On Generalized Wrońskians*

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Abstract

The Wroński determinant (*Wrońskian*), usually introduced in standard courses in Ordinary Differential Equations (ODE), is a very useful tool in algebraic geometry to detect ramification loci of linear systems. The present survey aims to describe some ”materializations” of the Wrońskian and of its close relatives, *the generalized Wrońskians*, in algebraic geometry. Emphasis will be put on the relationships between Schubert Calculus and ODE.

Contents

*Work partially sponsored by PRIN "Geometria sulle Varietà Algebriche" (Coordinatore A. Verra), Politecnico di Torino. The second author was sponsored by an INDAM-GNSAGA grant (2009) for Visiting Professors at the Politecnico di Torino.

Introduction

Let $\mathbf{f} := (f_0, f_1, \ldots, f_r)$ be an $(r+1)$-tuple of holomorphic functions in one complex variable. The *Wrońskian* of $\mathbf{f}$ is the holomorphic function $W(\mathbf{f})$ obtained by taking the determinant of the *Wroński matrix* whose entries of the j-th-row, $0 \le j \le r$, are the j-th derivatives of $(f_0, f_1, \ldots, f_r)$. The first appearance of Wrońskians dates back to 1812, introduced by J. M. Hoene–Wroński (1776–1853) in the treatise [28] – see also [45]. The ubiquity of the Wrońskian in nearly all the branches of mathematics, from analysis to algebraic geometry, from number theory to combinatorics, up to the theory of infinite dimensional dynamical systems, is definitely surprising if compared with its elementary definition. The present survey aims to outline links between some different Wrońskian materializations to make evident their common root. The emphasis will be put on the mutual relationships among linear Ordinary Differential Equations (ODEs), the theory of ramification loci of linear systems (e.g. Weierstrass points on curves) and the intersection theory of complex Grassmann varieties, ruled by the famous *Calculus* [51] elaborated in 1886 by H. C. H. Schubert (1848–1911), to which the Italians M. Pieri (1860–1913) and G. Z. Giambelli (1879–1953) contributed too – see [24, 40].

The notion of Wrońskian belongs to mathematicians' common background because of its most popular application, which provides a method (sketched in Section 2) to find a particular solution of a non-homogeneous linear ODE. It relies on the following key property of the Wrońskian of a fundamental system of solutions of a linear homogeneous ODE: *the derivative of the Wrońskian is proportional to the Wrońskian itself*, whose proof is due to J. Liouville (1809–1882) and N. H. Abel (1802–1829). This apparently innocuous property should be considered as the first historical appearance of Schubert Calculus. To see it, one must embed the Wroński determinant into a full family of *generalized Wrońskians*, already used in 1939 by F. H. Schmidt [50] to study Weierstrass points and, in recent times and with the same motivation, by C. Towse in [52]. For a sample of applications to number theory see also [3] and [35].

If $\boldsymbol{\lambda} = (\lambda_0 \ge \lambda_1 \ge \ldots \ge \lambda_r)$ is a *partition*, the *generalized Wrońskian* $W_{\boldsymbol{\lambda}}(\mathbf{f})$ is the determinant of the matrix whose j-th row, for $0 \le j \le r$, is the row of the derivatives of order $j + \lambda_{r-j}$ of $(f_0, f_1, \ldots, f_r)$. Clearly $W(\mathbf{f}) = W_0(\mathbf{f})$, where 0 stands for the *null partition* $(0, \ldots, 0)$. The derivative of $W(\mathbf{f})$, appeared in the proof of Liouville's–Abel's theorem, is the first example of a generalized Wrońskian, $W_{(1)}(\mathbf{f})$, corresponding to the partition $(1, 0, \ldots, 0)$. The bridge to Schubert Calculus is our generalization of Liouville's and Abel's theorem (see [22]): *Giambelli's formula for generalized Wrońskians holds*. More precisely, if $\mathbf{f}$ is a fundamental system of solutions of a linear ODE *with constant coefficients*, then $W_{\boldsymbol{\lambda}}(\mathbf{f})$ is proportional to the usual Wroński determinant, $W_{\boldsymbol{\lambda}}(\mathbf{f}) = \Delta_{\boldsymbol{\lambda}}(\bar{\mathbf{h}})W_0(\mathbf{f})$, where $\Delta_{\boldsymbol{\lambda}}(\bar{\mathbf{h}})$ is the Schur polynomial associated to a sequence $\bar{\mathbf{h}} = (h_0, h_1, \ldots)$ of explicit polynomial expressions in the coefficients of the given ODE and to the partition $\boldsymbol{\lambda}$ – see Section 7. If the characteristic polynomial of the linear differential equation splits into the product of distinct linear factors, then h_j is nothing else than the j-th complete symmetric polynomial in its roots.

Let us now change the landscape for a while. Take a smooth complex projective curve C of genus $g \geq 0$ and an isomorphism class $L \in Pic^d(C)$ of line bundles of degree d on C. A g^r_d on C is a pair (V, L), where V is a point of the Grassmann variety $G(r+1, H^0(L))$ parameterizing of $(r+1)$-dimensional vector subspaces of the global holomorphic sections of L. If $\mathbf{v} = (v_0, v_1, \ldots, v_r)$ is a basis of V, the *Wrońskian* $W(\mathbf{v})$ is a holomorphic section of the bundle $\mathcal{L}_{g,r,d} := L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}$ – see Section 3. It can be constructed by gluing together local Wrońskians $W(\mathbf{f})$, where $\mathbf{f} = (f_0, f_1, \ldots, f_r)$ is an $(r+1)$-tuple of holomorphic functions representing the basis $\mathbf{v}$ in some open set of C that trivializes L. As changing the basis of V amounts to multiply $W(\mathbf{v})$ by a non-zero complex number, one obtains a well defined point $W(V) := W(\mathbf{v}) \pmod{\mathbb{C}^*}$ in $\mathbb{P}H^0(\mathcal{L}_{g,r,d})$ called the *Wrońskian* of V. The *Wroński map* $G(r+1, H^0(L)) \to \mathbb{P}H^0(\mathcal{L}_{g,r,d})$ mapping V to $W(V)$ is a holomorphic map; two extremal cases show that, in general, it is neither injective nor surjective. Indeed, if C is hyperelliptic and $L \in Pic^2(C)$ is the line bundle defining its unique g^1_2, then $G(2, H^0(L))$ is just a point, and the Wroński map to $\mathbb{P}H^0(\mathcal{L}_{g,1,2})$ is trivially injective and not surjective. On the other hand, if $C = \mathbb{P}^1$ and $L = O_{\mathbb{P}^1}(d)$, then the Wroński map $G(r+1, H^0(O_{\mathbb{P}^1}(d))) \to \mathbb{P}H^0(\mathcal{L}_{0,r,d})$ is a finite surjective morphism whose degree is equal to the Plücker degree of the Grassmannian $G(r+1, d+1)$, thence in this case the Wroński map is not injective, cf. [9].

The problem of determining the pre-image of an element of $\mathbb{P}H^0(\mathcal{L}_{0,r,d})$ through the Wroński map defined on $G(r+1, H^0(O_{\mathbb{P}^1}(d)))$ leads to an intriguing mixing of Geometry, Analysis and Representation Theory. It turns out that certain *non-degenerate elements* of $G(r+1, H^0(O_{\mathbb{P}^1}(d)))$, defined through suitable *intermediate Wrońskians*, correspond to the so-called Bethe vectors appeared in representation theory of the Lie algebra $\mathfrak{sl}_{r+1}(\mathbb{C})$. The correspondence goes through critical points of a remarkable rational function related to Knizhnikov-Zamolodchikov equation on correlation functions of the conformal field theory, [37, 47, 48, 49]. Interestingly, the critical points of the mentioned rational function in the case $r = 1$ were examined in the XIX century, in works of Heine and Stieltjes on second order Fuchsian differential equations having a polynomial solution of a prescribed degree. Schubert Calculus on Grassmannians has been introduced even before. However, the relationship between these items - in the case $r = 1$ - was conceived a decade ago in [46, 49].

