INTERPOLATION POLYNOMIALS IN SEVERAL VARIABLES

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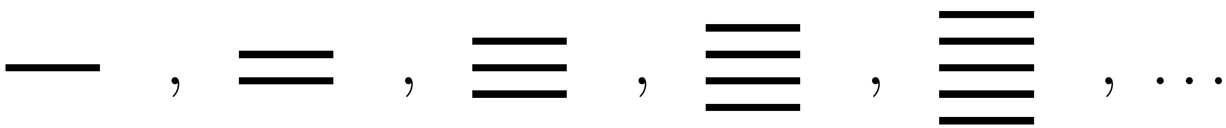
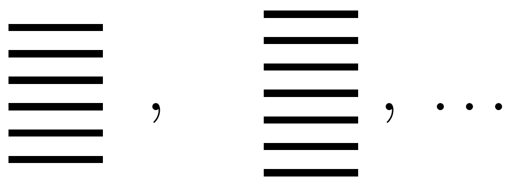
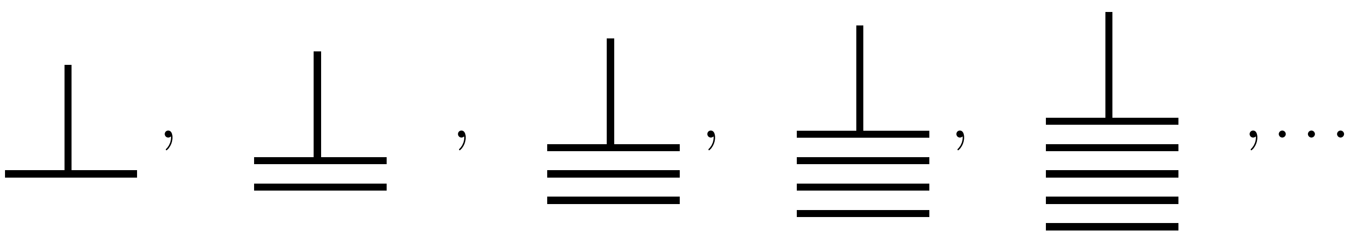


everybody continues with



However, the Yigu yanduan of Li Ye prefers :



Modern occidental science is supposed to begin with Galileo, who throws a ball from the Pisa tower and records its position at regular intervals of time :

$$I,\ IV,\ IX,\ XVI,\ XXV,\ XXXVI, \ldots$$

The law is a little more complicated, but philosophy comes to the rescue :

Quando, dunque, osservo che una pietra, che discende dall'alto a partire dalla quiete, acquista via nuovi incrementi di velocità, perché non dovrei credere che tali aumenti avvengano secondo la pi semplice e pi ovvia proporzione? Ora, se consideriamo attentamente la cosa, non troveremo nessun aumento o incremento più semplice di quello che aumenta sempre nel medesimo modo .... quel moto che in tempi eguali, comunque presi, acquista eguali aumenti di velocità.

In short, if it is not the increment of space which is uniform, it must be the increment of speed.

Galileo's method would indeed work for any polynomial law !

```
ACE> [seq( n^3-2*n+3,n=1..10)];
     [2, 7, 24, 59, 118, 207, 332, 499, 714, 983]
```

The solution of such a problem was already known at the very beginning of astronomy : compute differences, iterate.

$$
\begin{array}{cccccccccccccc}
 & 7 & & 24 & & 59 & & 118 & & 207 & & 332 & & 499 \\
5 & & 17 & & 35 & & 59 & & 89 & & 125 & & 167 & \\
 & 12 & & 18 & & 24 & & 30 & & 36 & & 42 & & \\
 & & 6 & & 6 & & 6 & & 6 & & 6 & & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & &
\end{array}
$$

From the first diagonal, one can reconstruct (and understand) the original sequence.

However, Chinese mandarins had more serious problems to solve than throwing balls.

For example, in the Jade Mirror of the four unknowns one finds

A mandarin recruits soldiers according to cubic numbers. He begins with a 3-feet cube. Then, he increases the side of the cube by one foot each time. Each soldiers receives a daily allowance of 250 sapeques. 23400 soldiers have been recruited, and the total expenditure was 23462 silver taels. In how many days were they recruited ?

