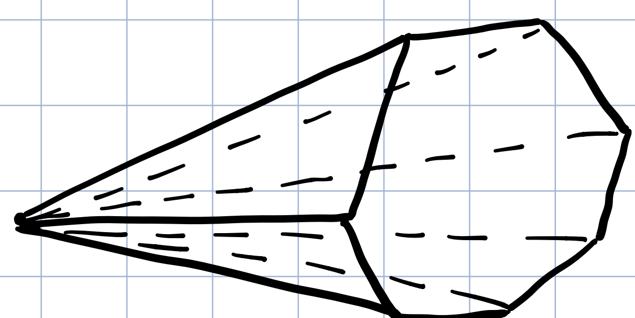
and s.t.

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum_{i \in \mathbb{N}} \mathbb{R}_{\geq 0}[C_i]$$

If X is Fano ($-K_X$ is ample)

Rmk: If $K_X \cdot C < 0$ for all curves C on X ✓

$\Rightarrow$ $\overline{NE}(X)$ is a rationally polyhedral cone.



- finitely many extremal rays
- finitely many birational. contrac.

We can now go back to X_k^n.
Let's start from $n = 2$.

$\mathbb{P}^2$. Fano $\overline{NE}(X) = \{aH \mid a \geq 0\}$

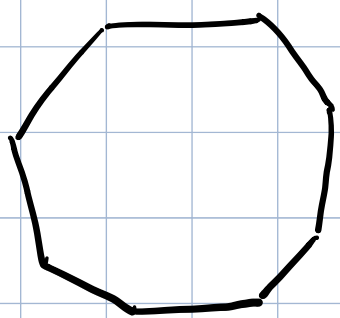
Toric 1-dimensional

$X^2_{k \geq 4}$ not toric

$X^2_{k \geq 9}$ not Fano

$\mathrm{Eff}(X_k^2)$ (We look at a cross section)

$k \leq 8$

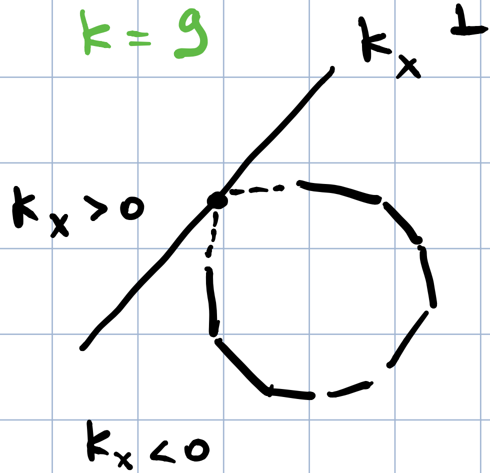


del Pezzo surface

$\Downarrow$

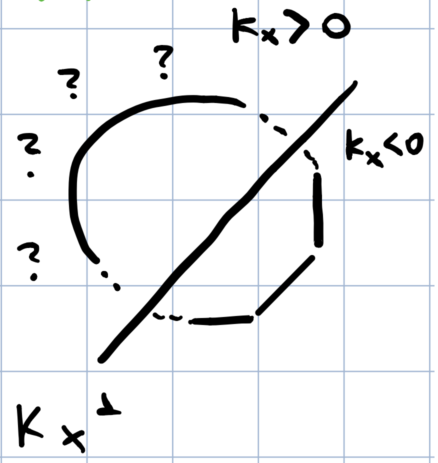
cone is rationally polyhedral.

$k = 9$



infinitely many extremal rays.

$k \geq 10$



Study of the K_X-positive mysterious part is connected to important conj in plane curve.

- Nagata conj.

¿ What is the picture in higher dim ?

We assume $n \geqslant 3$.

- $X^n_{k \geqslant 2}$ is not Fano

- $X^n_{k \geqslant n+2}$ is not toric.

We want to have a notion that captures the nice birational properties connected to having rational polyhedral effective cone.

Def (Mori Dream Space) introduced by Hu & Keel (2000)

Let X be a normal proj $\mathbb{Q}$-factorial variety. X is a Mori Dream Space (MDS) if

(i) $\mathrm{Pic}(X)$ is finitely generated.

(ii) $\mathrm{Nef}(X) = \langle D_1, \ldots, D_n \rangle$ is generated by finitely many semiample classes D_i.

(iii) There are finitely many small $\mathbb{Q}$-factorial modifications $f_i : X \dashrightarrow X_i$ st X_i satisfies (ii) and $\mathrm{Mov}(X) = \bigcup_i f_i^*(\mathrm{Nef}(X_i))$

$\hookrightarrow$ "finite picture"

Rmk: If X is a MDS, then $\mathrm{Eff}(X)$ is a rationally-polyhedral cone that admits a finite wall-chamber decomposition, where every chamber corresponds to a birational model of X.