In the real framework, the relationship between Wrońskians, Schubert Calculus and rational curves was discovered and studied by L. Goldberg, A. Eremenko& A. Gabrielov, V. Karlhamov & F. Sottile, and others – see [25, 10, 11, 30] and references therein.

More links between linear differential equations, projective curves and Schubert varieties appeared in a local context in the investigations of M. Kazarian on singularities of the boundary of fundamental systems of solutions of linear differential equations, [29].

Here, we take another point of view. A. Nigro proposes to extend the notion of ramification locus of a linear system on a curve to that of ramification locus of a holomorphic section of a Grassmann bundle [39]. The construction was motivated by the following observation (see also [8]): Let $\Gamma_{\mathtt{triv}}(\rho_{r,d})$ be the set of all the sections $\gamma : C \to G(r+1, J^dL)$ such that the pull back of the tautological bundle $\mathcal{S}_r$ over $G(r+1, J^dL)$ is trivial. Then each $g^r_d := (V, L)$ induces a holomorphic section $\gamma_V \in \Gamma_{\mathtt{triv}}(\rho_{r,d})$, via the bundle monomorphism $C \times V \to J^dL$ (cf. Section 5.3). The point is that the space $\Gamma_{\mathtt{triv}}(\rho_{r,d})$ is larger than the space of linear systems, and so the theory becomes richer. A distinguished subvariety indwells in $G(r+1, J^dL)$, called *Wroński subvariety* in [39]. It is a Cartier divisor which occurs as the zero locus of a certain *Wroński section* $\mathbb{W}$. The Wrońskian of any section $\gamma \in \Gamma_{\mathtt{triv}}(\rho_{r,d})$ is defined to be $W_0(\gamma) := \gamma^*\mathbb{W} \,(\mathrm{mod}\, \mathbb{C}^*)$; if $\gamma = \gamma_V$ for some $V \in G(r+1, H^0(L))$, it coincides with the usual Wrońskian of V – see Section 5. In particular, if $\mathcal{M}$ is a line bundle defining the unique g^1_2 over a hyperelliptic curve of genus $g \geq 2$, the extended Wroński map $\Gamma_{\mathtt{triv}}(\rho_{1,2}) \to \mathbb{P}H^0(\mathcal{M}^{\otimes 2} \otimes K)$ is dominant (see [8]), its behavior is closer to the surjectivity of the Wroński map defined on the space of g^r_ds on $\mathbb{P}^1$. The latter, in this case, coincides with $\Gamma_{\mathtt{triv}}(\rho_{r,d})$ modulo identification of V with γ_V.

In general, the construction works as follows. Let $\varrho : F \to X$ be a vector bundle of rank $d+1$ and $\varrho_{r,d} : G(r+1, F) \to X$ be the Grassmann bundle of $(r+1)$-dimensional subspaces of fibers of ϱ. Consider $0 \to \mathcal{S}_r \to \varrho^*_{r,d}F \to \mathcal{Q}_r \to 0$, the universal exact sequence over G, and denote by $\Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q}_r - \varrho^*_{r,d}F))$ the Schur polynomial, associated to the partition $\boldsymbol{\lambda}$, in the coefficients of the Chern polynomial of $\mathcal{Q}_r - \varrho^*_{r,d}F$. As is well known (see e.g. [15, Ch. 14]), the Chow group $A^*(G)$ of cycles modulo rational equivalence is a free $A^*(X)$-module generated by $\mathcal{B} := \{\Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q}_r - \varrho^*_{r,d}F)) \cap [G] \,|\, \boldsymbol{\lambda} \in \mathcal{P}^{(r+1)\times(d-r)}\}$, where $\mathcal{P}^{(r+1)\times(d-r)}$ denotes the set of the partitions $\boldsymbol{\lambda}$ such that $\lambda_0 \leq d-r$, and $\cdot \cap [G]$ denotes the *cap* product with the fundamental class of G. Let $F_\bullet := (F_i)_{d \geq i \geq 0}$ be a filtration of $F_\bullet$ by quotient bundles, such that F_i has rank i. Schubert varieties $\{\Omega_{\boldsymbol{\lambda}}(F_\bullet) \,|\, \boldsymbol{\lambda} \in \mathcal{P}^{(r+1)\times(d-r)}\}$ associated to $F_\bullet$ (the definition is in Section 4.4) play the role of *generalized Wroński subvarieties*. In particular $\Omega_{(1)}(F_\bullet)$ is what in [39] was called the $F_\bullet$*-Wroński subvariety* of G. It is a Cartier divisor, that is the zero locus of a section $\mathbb{W}$ of the bundle $\bigwedge^{r+1} \varrho^*_{r,d}F_{r+1} \otimes \bigwedge^{r+1} \mathcal{S}_r^\vee$ over G. We say that $\mathbb{W}$ is the $F_\bullet$*-Wrońskian*. If $\gamma : X \to G$ is a holomorphic section, its Wrońskian is, by definition, $W_0(\gamma) := \gamma^*\mathbb{W} \in H^0(\bigwedge^{r+1} F \otimes \bigwedge^{r+1} \gamma^*\mathcal{S}_r^\vee)$. Its class in $A_*(X)$ is nothing else than $\gamma^*[\Omega_{(1)}(F_\bullet)] \cap [G]$. The *generalized Wroński class* of γ in $A_*(X)$ is $\gamma^*[\Omega_{\boldsymbol{\lambda}}(F_\bullet)] \cap [X]$, which is the class of $\gamma^{-1}(\Omega_{\boldsymbol{\lambda}}(F_\bullet))$, provided that the codimension of the locus coincides with the expected codimension $|\boldsymbol{\lambda}| := \lambda_0 + \ldots + \lambda_r$. Recall that $[\Omega_{\boldsymbol{\lambda}}(F_\bullet)]$ can be easily computed as an explicit linear combination of the elements of the basis $\mathcal{B}$ above, for instance by the recipe indicated in Section 4, especially Theorem 4.13.

Let now $\varepsilon_i := c_i(\mathcal{S}_r) \in A^*(G)$ be the Chern classes of the tautological bundle $\mathcal{S}_r \to G$. Consider a basis $\mathbf{v} := (v_0, v_1, \ldots, v_r)$ of solutions of the differential equation

$$y^{(r+1)} - \varepsilon_1 y^{(r)} + \ldots + (-1)^{r+1}\varepsilon_{r+1} y = 0, \tag{1}$$

taken in the algebra $(A^*(G) \otimes \mathbb{Q})[[t]]$ of formal power series in an indeterminate t with coefficients in the Chow ring of G with rational coefficients. In Section 7.12 we show that, for each partition $\boldsymbol{\lambda} \in \mathcal{P}^{(d-r)\times(r+1)}$,

$$\Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q}_r - \rho_{r,d}^* F)) = \frac{W_{\boldsymbol{\lambda}}(\mathbf{v})}{W_0(\mathbf{v})},$$

i.e. each element of the $A^*(X)$-basis of the Chow ring of G is the quotient of generalized Wrońskians associated to a fundamental system of solutions of an ordinary linear ODE with constant coefficients taken in $A^*(G)$. This will be a consequence of *Giambelli's formula* for generalized Wrońskians, proven in [22], which so provides another clue of the ubiquity of Wrońskians in mathematics.

