Pointwise universal Gysin formulae and positivity of some characteristic forms

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$E^{r \geq 2} \longrightarrow X^n$ complex mfd

$0 = \rho_0 < \rho_1 < \cdots < \rho_m = r$ $\boxed{m \leq r}$

$\pi : \mathbb{F}_\rho(E) \longrightarrow X$

$U_{\rho,j} \longrightarrow \mathbb{F}_\rho(E)$ $j = 0, \ldots, m$ $\quad rk\, U_{\rho,j} = \rho_j$

$\cap$ $\pi^* E$

$$Q_{\rho,j} = \det \begin{pmatrix} U_{\rho,m-j+1} & \\ & U_{\rho,m-j} \end{pmatrix}$$

$(a_1, \ldots, a_m) \Rightarrow Q_\rho^{\underline{a}} = Q_{\rho,1}^{\otimes a_1} \otimes \cdots \otimes Q_{\rho,m}^{\otimes a_m}$

Es: $\rho = (0, 1, r)$ $m = 2$

$\mathbb{F}_\rho(E) = P(E) \to X$

$U_0 = P(E) \times \{0\}$

$$0 \to \mathcal{O}_E(-1) \to \pi^* E \to H \to 0$$

$U_1 = \mathcal{O}_E(-1)$

$$Q_1 = \det\left(\pi^* E \big/ \mathcal{O}_E(-1) \right) = \underbrace{\det H}$$

$U_2 = \pi^* E$

$$= \mathcal{O}_E(1) \otimes \pi^* \det E$$

$$Q_2 = \mathcal{O}_E(-1)$$

$$Q_1^{\otimes a_1} \otimes Q_2^{\otimes a_2} \simeq \mathcal{O}_E(a_1 - a_2) \otimes \left(\pi^* \det E \right)^{\otimes a_1}$$

$\mathbb{F}_\rho(E) \to X$

F homogeneous polynomial in the variables $\xi_1, \dots, \xi_r$ Chern roots of E (F has the appropriate symmetries i.e.

$F(\xi_1, \dots, \xi_r) \in H^*(\mathbb{F}_\rho(E))$)

$$\pi_* F(\xi_1, \dots, \xi_r) = \Phi(c_1(E), \dots, c_r(E))$$

Hypothesis $F(\xi_1, \dots, \xi_r) = \widetilde{F}(c_1(Q_{\rho,1}), \dots, c_1(Q_{\rho,m}))$

$(E, h) \longrightarrow X$ h hermitian metric

induces hermitian metrics on all of the $Q_{p,j}$'s.

$$\Xi_{p,j} = \frac{i}{2\pi} \Theta(Q_{p,j}, h) \in c_1(Q_{p,j})$$

$$F(\Xi_{p,1}, \ldots, \Xi_{p,m}) \quad \text{degree } 2(d_p + k)$$

$\uparrow$ relative dimension of $\mathbb{F}_p(E) \xrightarrow{\pi} X$

$$\pi_* F(\Xi_{p,1}, \ldots, \Xi_{p,m}) \quad (k,k)\text{-differential form on } X$$

$$\cap$$

$$\Phi(c_1(E), \ldots, c_r(E)) \in H^{k,k}(X).$$

$$(E, h) \rightsquigarrow \frac{i}{2\pi}\Theta(E,h) \rightsquigarrow c_k(E,h) = \mathrm{Tr}\left(\Lambda^k \frac{i}{2\pi}\Theta\right)$$

$$\cap \; c_k(E)$$

$$\Phi\left(c_1(E,h), \ldots, c_r(E,h)\right) \in \Phi(c_1(E), \ldots, c_r(E))$$

<u>Theorem</u> (FAGIOLI, – '20):

$$\pi_* F(\Xi_{p,1}, \ldots, \Xi_{p,m}) = \Phi\left(c_1(E,h), \ldots, c_r(E,h)\right)$$

Application towards Griffiths' conjecture

$(E, h) \longrightarrow X$

$\rightsquigarrow \frac{i}{2\pi} \Theta(E, h)$ (1,1)-form with values in Herm(E).

<u>Def</u>: (E,h) is <u>Griffith's positive</u> if $\forall\, v \in T_x$

$\forall\, s \in E$ you have

$$\langle \Theta(E,h)(v, \bar{v}) \cdot s, s \rangle_h \geq 0$$

$(=0$ iff $v \otimes s = 0)$

hermitian form on $T_x \otimes E \ni v \otimes s$

Positivity for differential forms.

u (k,k)-differential form $u = \bar{u}$

u is Hermitian-positive if the following induced hermitian form on $\Lambda^{n-k,0} T_x^*$ is positive (semi)-definite

$$\eta \longmapsto u \wedge i^{(n-k)^2} \eta \wedge \bar{\eta}$$

Conj (GRIFFITHS' '69): $(E,h) \geq_{Grif} 0$ then every positive linear combination of Schur polynomials in the Chern forms of (E,h) is a positive form.

<u>Known</u>: • c_1 is OK

• c_2 for rank 2 v.b. (weaker positivity)

• '12 GULER $(-1)^k S_k(E,h)$

• Ping LI, S. FINSKI full conjecture but with stronger positivity assumptions on (E,h)

$(E,h) \geq_{Grif} 0$ consider all forms of type

$$\pi_* \left(a_1 \Xi_{p,1} + \cdots + a_m \Xi_{p,m} \right)^{d_p+k}$$

$a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$.

when p, m, k, vary.

<u>Theorem</u> (FAGIOLI, – '20): These forms are (strongly) positive and positive linear combination of Schur forms. (uses Fulton–Lazarsfeld Theorem)

$\underline{\text{Es}}$: Rank 3 $\quad (0,1,r) = \rho$

$$\pi_* \, c_1(Q_1)^5 = 4c_1^3 - 3c_1c_2 - c_3 = \underbrace{s_3}_{<0} - 5 s_1 s_2$$

$$a_1 = 1 \geq a_2 = 0$$

Sketch of the proof : (very special case)

$$\mathbb{P}(E) \xrightarrow{\pi} X$$

$$\uparrow$$

$$\mathcal{O}_E(1)$$

$$\Xi = \frac{i}{2\pi} \, Ⓡ(\mathcal{O}_E(1), h)$$

$$\Xi = \Xi^{\text{vert}} + \Xi^{\text{hor}}$$

Ξ^{vert} ← Fubini-Study

Ξ^{hor} ← Griffiths curvature of E

$$\pi_* \Xi^{r-1+k} = \pi_* \left(\Xi^{\text{vert}} + \Xi^{\text{hor}} \right)^{r-1+k}$$

$$= \text{poly in } \left(c_\bullet(E,h) \right)$$

universal !

$$\in (-1)^k s_k(E)$$

$$\pi_* \Xi^{r-1+k} - (-1)^k s_k(E,h)$$

$$= \sum g_J \, c_1^{j_1} \wedge \cdots \wedge c_r^{j_r} = G$$

X proj $E = A^{\otimes m_1} \oplus \cdots \oplus A^{\otimes m_r}$

where $A \to X$ complex line bundle.

$$G(c_1(E,h), \ldots, c_r(E,h))$$

$$= \Big[\underbrace{\sum_J g_J \prod_{s=1}^{r} \Big(\sum_{k_1 < \cdots < k_s} m_{k_1} \cdots m_{k_s} \Big)^{j_s}}_{p(m_1, \ldots, m_r)} \Big] c_1(A,h)^k$$

$$p(m_1, \ldots, m_r)\, c_1(A)^k = [G(c_\cdot(E,h)] = 0$$

$m \in \mathbb{N}^r$