Mori Chamber decomp. (MCD)

Prototype examples of Mori Dream Space:

- (Log) Fano varieties (BCHM)
- Toric varieties

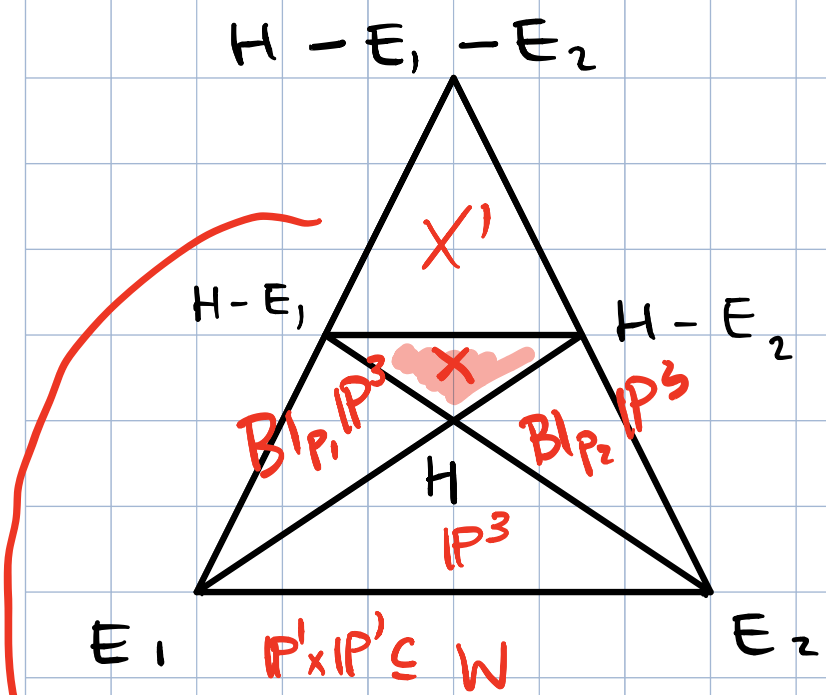
Example: $X = X_2^3 = Bl_{p_1,p_2}\mathbb{P}^3 \xrightarrow{\pi} \mathbb{P}^3$

$\mathrm{Pic}(X) = \langle H, E_1, E_2 \rangle$, where

$H = \pi^* \mathcal{O}_{\mathbb{P}^3}(1)$

E_1, E_2 = exceptional divisors.

Cross section of $\mathrm{Eff}(X_3^2)$



$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ \tilde{\ell}_{12} \subseteq X \arrow[rr, dashed, leftrightarrow] & & X' \supseteq \tilde{\ell} \end{tikzcd}$$

flop of $\tilde{\ell}_{12}$

· $D \in \mathrm{Eff}(X)$

$R(X,D) = \bigoplus_{n \in \mathbb{Z}} H^0(X, nD)$

$\Rightarrow$ f.g.

X is MDS

$f_D : X \dashrightarrow \mathrm{Proj}(R(X,D)) = X_D$

D_1 and D_2 in the same chamber

$$\begin{tikzcd} & X \arrow[dl, dashed, "f_{D_1}"'] \arrow[dr, dashed, "f_{D_2}"] \arrow[d, phantom, "G"] & \\ X_{D_1} \arrow[rr, phantom, "\cong"] & {} & X_{D_2} \end{tikzcd}$$

To summarize:

- finite decomposition:

We can read from the decomposit:

- birational contractions
- birational models
- birational maps connecting different models.

Question: For which values of n and k is X^n_k a MDS?

Theorem ("⇒" Mukai 2001, "⇐" Castravet-Tevelev 2005)

X^n_k is MDS $\iff$ $\frac{1}{2} + \frac{1}{n+1} + \frac{1}{k-n-1} > 1$

Concretely :

$$(*)\begin{cases} \cdot\ n=2 & k\leq 8 \quad \text{(del Pezzo)} \\ \cdot\ n=3 & k\leq 7 \\ \cdot\ n=4 & k\leq 8 \\ \cdot\ n\geq 5 & k\leq n+3 \end{cases}$$

Few words on the proof :
Studying intersection theory on Blow-ups, it is possible to define a pairing on $\mathrm{Pic}(X_k^n)$:

$$H^2 = n-1$$
$$H.E_i = 0$$
$$E_i \cdot E_j = -\delta_{ij}$$

Wrt this pairing $K_X^\perp$ is a unimodular lattice. It contains a root system. Let W be the group of reflections associated to the root system.
$(*) \iff W$ is finite.

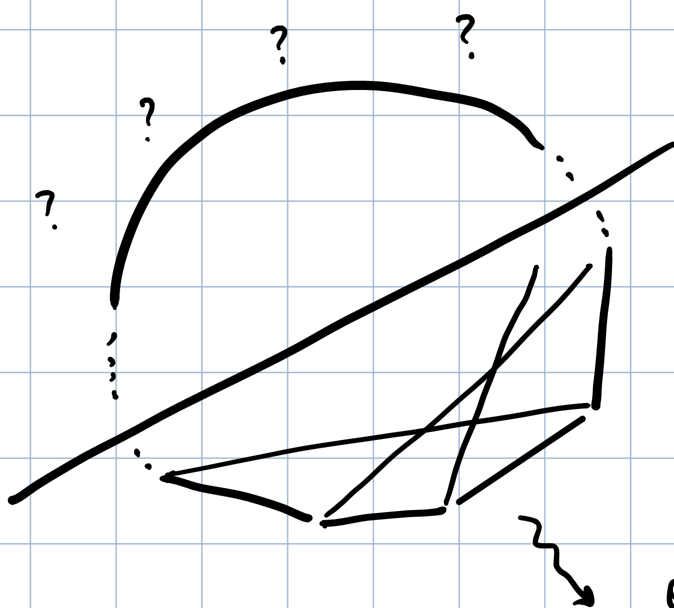
Suppose $n\geq 5$ and $k\geq n+4$.