The survey was written with an eye on a wide range of readers, not necessarily experts in algebraic geometry. We thank the referees for substantial efforts to improve the presentation.

1 Wrońskians, in General

1.1 In the next two sections let $\mathbb{K}$ be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$ together with their usual euclidean topologies. If $U \subseteq \mathbb{K}$ is an open connected subset of $\mathbb{K}$, we shall write $\mathcal{O}(U)$ for the $\mathbb{K}$-algebra of *regular* $\mathbb{K}$-valued functions defined over U: here *regular* means either C^∞ differentiable if $\mathbb{K} = \mathbb{R}$ or complex holomorphic if $\mathbb{K} = \mathbb{C}$. Let

$$\mathbf{v} := (v_0, v_1, \ldots, v_r) \in \mathcal{O}(U)^{r+1}. \tag{2}$$

If t is a local parameter on U, we denote by $D : \mathcal{O}(U) \to \mathcal{O}(U)$ the usual derivation d/dt. The *Wroński matrix* associated to the $(r+1)$-tuple (2) is the matrix valued regular function:

$$WM(\mathbf{v}) := \begin{pmatrix} \mathbf{v} \\ D\mathbf{v} \\ \vdots \\ D^r\mathbf{v} \end{pmatrix} = \begin{pmatrix} v_0 & v_1 & \ldots & v_r \\ Dv_0 & Dv_1 & \ldots & Dv_r \\ \vdots & \vdots & \ddots & \vdots \\ D^r v_0 & D^r v_1 & \ldots & D^r v_r \end{pmatrix}.$$

The determinant $W_0(\mathbf{v}) := \det(WM(\mathbf{v}))$ is the *Wrońskian* of $\mathbf{v} := (v_0, v_1, \ldots, v_r)$. It will be often written in the form:

$$W_0(\mathbf{v}) := \mathbf{v} \wedge D\mathbf{v} \wedge \ldots \wedge D^r\mathbf{v}. \tag{3}$$

In this paper, however, we want to see Wrońskians as a part of a full family of natural functions generalizing them. They will be called, following the few pieces of literature where they have already appeared ([3], [52]) *generalized Wrońskians*.

1.2 Generalized Wrońskians. Let $r \geq 0$ be an integer. A partition $\boldsymbol{\lambda}$ of length at most $r+1$ is an $(r+1)$-tuple of non-negative integers in the non-increasing order:

$$\boldsymbol{\lambda}: \quad \lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_r \geq 0. \tag{4}$$

The *weight* of $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_r)$ is $|\boldsymbol{\lambda}| := \sum_{i=0}^{r} \lambda_j$, that is $\boldsymbol{\lambda}$ is a partition of the integer $|\boldsymbol{\lambda}|$. In this paper we consider only partitions of length $r+1$. To each partition one may associate a Young–Ferrers diagram, an array of left justified rows, with λ_0 boxes in the first row, λ_1 boxes in the second row, …, and λ_r boxes in the $(r+1)$-th row. We denote by $\mathcal{P}^{(r+1)\times(d-r)}$ the set of all partitions whose Young diagram is contained in the $(r+1)\times(d-r)$ rectangle, i.e. the set of all partitions $\boldsymbol{\lambda}$ such that

$$d - r \geq \lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_r \geq 0.$$

If the last $r-h$ entries of $\boldsymbol{\lambda} \in \mathcal{P}^{(r+1)\times(d-r)}$ are zeros, then we write simply $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_h)$, omitting the last zero parts. For more on partitions see [34].

1.3 Definition. *Let* $\mathbf{v}$ *as in (2) and* $\boldsymbol{\lambda}$ *as in (4). The* generalized Wroński matrix *associated to* $\mathbf{v}$ *and to the partition* $\boldsymbol{\lambda}$ *is, by definition,*

$$WM_{\boldsymbol{\lambda}}(\mathbf{v}) := \begin{pmatrix} D^{\lambda_r}\mathbf{v} \\ D^{1+\lambda_{r-1}}\mathbf{v} \\ \vdots \\ D^{r+\lambda_0}\mathbf{v} \end{pmatrix} := \begin{pmatrix} D^{\lambda_r}v_0 & D^{\lambda_r}v_1 & \ldots & D^{\lambda_r}v_r \\ D^{1+\lambda_{r-1}}v_0 & D^{1+\lambda_{r-1}}v_1 & \ldots & D^{1+\lambda_{r-1}}v_r \\ \vdots & \vdots & \ddots & \vdots \\ D^{r+\lambda_0}v_0 & D^{r+\lambda_0}v_1 & \ldots & D^{r+\lambda_0}v_r \end{pmatrix}.$$

The $\boldsymbol{\lambda}$-generalized Wrońskian *is the determinant of the generalized Wroński matrix:*

$$W_{\boldsymbol{\lambda}}(\mathbf{v}) := \det WM_{\boldsymbol{\lambda}}(\mathbf{v}).$$

Coherently with (3) we shall write the $\boldsymbol{\lambda}$-generalized Wrońskian in the form:

$$W_{\boldsymbol{\lambda}}(\mathbf{v}) := D^{\lambda_r}\mathbf{v} \wedge D^{1+\lambda_{r-1}}\mathbf{v} \wedge \ldots \wedge D^{r+\lambda_0}\mathbf{v}. \tag{5}$$

The usual Wrońskian corresponds to the partition of 0, that is $W(\mathbf{v}) \equiv W_0(\mathbf{v})$.

1.4 Remark. Notation (3) and (5) is convenient because the derivative of any generalized Wrońskian can be computed via Leibniz's rule with respect to the product "$\wedge$":

$$\begin{aligned} D(W_{\boldsymbol{\lambda}}(\mathbf{v})) &= D(D^{\lambda_r}\mathbf{v} \wedge D^{1+\lambda_{r-1}}\mathbf{v} \wedge \ldots \wedge D^{r+\lambda_0}\mathbf{v}) = \\ &= \sum_{\substack{i_0+i_1+\ldots+i_r=1 \\ i_j \geq 0}} D^{i_0+\lambda_r}\mathbf{v} \wedge D^{1+i_1+\lambda_{r-1}}\mathbf{v} \wedge \ldots \wedge D^{r+i_r+\lambda_0}\mathbf{v}. \end{aligned}$$

A simple induction shows that *any* derivative of $W_{\boldsymbol{\lambda}}(\mathbf{v})$ is a $\mathbb{Z}$-linear combination of generalized Wrońskians. Recall, as in Section 1.2, that partitions can be described via Young–Ferrers diagrams, and that a standard *Young tableau* is a numbering of the boxes of the

Young–Ferrers diagram of $\boldsymbol{\lambda}$ with integers $1, \ldots, |\boldsymbol{\lambda}|$ arranged in an increasing order in each column and each row [17].

The following observation has convinced us that the Schubert Calculus can be recast in terms of Wrońskians, see Section 7.

1.5 Theorem. *We have*

$$D^h W(\mathbf{v}) = \sum_{|\boldsymbol{\lambda}|=h} c_{\boldsymbol{\lambda}} W_{\boldsymbol{\lambda}}(\mathbf{v}),$$

where $c_{\boldsymbol{\lambda}}$ is the number of the standard Young tableaux of the Young–Ferrers diagram $\boldsymbol{\lambda}$. ■

The coefficients $c_{\boldsymbol{\lambda}}$'s and their interpretation in terms of Schubert Calculus are very well known; in particular, they can be calculated by the *hook formula*:

$$c_{\boldsymbol{\lambda}} = \frac{|\lambda|!}{k_1 \cdot \ldots \cdot k_{|\lambda|}},$$

where the k_j's, $1 \leq j \leq |\boldsymbol{\lambda}|$, are the *hook lengths* of the boxes of $\boldsymbol{\lambda}$, see [17, p. 53].