Notons f(n) le nombre de personnes recrutées le n-ième jour.

n	f(n)	**différence**	**2^e différence**	**3^e différence**	**4^e différence**
0	0				
		$3^3 = 27$ (Δ_1)			
1	27		37 (Δ_2)		
		$4^3 = 64$		24 (Δ_3)	
2	91		61		6 (Δ_4)
		$5^3 = 125$		30	
3	216		91		6
		$6^3 = 216$		36	
4	432		127		
		$7^3 = 343$			
5	775				
...	...				

La quatrième différence étant constante, on aura comme interpolation polynomiale :

$$f(n) = n\Delta_1 + n(n-1)\Delta_2/2! + n(n-1)(n-2)\Delta_3/3! + n(n-1)(n-2)(n-3)\Delta_4/4!$$

Dans le cas présent, f(n) = 23 400.

On obtient $n = 15$ comme une solution de cette équation.

Issue des problèmes d'interpolation en astronomie, l'« art de la différence pour le recrutement » utilise les différences d'ordre 4.

Ici, le « côté de 3 pieds » n'a rien à voir avec une longueur : les hommes étant recrutés suivant les nombres cubiques, le premier jour, $3^3 = 27$ hommes sont recrutés ; on ajoute ensuite un pied, c'est-à-dire que le deuxième jour, $4^3 = 64$ hommes sont recrutés. La méthode de résolution est introduite par la formule : « *L'art dit* : ». Elle correspond à la formule d'interpolation utilisant les différences d'ordre 4.

La formule d'interpolation inventée sous les Sui par Liu Zhuo s'arrêtait à l'ordre 2 ; dans le *Miroir de jade des quatre inconnues*, elle est

However, comets are not likely to appear at regularly spaced times. How to treat them ? This is Newton who, while working on the Principia found how to transform a discrete set of data into an algebraic function :

normalize differences

by

dividing them

by the interval of time

Here is the preceding set of data, minus three observations :

$$7 \qquad \blacksquare \qquad 59 \qquad \blacksquare \qquad \blacksquare \qquad 332 \qquad 499$$

$$5 \qquad \frac{52}{2} = 26 \qquad \frac{273}{3} = 91 \qquad 167$$

$$\frac{21}{3} = 7 \qquad \frac{65}{5} = 13 \qquad \frac{76}{4} = 19$$

$$\frac{6}{6} = 1 \qquad \frac{6}{6}$$

$$0$$

According to Newton, the comet position $f(t)$, known at times $t_0, t_1, \ldots$, is :

$$f(t) = f(t_0) + f^{\partial}(t - t_0) + f^{\partial\partial}(t - t_0)(t - t_1) + \cdots$$

obtaining the coefficients f^{∂}, $f^{\partial\partial}, \ldots$ thanks to the preceding method of dividing differences.

In more algebraic terms: one starts from a function of $x_1, x_2, \ldots$ (the interpolation points), and for each pair x_i, x_{i+1}, one defines an operator on polynomials (a divided difference) :

$$f \to f\partial_i := \frac{f(\ldots, x_i, x_{i+1}, \ldots) - f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}$$

and one iterates (functions of x_1 are functions of degree 0 in $x_2, x_3, \ldots$, the Newton calculus is indeed a multivariate calculus !).

More generally, and more simply, one uses operators $T_1, T_2, \ldots$ on polynomials. Each T_i acts on x_i, x_{i+1} only, and commutes with multiplication with symmetric functions in x_i, x_{i+1}. It is therefore sufficient to define its action on a basis, say $1, x_{i+1}$.

Here are 5 examples :

$$
\begin{array}{c|cccccc}
1 & 1 & 0 & 1 & 1 & t \\
x_{i+1} & x_i & -1 & 0 & x_i + x_{i+1} - 1 & x_i \\
 & \mathfrak{S}_n & \textit{Schubert} & \textit{Demazure} & \textit{Grothendieck} & \textit{Macdonald}
\end{array}
$$

I shall only speak of Schubert and Macdonald.

What is the problem ?

- Find linear bases of the ring of polynomials in $x_1, \ldots, x_n$
- Generate their elements
- Expand every polynomial in these bases
- Recover the multiplicative structure

Starting point: monomial, denoted exponentially :

$$\{x^v = x_1^{v_1} \cdots x_n^{v_n} : v \in \mathbb{N}^n\}.$$

Two other bases (triangular, but not with respect to the same order) :

Schubert $\{Y_v : v \in \mathbb{N}^n\}$ & Macdonald $\{M_v : v \in \mathbb{N}^n\}$.

One specializes into spectral vectors :

$\langle v \rangle$ (components are variables y_j, for Schubert),

$\langle v \rangle$ (components are some $t^i q^j$, for Macdonald).