X_k^n is not a MDS.

Theorem (Araujo, Castravet, Kaur, M)

$X^{n \geq 5}_{n+4}$

Cross section of $\mathrm{Eff}(X^{n\geq 5}_{n+4})$



On the negative part of the cone we have a "MDS region".

$$\mathrm{Eff}(X^n_k) \cap (K_X \leq 0) = \langle e_i \rangle$$

infinitely many excep divisors.

and there are countably many smooth - modifications $f : X \dashrightarrow X_i$

st $\quad f : X \dashrightarrow X_i \quad$ st

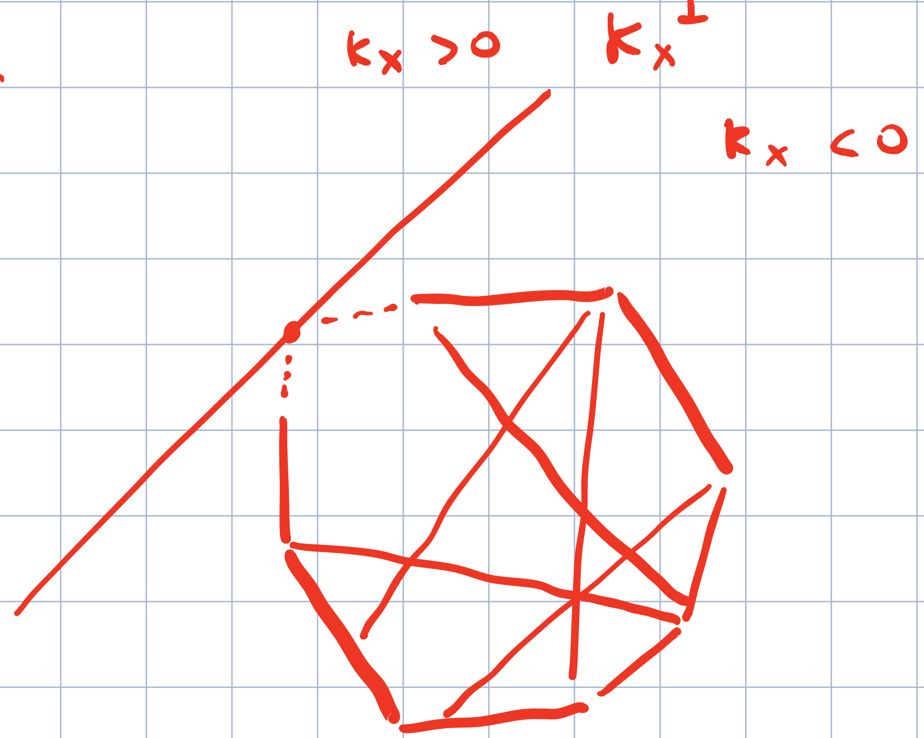
$$\mathrm{Mov}(X) \cap (K_X < 0) = \bigcup_i f_i^*(\mathrm{Nef}(X_i))$$

If $\quad f_i^* \mathrm{Nef}(X_i) \subsetneq (\overline{K_X < 0})$

then $\quad f_i^*(\mathrm{Nef}(X_i)) = \langle D_1, \dots, D_n \rangle$
semiample.

Special case:

X_g^5



Infinitely many exceptional divisors.

$\hookrightarrow$ Similar picture to X_g^2.

§ Ingredients of the proof

- Gale-duality :

Association of

$r+s+2$ pts in $\mathbb{P}^r \longleftrightarrow r+s+2$ pts in $\mathbb{P}^s$

simple linear algebra recipe.

Mukai : He realized that in some cases there is a very nice realization of this duality.

- 8 pts in $\mathbb{P}^2 \longleftrightarrow$ 8 pts in $\mathbb{P}^4$

X^4_8 = Moduli space of Gieseker stable torsion free sheaves on X^2_8.
  - wrt $L = -K_S + 2e_1$
  - $rk = 2$, $c_1 = -K_S$, $c_2 = 2$.

[Mukai], [Casagrande-Fanelli-Codogni]

- $X^{n \geq 5}_{n+4}$

Our case:

$n+4$ in $\mathbb{P}^n \longleftrightarrow n+4$ in $\mathbb{P}^2$

X^n_{n+4} = Moduli space of Gieseker-stable torsion free sheaves on X^2_{n+4}.

- $rk = 2$, $c_1 = -K_S$, $c_2 = 2$.
- explicit polarization.