2 Wrońskians and Linear ODEs

Wrońskians are usually introduced when dealing with linear *Ordinary Differential Equations* (ODEs).

2.1 We use notation of Section 1.1. For $\mathbf{a}(t) = (a_1(t), \ldots, a_{r+1}(t)) \in \mathcal{O}(U)^{r+1}$ and $f \in \mathcal{O}(U)$, consider the linear ODE

$$D^{r+1}x - a_1(t)D^r x + \ldots + (-1)^{r+1}a_{r+1}(t)x = f \tag{6}$$

and the corresponding linear differential operator $P_{\mathbf{a}}(D) \in End_{\mathbb{K}}(\mathcal{O}(U))$,

$$P_{\mathbf{a}}(D) := D^{r+1} - a_1(t)D^r + \ldots + (-1)^{r+1}a_{r+1}(t). \tag{7}$$

The set of solutions, $\mathcal{S}_{f,\mathbf{a}}$, of (6) is an affine space modelled over $\mathbb{K}^{r+1}$: if x_p is a *particular solution* , then

$$\mathcal{S}_{f,\mathbf{a}} = x_p + \ker P_{\mathbf{a}}(D).$$

The celebrated Cauchy theorem ensures that given a column $\mathbf{c} = (c_j)_{0 \leq j \leq r} \in \mathbb{K}^{r+1}$, there exists a unique element $x_{\mathbf{c}} \in \ker P_{\mathbf{a}}(D)$ such that $D^j f(0) = c_j$, for all $0 \leq j \leq r$. Assume now that $\mathbf{v}$ as in (2) is a basis of $\ker P_{\mathbf{a}}(D)$. A particular solution of (6) can be found through the method of *variation of arbitrary constants*. Assume that

$$\mathbf{c} = \mathbf{c}(t) = \begin{pmatrix} c_0(t) \\ c_1(t) \\ \vdots \\ c_r(t) \end{pmatrix} \in \mathcal{O}(U)^{r+1},$$

and look for a solution of (6) of the form

$$x_p := (\mathbf{v} \cdot \mathbf{c})(t) = \mathbf{v}(t) \cdot \mathbf{c}(t) = \sum_{i=0}^{r} c_i(t) v_i(t)\,,$$

where '$\cdot$" stands for the usual row-by-column product. The condition that $D^j \mathbf{v} \cdot D\mathbf{c} = 0$ for all $0 \leq j \leq r$ means that $D^j x_p = D^j \mathbf{v} \cdot \mathbf{c}$ for all $0 \leq j \leq r$ and $D^{r+1} x_p = D^{r+1} \mathbf{v} \cdot \mathbf{c} + D^r \mathbf{v} \cdot D\mathbf{c}$. The equality

$$P_{\mathbf{a}}(D) x_p = f$$

implies, by substitution, the equation

$$D^r \mathbf{v} \cdot D\mathbf{c} = f.$$

The unknown functions $\mathbf{c} = \mathbf{c}(t)$ must then satisfy the differential equations:

$$WM(\mathbf{v}) \begin{pmatrix} Dc_0 \\ Dc_1 \\ \vdots \\ Dc_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f \end{pmatrix}.$$

The key remark is that the *Wroński matrix is invertible in* $\mathcal{O}(U)$. Thus we get a system of first order ODEs,

$$D\mathbf{c} = (WM(\mathbf{v}))^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f \end{pmatrix},$$

which can be solved by usual methods.

To show the invertibility, one usually shows that if the Wroński matrix does not vanish at some point of U, then it does vanish nowhere on U (recall that U is a connected open set). Assume $W_0(\mathbf{v})(P) \neq 0$ for some $P \in U$. Let us choose a local parameter t on U which is 0 at P, identifying the open set U with a connected neighborhood of the origin. Computing the derivative of the Wrońskian one discovers the celebrated

2.2 Liouville's Theorem ([4, p. 195, §27.6]). *The Wrońskian $W = W_0(\mathbf{v})$ satisfies the differential equation:*

$$DW = a_1 W. \tag{8}$$

The proof of Theorem 2.2 is as follows. By defining $D\mathbf{v}$ as the row whose entries are the derivatives of the entries of $\mathbf{v}$, one notices that

$$P_{\mathbf{a}}(D)\mathbf{v} = (P_{\mathbf{a}}(D)v_0, P_{\mathbf{a}}(D)v_1, \ldots, P_{\mathbf{a}}(D)v_r) = \mathbf{0}\,.$$

Thence $D^{r+1}\mathbf{v} = a_1(t)D^r\mathbf{v} - a_2(t)D^{r-1}\mathbf{v} + \ldots + (-1)^r a_{r+1}(t)\mathbf{v}$ and one gets

$$\begin{aligned} DW_0(\mathbf{v}) &= D(\mathbf{v} \wedge D\mathbf{v} \wedge \ldots \wedge D^r\mathbf{v}) = \mathbf{v} \wedge D\mathbf{v} \wedge \ldots \wedge D^{r-1}\mathbf{v} \wedge D^{r+1}\mathbf{v} \\ &= \mathbf{v} \wedge D\mathbf{v} \wedge \ldots \wedge (a_1(t)D^r\mathbf{v} - a_2(t)D^{r-1}\mathbf{v} + \ldots + (-1)^r\mathbf{v}) = \\ &= a_1(t)\mathbf{v} \wedge D\mathbf{v} \wedge \ldots \wedge D^r\mathbf{v} = a_1(t)W_0(\mathbf{v}). \end{aligned}$$

The Wrońskian then takes the form (Abel's formula):

$$W_0(\mathbf{v}) = W_0(\mathbf{v})(0) \cdot \exp(\int_0^t a(u)du), \tag{9}$$

where $W_0(\mathbf{v})(0)$ denotes the value of the Wrońskian at $t = 0$. Equation (9) shows that if $W(\mathbf{v})(0) \neq 0$ then $W(\mathbf{v})(t) \neq 0$ for all $t \in U$. We shall see in Section 7 why the proof of Liouville's theorem is a first example of the *Schubert Calculus* formalism governing the intersection theory on Grassmann Schemes.

2.3 Generalized Wronkians of Solutions of ODEs. Using *generalized Wrońskians* as in 1.2, Liouville's theorem (8) can be rephrased as

$$W_{(1)}(\mathbf{v}) = a_1(t)W_0(\mathbf{v})\,,$$

and generalized as follows.

2.4 Proposition. *Let* $1^k := (1, 1, \ldots, 1)$ *be the* primitive partition *of the integer* $1 \leq k \leq r+1$. *If* $\mathbf{v} := (v_0, v_1, \ldots, v_r)$ *is a basis of* $\ker P_{\mathbf{a}}(D)$ *then*

$$W_{(1^k)}(\mathbf{v}) = a_k(t)W(\mathbf{v}). \tag{10}$$

Indeed, consider (7). If $v \in \ker P_{\mathbf{a}}(D)$, then it is a $\mathbb{K}$-linear combination of $v_0, v_1, \ldots, v_r$, and hence the Wrońskian of these $r+2$ functions vanishes:

$$W(v, v_0, v_1, \ldots, v_r) = 0.$$

By expanding the Wrońskian along the first column one obtains

$$W(\mathbf{v})D^{r+1}v - W_{(1)}(\mathbf{v})D^r v + \ldots + (-1)^{r+1}W_{(1^{r+1})}(\mathbf{v})v = 0, \tag{11}$$

and combining with $P_{\mathbf{a}}(D) = 0$ this implies

$$\sum_{k=1}^{r+1}(-1)^k(W_{(1^k)}(\mathbf{v}) - a_k(t)W(\mathbf{v}))D^k v = 0. \tag{12}$$

For general $v \in \ker P_{\mathbf{a}}(D)$, the $(r+1)$-tuple $(v, Dv, \ldots, D^r v)$ is linearly independent, and then (12) implies (10) for all $1 \leq k \leq r+1$.