Definition: Y_v et M_v are the only polynomials of degree $|v|$ such that

$$Y_v(\langle u \rangle) = 0 \text{ and } M_v(\langle u \rangle) = 0,$$

$$\text{for each } u : \ |u| \leq |v|, \, u \neq v \ \ ,$$

plus normalization conditions

The number of vanishing conditions is equal to the dimension of the space minus 1, no wonder that such polynomials exist ! But this the specific choice of the spectral vectors which makes all the beauty and fruitfulness of the theory !

To define the spectral vectors in the Schubert case, one needs a bijection between integral vectors and permutations, the code of a permutation: Given $\sigma \in \mathfrak{S}_N$, its code v is the sequence of number of inversions due to $\sigma_1, \sigma_2, \ldots$, i.e.

$$v_i := \#\{j : j > i \,\&\, \sigma_i > \sigma_j\}$$

Then, one defines

$$\langle v \rangle = [y_{\sigma_1}, y_{\sigma_2}, y_{\sigma_3}, \ldots,]$$

i.e. instead of v, one will specialize into the permutation such that v is its code.

To normalize, one chooses the inversions of the permutation, i.e. one requires that

$$Y_v\big(\langle v\rangle\big) = \Cap(v) := \prod_{i<j,\, \sigma_i>\sigma_j} (y_{\sigma_i} - y_{\sigma_j})$$

For example, $\sigma = [3, 5, 1, 4, 2]$ has code $v = [2, 3, 0, 1, 0]$ and

$$\Cap(v) = (y_3 - y_1)(y_3 - y_2)(y_5 - y_1)(y_5 - y_4)(y_5 - y_2)(y_4 - y_2)$$

Supposing known Y_v, with $v_i > v_{i+1}$, one will deduce Y_u, with $u = [v_1, \ldots, v_{i-1}, v_{i+1}, v_i-1, \ldots, v_n]$.

Indeed, one writes $Y_v = f + x_i g$, $f, g \in \mathfrak{Sym}(x_i, x_{i+1})$.

The equations $Y_v\big(\langle v\rangle\big) = \Cap(v)$, $Y_v(\langle(u)) = 0$, which are $\Cap(v) = f\big(\langle v\rangle\big) + \langle u\rangle_i g\big(\langle v\rangle\big)$, $0 = f\big(\langle v\rangle\big) + \langle u\rangle_{i+1} g\big(\langle v\rangle\big)$ imply that g be such that $g\big(\langle v\rangle\big) = g\big(\langle u\rangle\big) = \Cap(u)$, and it is not difficult to check that the vanishing conditions are still satisfied.

In summary, $g = Y_v \partial_i$ is the new Schubert polynomial, and divided differences provide a recursion between Schubert polynomials.

Initial case: dominant vectors, i.e. $v_1 \geq v_2 \geq \cdots \geq v_n$.
One defines

$$Y_v = \prod_{i=1}^{n} \prod_{j=1}^{v_i} (x_i - y_j)$$

Since this is a product of linear factors, it is not difficult to check the vanishing conditions, together with the normalization condition.

For $v \in \mathbb{N}^n$, let ∂^v be a product of divided differences such that $Y_v \partial^v = Y_{0\ldots0}$. Then, for every other Y_u, either $Y_u \partial^v$ is 0 or equal to Y_w, with $w \neq [0 \ldots 0]$. This elementary observation suffices to extend Newton's interpolation to the case of several variables.

Theorem. For every $f \in \mathfrak{Pol}(\mathbf{x}, \mathbf{y})$, one has

$$f(\mathbf{x}) = \sum_{v \in \mathbb{N}^n} f(\mathbf{x}) \partial^v \Big|_{\mathbf{x}=\mathbf{y}} Y_v \, .$$

Proof: Sufficient to test the statement on the Schubert basis. Since one specializes into $\mathbf{x} = \mathbf{y} = \langle 0 \ldots 0 \rangle$, only the term $Y_{0\ldots0}(\langle 0 \ldots 0 \rangle) = 1$ survives !

We need new spectral vectors for Macdonald.

When $\lambda \in \mathbb{N}^n$ is dominant,
the spectral vector $\langle\lambda\rangle$ is $[t^{n-1}q^{\lambda_1}, \ldots, t^0 q^{\lambda_n}]$.