2.5 A natural question arises: Can we conclude that any generalized Wrońskian $W_{\boldsymbol{\lambda}}(\mathbf{v})$ associated to a basis of $\ker P_{\mathbf{a}}(D)$ is a multiple of the Wrońskian $W_0(\mathbf{v})$? The answer is obviously yes. In fact whenever one encounters one exterior factor in the generalized Wrońskian of the form $D^{j+\lambda_{r-j}}\mathbf{v}$ with $j+\lambda_{r-j}\geq r+1$, one uses the differential equation to express $D^{j+\lambda_{r-j}}\mathbf{v}$ as a linear combination of lower derivatives of the vector $\mathbf{v}$, with coefficients polynomial expressions in $\mathbf{a}$ and its derivatives,

$$W_{\boldsymbol{\lambda}}(\mathbf{v}) = G_{\boldsymbol{\lambda}}(\mathbf{a}, D\mathbf{a}, D^2\mathbf{a}, \ldots)W(\mathbf{v}).$$

The coefficient $G_{\boldsymbol{\lambda}}(\mathbf{a}, D\mathbf{a}, D^2\mathbf{a}, \ldots)$ assumes a particular interesting form in the case the coefficients $\mathbf{a}$ of the equation are constant (so $D^i\mathbf{a}=0$, for $i>0$). We will address this case in Section 7.

3 Wroński Sections of Line Bundles

3.1 A *holomorphic vector bundle* of rank $d+1$ on a smooth complex projective variety X is a holomorphic map $\varrho : F \to X$, where the complex manifold F is locally a product of X and a complex $(d+1)$-dimensional vector space, cf. [26, page 69]. For $P \in X$, we denote by $F_P := \varrho^{-1}(F) \subset F$ the fiber.

Consider the vector space $H^0(F) := H^0(X, F)$ of global holomorphic sections of F (omitting the base variety when clear from the context). For $s \in H^0(X, F)$ we will denote the value of s at $P \in X$ by $s(P) \in F_P$. The image of s in the stalk of the *sheaf of sections of* F at P will be denoted by s_P.

A *line bundle* over X is a vector bundle of rank 1. The set of isomorphism classes of line bundles on X is a group under the tensor product; this group is denoted by $Pic(X)$. If $\pi : X \to S$ is a proper flat morphism, then we define a *relative line bundle* as an equivalence class of line bundles on X, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are declared equivalent if $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1} \cong \pi^*\mathcal{N}$, for some $\mathcal{N} \in Pic(S)$. The group of isomorphism classes of relative line bundles on X is denoted by $Pic(X/S) := Pic(X)/\pi^*Pic(S)$.

3.2 In the attempt to keep the paper self-contained, we recall a few basic notions about line bundles on a smooth projective complex curve. From now on, we denote the curve by C. It will often be identified with a *compact Riemann surface*, i.e. with a complex manifold of complex dimension 1 equipped with a holomorphic atlas $\mathfrak{A} := \{(U_\alpha, z_\alpha)\,|\,\alpha \in \mathcal{A}\}$, where z_α is a local coordinate on an open U_α. In this context, denote by $\mathcal{O}_C$ the sheaf of holomorphic functions on C: for $(U_\alpha, z_\alpha) \in \mathfrak{A}$ the sheaf $\mathcal{O}_C(U_\alpha)$ is the $\mathbb{C}$-algebra of complex holomorphic functions in z_α.

The *canonical line* bundle of C is the line bundle $K \to C$ whose transition functions are the derivatives of the coordinate changes,

$$\kappa_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*\,, \quad \kappa_{\alpha\beta} = dz_\alpha/dz_\beta\,.$$

The holomorphic functions $\{\kappa_{\alpha\beta}\}$ obviously form a *cocycle*: $\kappa_{\alpha\beta}\kappa_{\beta\gamma} = \kappa_{\alpha\gamma}$. A global holomorphic section $\omega \in H^0(C, K)$ is a *global holomorphic differential*, i.e. a collection $\{f_\alpha\, dz_\alpha\}$, where $f_\alpha \in \mathcal{O}(U_\alpha)$ and $f_{\alpha|_{U_\alpha\cap U_\beta}} = \kappa_{\alpha\beta} f_{\beta|_{U_\alpha\cap U_\beta}}$. We shall write $f_\alpha\, dz_\alpha = \omega_{|U_\alpha}$. The integer $g = h^0(K) := \dim_{\mathbb{C}} H^0(K)$ is the *genus of the curve*.

3.3 Jets of line bundles. Let $\pi : \mathfrak{X} \longrightarrow S$ be a proper flat family of smooth projective curves of genus $g \geq 1$ parameterized by some smooth scheme S. Let $\mathfrak{X} \times_S \mathfrak{X} \to S$ be the 2-fold fiber product of $\mathfrak{X}$ over S and let $p, q : \mathfrak{X} \times_S \mathfrak{X} \to \mathfrak{X}$ be the projections onto the first and the second factor respectively. Denote by $\delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ be the diagonal morphism and by $\mathcal{I}$ the ideal sheaf of the diagonal in $\mathfrak{X} \times_S \mathfrak{X}$. The *relative canonical bundle* of the family π is by definition $K_\pi := \delta^*(\mathcal{I}/\mathcal{I}^2)$. For each $\mathcal{L} \in Pic(\mathfrak{X}/S)$, see 3.1, and each $h \geq 0$ let

$$J^h\mathcal{L} := p_*\left(\frac{O_{\mathfrak{X}\times_S\mathfrak{X}}}{\mathcal{I}^{h+1}} \otimes q^*\mathcal{L}\right) \tag{13}$$

be the bundle of *jets* (or *principal parts*) of $\mathcal{L}$ of order h. As $\mathfrak{X}$ is smooth, $J^h\mathcal{L}$ is a vector bundle on $\mathfrak{X}$ of rank $h+1$.

By definition, $J^0\mathcal{L} = \mathcal{L}$. Set, by convention, $J^{-1}\mathcal{L} = 0$ – the vector bundle of rank 0. The fiber of $J^h\mathcal{L}$ over $P \in \mathfrak{X}$ – a complex vector space of dimension $h+1$ – will be denoted by $J^h_P\mathcal{L}$. The obvious exact sequence

$$0 \longrightarrow \frac{\mathcal{I}^h}{\mathcal{I}^{h+1}} \longrightarrow \frac{O_{\mathfrak{X}\times_S\mathfrak{X}}}{\mathcal{I}^{h+1}} \longrightarrow \frac{O_{\mathfrak{X}\times_S\mathfrak{X}}}{\mathcal{I}^{h}} \longrightarrow 0,$$

gives rise to an exact sequence (see [32, p. 224] for details):

$$0 \longrightarrow \mathcal{L} \otimes K_\pi^{h+1} \longrightarrow J^h\mathcal{L} \xrightarrow{t_{h,h-1}} J^{h-1}\mathcal{L} \longrightarrow 0. \tag{14}$$

If $\pi_0 : C \to \{pt\}$ is a trivial family over a point, i.e. reduced to a single curve, and if L is any line bundle, then the exact sequence (14) for J^hL remains the same: in this case the relative canonical bundle coincides with the canonical bundle of the curve.

3.4 In notation of Section 3.2, let $v = (v_\alpha)$ be a non-zero holomorphic section of a line bundle L, i.e. $v_\alpha \in \mathcal{O}(U_\alpha)$ and $v_\alpha = \ell_{\alpha\beta}\cdot v_\beta$ on $U_\alpha \cap U_\beta$, where $\{\ell_{\alpha\beta}\}$ are transition functions. Let (U_α, z_α) be a coordinate chart of C trivializing L. Denote by $D_\alpha : \mathcal{O}(U_\alpha) \to \mathcal{O}(U_\alpha)$ the derivation d/dz_α and by D^j_α the j-th iterated of D_α. Then

$$D_h v = \left\{ \begin{pmatrix} v_\alpha \\ D_\alpha v_\alpha \\ \vdots \\ D^h_\alpha v_\alpha \end{pmatrix} \,|\, \alpha \in \mathcal{A} \right\} \tag{15}$$

is a section of J^hL – see [8]. It may thought of as a *global derivative* of order h of the section v. In fact it is a local representation of v together with its first h derivatives.

The truncation morphism occurring in (14), $t_{h,h-1} : J^hL \to J^{h-1}L$, is defined in such a way that $t_{h,h-1}(D_hv(P)) = (D_{h-1}v)(P)$. See [8] for further details.

3.5 One says that $v \in H^0(L)$ vanishes at $P \in C$ with multiplicity at least $h+1$ if $(\mathcal{D}_h v)(P) = 0$. Concretely, if $v_\alpha \in \mathcal{O}_C(U_\alpha)$ is the local representation of v in the open set U_α, then v vanishes at $P \in U_\alpha$ with multiplicity at least $h+1$ if v_α vanishes at P together with all of its first h derivatives. The fact that $D_h v$ is a section of $J^h L$ says that the definition of vanishing at a point P does not depend on the open set U_α containing it.

We also say that the order of v at P is $h \geq 0$ if $D_{h-1}v(P) = 0$ and $D_h v(P) \neq 0$. To each $0 \neq v \in H^0(L)$ one may attach a divisor on C:

$$(v) = \sum_{P \in C} (ord_P v) P \,. \tag{16}$$

The sum (16) is finite because v is locally a holomorphic function and hence its zeros are isolated and the compactness of C implies that they are finitely many. The *degree* of v is $\sum_{P \in C} ord_P v \geq 0$. This number does not depend on a holomorphic section of L, and by definition is the *degree* of L. The degree of the canonical bundle is $2g-2$ [1, p. 8]. The set of isomorphism classes of line bundles of degree d is denoted by $Pic^d(C)$. If $\pi : \mathfrak{X} \longrightarrow S$ is a smooth proper family of smooth curves of genus g, then $Pic^d(\mathfrak{X}/S)$ denotes the relative line bundles of relative degree d. A bundle $\mathcal{L} \in Pic(\mathfrak{X}/S)$ has relative degree d if $\deg(\mathcal{L}_{|\mathfrak{X}_s}) = d$ for each $s \in S$.

3.6 If U is a (finite dimensional complex) vector space, $G(k, U)$ will denote the Grassmannian parameterizing the k-dimensional vector subspaces of U. Let $g^r_d(L)$ be a point on chart C of $G(r+1, H^0(L))$, where $L \in Pic^d(C)$. We write g^r_d for $g^r_d(L)$ and some $L \in Pic^d(C)$. If $E = \sum e_P P$ is an effective divisor on C, and V is a $g^r_d(L)$, let

$$V(-E) := \{v \in V \,|\, ord_P v \geq e_P\},$$

Clearly $V(-E)$ is a vector subspace in V; it is not empty because it contains at least the zero section. If $\dim V(-P) = r$ for all $P \in C$, then the $g^r_d(L)$ is said to be *base point free*. It is very ample if $\dim_{\mathbb{C}} V(-P-Q) = r-1$ for all $(P,Q) \in C \times C$. If V is base point free and $\mathbf{v} := (v_0, v_1, \ldots, v_r)$ is a basis of V, the map

$$\begin{cases} \phi_{\mathbf{v}} & : & C & \longrightarrow & \mathbb{P}^r \\ & & P & \longmapsto & (v_0(P) : v_1(P) : \ldots : v_r(P)) \end{cases} \tag{17}$$

is a morphism whose image is a projective algebraic curve of degree d. Although the complex value of a section at a point is not well defined, the ratio of two sections is. Thus the map (17) is well defined. If V is very ample, (17) is an embedding, i.e. a biholomorphism onto its image.

3.7 Let $\boldsymbol{\omega} := (\omega_0, \omega_1, \ldots, \omega_{g-1})$ be a basis of $H^0(K)$. The map

$$\phi_{\boldsymbol{\omega}} := (\omega_0 : \omega_1 : \ldots : \omega_{g-1}) : C \to \mathbb{P}^{g-1}$$

sending $P \mapsto (\omega_0(P) : \omega_1(P) : \ldots : \omega_{g-1}(P))$ is the *canonical morphism*, that is, its image in $\mathbb{P}^{g-1}$ is a curve of degree $2g-2$. If the canonical morphism is not an embedding, the curve is called *hyperelliptic*.

3.8 Definition. *Let V be a $g^r_d(L)$. A point $P \in C$ is a V-ramification point if there exists $0 \neq v \in V$ such that $D_r v(P) = 0$, i.e. iff there exists a non-zero $v \in V$ vanishing at P with multiplicity $r+1$ at least.*

Ramification points of a g^r_d can be detected as zero loci of suitable Wrońskians. Let

$$\mathbf{v} := (v_0, v_1, \ldots, v_r)$$

be a basis of V and let $v_{i,\alpha} : U_\alpha \to \mathbb{C}$ be holomorphic functions representing the restriction of the section v_i to U_α, for $0 \leq i \leq r$. If $P \in U_\alpha$ is a V-ramification point, let $v = \sum_{i=0}^{r} a_i v_i$ be such that $D_r v(P) = 0$. The last condition translates into the following linear system:

$$WM_\alpha(\mathbf{v}) \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_r \end{pmatrix} := \begin{pmatrix} v_{0,\alpha} & v_{1,\alpha} & \ldots & v_{r,\alpha} \\ D_\alpha v_{0,\alpha} & D_\alpha v_{1,\alpha} & \ldots & D_\alpha v_{r,\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ D^r_\alpha v_{0,\alpha} & D^r_\alpha v_{1,\alpha} & \ldots & D^r_\alpha v_{r,\alpha} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{18}$$