Otherwise, if $v = \lambda\sigma$ (σ minimal), one defines

$$\langle v\rangle = \langle\lambda\rangle\sigma.$$

For example, $\langle 3, 3, 0\rangle = [q^3t^2, q^3t^1, q^0t^0]$,
$\langle 3, 0, 3\rangle = [q^3t^2, q^0t^0, q^3t^1]$, $\langle 0, 3, 3\rangle = [q^0t^0, q^3t^2, q^3t^1]$.

Instead of divided differences, one uses the Hecke algebra which acts by $1T_i = t$, $x_{i+1}T_i = x_i$.

The operators T_i are not sufficient, one needs an affine operation. One takes an infinite set of variables x_i, putting $x_{i+rn} = q^r x_i$, with a second parameter q.
Similarly, $v \in \mathbb{N}^n$ must be thought as an infinite vector : $v_{i+rn} = v_i + r$, $r \in \mathbb{Z}$.

One now has a translation τ and its inverse $\bar{\tau} = \tau^{-1}$:

$$\tau : \; x_i \longrightarrow x_{i+1} \; , \; v_i \longrightarrow v_{i+1}$$

Definition. The Macdonald polynomial M_v, $v \in \mathbb{N}^n$ is the only polynomial of degree $|v|$ such that

$$\begin{aligned} M_v(\langle u \rangle) &= 0\,,\ u \neq v,\ |u| \leq |v| \\ M_v &= x^v q^{-\sum_i \binom{v_i}{2}} + \cdots \end{aligned}$$

Existence and unicity are proved by studying the compatibility of vanishing conditions with respect to the action of T_i or τ.

One writes $M_v = f + x_{i+1} g$, with $f, g \in \mathfrak{Sym}(x_i, x_{i+1})$. Since $M_v(\langle v \rangle s_i) = 0$, $M_v(\langle v \rangle) \neq 0$, there is unique constant c such that $T_i + c$ exchanges the two specializations :

$$M_v \to F := tf + x_i g + c(f + x_{i+1} g)$$

$$\text{and} \quad F(\langle v \rangle s_i) \neq 0,\ F(\langle v \rangle) = 0\,.$$

In final :

$$\boxed{M_{vs_i} = M_v \left(T_i + \frac{t-1}{\langle v \rangle_{i+1} {\langle v \rangle_i}^{-1} - 1} \right)}$$

The affine operation is no more complicated to follow. The polynomial $M_v\bar{\tau}$ inherits all the vanishings of M_v. However $v = [v_1, \ldots, v_n] \to v\tau = [v_2, \ldots, v_n, v_1{+}1]$ increases degree, $M_{v\tau}$ has more vanishings to satisfy, but this is provided to by the linear factor.

In final

$$\boxed{M_{v\tau} = M_v\,\bar{\tau}\,(x_n - 1)}$$

For example,

$$M_{053}(x_1, x_2, x_3) = M_{205}(x_3/q, x_1, x_2)\,(x_3 - 1)$$

What kind of applications ? I shall give one to physics, to illustrate that vanishing conditions are not restricted to the mathematical world.

One wants to describe the space of polynomials in degree 6 in $x_1, \ldots, x_6$, which vanish in all triples

$$[x_i, x_j, x_k] = [t^2, t, 1] \,, \; i < j < k$$

Answer : The space is 5-dimensional, with basis $M_{210210}, M_{212010}, M_{221010}, M_{212100}, M_{221100}$, specialized in $q = 1/t^3$.

Indeed, one finds that

$$M_{210210}\Big|_{q=1/t^3} = \Delta_t(x_1, x_2, x_3)\Delta_t(x_4, x_5, x_6)$$

with $\Delta_t := \prod_{j>i}(tx_j - x_i)$ the RHS satisfying the required vanishing conditions to be a Macdonald polynomial, though it is homogeneous (some care needed, q is not generic!).

The Hecke algebra generate then a 5-dimensional space which is an irreducible representation (deforming the Specht representation of the symmetric group corresponding to the partition $[2, 2, 2]$). Because of the specialization of q, the usual vanishing conditions on Macdonald polynomials imply the vanishing on triples $[t^2, t, 1]$, and conversely for this degree.

The physical model which is supposed to be studied is the XXZ spin chain model with periodic boundary conditions, or the Quantum Hall effect, as well as polynomials solutions of the Quantum Knizhnik-Zamolodchikov equation. You can choose ! I prefer the formulation:

“studying the rule $1T_i = t$, $x_{i+1}T_i = x_i$. ”