It admits a non-trivial solution if and only if the determinant

$$W_0(\mathbf{v}_\alpha) = \mathbf{v}_\alpha \wedge D_\alpha \mathbf{v}_\alpha \wedge \ldots \wedge D^r_\alpha \mathbf{v}_\alpha \in \mathcal{O}_C(U_\alpha)$$

vanishes at P. It is easy to check that on $U_\alpha \cap U_\beta$ one has (see e.g. [14, Ch. 2-18] or [8])

$$W_0(\mathbf{v}_\alpha) = {\ell_{\alpha\beta}}^{r+1} (\kappa_{\alpha\beta})^{\frac{r(r-1)}{2}} W_0(\mathbf{v}_\beta),$$

and thus the data $\{W_0(\mathbf{v}_\alpha) \mid \alpha \in \mathcal{A}\}$ glue together to give a global holomorphic section

$$W_0(\mathbf{v}) \in H^0(C, L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}), \tag{19}$$

said to be the *Wrońskian of the basis* $\mathbf{v}$ of V. The Wrońskian of any such a basis cannot vanish identically. Indeed, write the section $W_0(\mathbf{v})$ as

$$W_0(\mathbf{v}) := D_r v_0 \wedge \ldots \wedge D_r v_r,$$

where $D_r v$ is as in (15), i.e. $D_r v_j$ is locally represented by the j-th row of the matrix (18). Assume that $W_0(\mathbf{v}_\alpha)$ vanished everywhere along the smooth connected curve C. Then the sections $D_r v_j$, for $0 \leq j \leq r$, corresponding to the columns of the matrix (18), are linearly dependent, that is, up to a basis renumbering,

$$D_r v_0 = a_1 D_r v_1 + \ldots + a_r D_r v_r \in H^0(J^r L).$$

However $D_r : H^0(L) \to H^0(J^rL)$ is a section associated to the surjection $H^0(J^rL) \to H^0(L)$, induced by the truncation map $J^rL \to L \to 0$ (see e.g. [8, Section 2.7]) and one would get the non-trivial linear relation

$$v_0 = a_1v_1 + \ldots + a_rv_r \in H^0(L),$$

against the assumption that $(v_0, v_1, \ldots, v_r)$ is a basis of V.

As a consequence the ramification locus of the given g^r_d occurs in codimension 1. The construction does not depend on the choice of a basis $\mathbf{v}$ of V. Indeed, if $\mathbf{u}$ were another one, then $\mathbf{u} = A\mathbf{v}$ for some invertible $A \in Gl_{r+1}(\mathbb{C})$, and thence $W_0(\mathbf{u}) = \det(A)W_0(\mathbf{v})$. Thus any basis of V defines the same point of $\mathbb{P}H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}})$, which we denote by $W_0(V)$.

3.9 The Wroński Map. We have so constructed a map:

$$\begin{cases} G(r+1, H^0(L)) & \longrightarrow & \mathbb{P}H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}) \\ \\ V & \longmapsto & W_0(V) \end{cases} \tag{20}$$

which associates to each $g^r_d(L)$ its Wrońskian $W_0(V)$. Adopting the same terminology used in the literature when $C = \mathbb{P}^1$ and $L := O_{\mathbb{P}^1}(d)$ (see e.g. [10], [11]), the map (20) will be called *Wroński map*. Its behavior depends on the curve and on the choice of the linear system. It is, in general, neither injective nor surjective as the following two extremal cases show. If $C = \mathbb{P}^1$, the unique bundle of degree d is $O_{\mathbb{P}^1}(d)$, $K = O_{\mathbb{P}^1}(-2)$ and the the Wroński map

$$G(r+1, H^0(O_{\mathbb{P}^1}(d))) \longrightarrow \mathbb{P}H^0(O_{\mathbb{P}^1}((r+1)(d-r))),$$

in this case defined between two varieties of the same dimension, is a finite surjective morphism of degree equal to the Plücker degree of the Grassmannian $G(r+1, d+1)$. In particular it is not injective – see [9, 48] and [10, 11] over the real numbers. At a general point of $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}((r+1)(d-r)))$ (represented by a form of degree $(r+1)(d-r)$) there correspond as many distinct linear systems V as the degree of the Grassmannian. For a closer analysis of the fibers of such a morphism see [47].

On the other hand if C is hyperelliptic and $\mathcal{M} \in Pic^2(C)$ is the line bundle defining its unique g^1_2, cf. Section 3.7, then $G(2, H^0(\mathcal{M}))$ is just a point and the Wroński map:

$$G(2, H^0(\mathcal{M})) \to \mathbb{P}H^0(\mathcal{M}^{\otimes 2} \otimes K)$$

is trivially injective and not surjective, as by Riemann-Roch formula $h^0(\mathcal{M}^{\otimes 2} \otimes K) > 1$.

Later on we shall see how to make the situation more uniform, by enlarging in a natural way the notion of linear system on a curve. It will be one of the bridges connecting this part of the survey with the first one, regarding Wrońskians of differential equations.

3.10 The V-weight of a point. Let V be a g^r_d and $P \in C$. The V*-weight* at P is:

$$wt_V(P) := ord_P W_0(V) = ord_P W_0(\mathbf{v}),$$

for some basis $\mathbf{v}$ of V. The *total weight* of the V-ramification points is:

$$wt_V = \sum_{P \in C} wt_V(P),$$

where the above sum is clearly finite. The *total weight* coincides with the degree of the bundle $L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}$, i.e. the degree of its first Chern class:

$$\begin{aligned} \mathrm{wt}_V &= \int_C c_1(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}) \cap [C] = \\ &= (r+1) \int_C (c_1(L) \cap [C]) + \frac{r(r+1)}{2} \int c_1(K) \cap [C] = \\ &= (r+1)d + (g-1)r(r+1), \end{aligned} \tag{21}$$

which is the so-called *Brill–Segre formula*. For example, a smooth plane curve of degree d can be thought of as an abstract curve (compact Riemann surface) embedded in $\mathbb{P}^2$ via some $V \in G(3, H^0(L))$ for some $L \in Pic^d(C)$:

$$(v_0 : v_1 : v_2) : C \longrightarrow \mathbb{P}^2$$

where $\mathbf{v} := (v_0, v_1, v_2)$ is a basis of V. The V-ramification points correspond, in this case, to flexes of the image of C in $\mathbb{P}^2$. According to the genus-degree formulae, the total number of flexes, keeping multiplicities into account, is given by (21) for $r = 2$

$$f = 3d(d-2),$$

which is one of the famous *Plücker formulas* for plane curves.

3.11 Wrońskians on Gorenstein Curves. Let C be an irreducible plane curve of degree d with δ nodes and κ cusps. Using the extension of the Wrońskian of a linear system defined on a Gorenstein curve, due to Widland and Lax [53], the celebrated Plücker formula

$$f = 3d(d-2) - 6\delta - 8\kappa$$

can be obtained from the tautological identity (see [18] for details):

♯(smooth V–ramification points) =
=♯(ramification points) - ♯(singular ramification points).

For more on jets and Wrońskians on Gorenstein curves see [12] and [13].

3.12 The V-weight of a point P coincides with the weight of its *order partition*. We say that $n \in \mathbb{N}$ is a V-order at $P \in C$ if there exists $v \in V$ such that $ord_P v = n$. Each point possesses only $r+1$ distinct V-orders. In fact n is a V-order if $\dim V(-nP) > \dim V(-(n+1)P)$. We have the following sequence of inequalities:

$$r+1 = \dim V \geq \dim V(-P) \geq \dim V(-2P) \geq \ldots \geq \dim V(-dP) \geq \dim V(-(d+1)P) = 0$$

The last dimension is zero because the unique section of V vanishing at P with multiplicity $d+1$ is zero. At each step the dimension does not drop more than one unit and then there must be precisely $r+1$ jumps. If

$$0 \leq i_0 < i_1 < \ldots < i_r \leq d$$

is the order sequence at some $P \in C$, the *V-order partition at P* is

$$\boldsymbol{\lambda}(P) = (i_r - r, i_{r-1} - (r-1), \ldots, i_1 - 1, i_0).$$

One may choose a basis $(v_0, v_1, \ldots, v_r)$ of V such that $ord_P v_j = i_j$. The use of such a basis shows that the Wrońskian $W_0(\mathbf{v})$ vanishes at P with multiplicity

$$wt_V(P) = \sum_{j=0}^{r} (i_j - j) = |\boldsymbol{\lambda}(P)|$$

The following result is due to [42] (unpublished) and to [52].

3.13 Proposition. *Partition $\boldsymbol{\lambda}$ is the V-order partition at $P \in C$ if and only if $W_{\boldsymbol{\mu}}(\mathbf{v}_\alpha)(P) = 0$, for all $\boldsymbol{\mu} \neq \boldsymbol{\lambda}$ such that $|\boldsymbol{\mu}| \leq |\boldsymbol{\lambda}|$, and $W_{\boldsymbol{\lambda}}(\mathbf{v}_\alpha)(P) \neq 0$ (here $\mathbf{v}_\alpha$ is any local representation of a basis of V around P).*

In this case $W_0(V)$ vanishes at P with multiplicity exactly $|\boldsymbol{\lambda}|$.

3.14 A more intrinsic way to look at Wrońskians and ramification points, which can be generalized to the case of families of curves, is as follows. For $V \in G(r+1, H^0(L))$ one considers the vector bundle map

$$\mathcal{D}_r : C \times V \longrightarrow J^r L \tag{22}$$

defined by $\mathcal{D}_r(P, v) = D_r v(P) \in J^r_P L$. Both bundles have rank $r+1$ and since V has only finitely many ramification points, there is a non-empty open subset of C where the map $\mathcal{D}_r$ has the maximal rank $r+1$. Then $P \in C$ is a V-ramification point if $\mathrm{rk}_P \mathcal{D}_r \leq r$. The rank of $\mathcal{D}_r$ is smaller than the maximum if and only if the determinant map of (22)

$$\bigwedge^{r+1} \mathcal{D}_r : O_C \longrightarrow \bigwedge^{r+1} J^r L$$

vanishes at P. The section $\bigwedge^{r+1}\mathcal{D}_r \in H^0(\bigwedge^{r+1} J^r L) = H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}})$ is precisely the *Wroński section*, which vanishes precisely where the map $\mathcal{D}_r$ drops rank. If $v = a_0 v_0 + \ldots + a_r v_r$, with respect to the basis $\mathbf{v} = (v_0, v_1, \ldots, v_r)$ of V, then $D_r v = a_0 D_r v_0 + a_1 D_r v_1 + \ldots + a_r D_r v_r$. On a trivializing open set U_α of C one has the expression:

$$(D_r v)_{|U_\alpha} = \begin{pmatrix} a_0 v_{0,\alpha} + a_1 v_{1,\alpha} + \ldots + a_r v_{r,\alpha} \\ a_0 D_\alpha v_{0,\alpha} + a_1 D_\alpha v_{1,\alpha} + \ldots + a_r D_\alpha v_{r,\alpha} \\ \vdots \\ a_0 D^r_\alpha v_{0,\alpha} + a_1 D^r_\alpha v_{1,\alpha} + \ldots + a_r D^r_\alpha v_{r,\alpha} \end{pmatrix} = W_0(\mathbf{v}_\alpha) \cdot \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_r \end{pmatrix}.$$

In other words, the local representation of the map $\mathcal{D}_r$ is:

$$W_0(\mathbf{v}_\alpha) : U_\alpha \times \mathbb{C}^{r+1} \longrightarrow U_\alpha \times \mathbb{C}^{r+1}$$

from which:

$$\det(\mathcal{D}_{r|U_\alpha}) = \mathbf{v}_\alpha \wedge D_\alpha \mathbf{v}_\alpha \wedge \ldots \wedge D^r_\alpha \mathbf{v}_\alpha,$$

i.e. $\bigwedge^{r+1}\mathcal{D}_r$ is represented by the *Wrońskian* $W_0(\mathbf{v})$. Changing the basis $\mathbf{v}$ of V, the Wroński section gets multiplied by a non-zero complex number and hence:

$$\bigwedge^{r+1}\mathcal{D}_r \bmod \mathbb{C}^* = W_0(V) \in \mathbb{P}H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}})$$

i.e. precisely the Wrońskian associated to the linear system V.

3.15 How do generalized Wrońskians come into play in this picture? Here the question is more delicate. We have already mentioned that if the V-order partition of a point P is $\boldsymbol{\lambda}(P)$ then the generalized Wrońskians $W_{\boldsymbol{\mu}}(V)$ must vanish for all $\boldsymbol{\mu}$ such that $|\boldsymbol{\mu}| < |\boldsymbol{\lambda}(P)|$ and $W_{\boldsymbol{\lambda}}(P) \neq 0$. It is however well known that the general g^r_d on a general curve C has only simple ramification points, i.e. all the points have weight 1. This says that if a g^r_d has a ramification point with weight bigger than 1, the generalized Wrońskians do not impose independent conditions, as the locus occurs in codimension 1 while the expected codimension is bigger than 1.

To look for more geometrical content one can move along two directions. The first, that we just sketch here, consists in considering families of curves.

Let $\pi : \mathfrak{X} \longrightarrow S$ be a proper flat family of smooth curves of genus g and let $(\mathcal{V}, \mathcal{L})$ be a relative g^r_d, i.e. $\mathcal{V}$ is a locally free subsheaf of $\pi_*\mathcal{L}$ and $\mathcal{L} \in Pic^d(\mathfrak{X}/S)$. One can then study the ramification locus of the relative g^r_d which fiberwise cuts the ramification locus of $\mathcal{V}_s \in G(r+1, H^0(\mathcal{L}_{|\mathfrak{X}_s}))$ through the degeneracy locus of the map

$$\mathcal{D}_r : \pi^*\mathcal{V} \longrightarrow J^r_\pi \mathcal{L},$$

where $J^r_\pi\mathcal{L}$ denotes the jets of $\mathcal{L}$ *along the fibers* (see e.g. [23]). The map above induces a section $\mathcal{O}_{\mathfrak{X}} \to \bigwedge^{r+1} J^r_\pi \mathcal{L} \otimes \bigwedge^{r+1} \pi^*\mathcal{V}$, which is the relative Wrońskian $W_0(\mathcal{V})$ of the family.

Because of the exact sequence (14):

$$\bigwedge^{r+1} J^r_\pi\mathcal{L} \otimes \bigwedge^{r+1} \pi^*\mathcal{V} = \mathcal{L}^{\otimes r+1} \otimes K_\pi^{\otimes\frac{r(r+1)}{2}} \otimes \bigwedge^{r+1}\pi^*\mathcal{V}\,.$$

In this case the class in $A_*(\mathfrak{X})$ of the ramification locus of $\mathcal{V}$ is

$$[Z(W_0(\mathcal{V}))] = c_1(\mathcal{L}^{\otimes r+1} \otimes K_\pi^{\otimes\frac{r(r+1)}{2}}) - \pi^* c_1(\mathcal{V})\,.$$

A second approach to enrich the phenomenology of ramification points consists in keeping the curve fixed and varying the linear system. This is the only possible approach with curves of genus 0: all the smooth rational curves are isomorphic, and all the g^r_ds, with base points or not, are parameterized by the Grassmannian $G(r+1, H^0(O_{\mathbb{P}^1}(d)))$. Here the situation is as nice as one would desire: all what may potentially occur it occurs indeed. For instance, if $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_h$ are partitions such that $\sum|\boldsymbol{\lambda}_i| = (r+1)(d-r)$ (= the total weight of the ramification points of a g^r_d) and $P_1, \ldots, P_h$ are arbitrary points on $\mathbb{P}^1$ one can count the number of all of the linear system such that the V order partition at P_i is precisely $\boldsymbol{\lambda}_i$. However if C has higher genus, such a kind of analysis is not possible anymore. For instance the general curve C of genus $g \geq 2$ has only simple Weierstrass points, i.e. all have weight 1, but each curve carries one and only one canonical system. The picture holding for linear systems on the projective line can be generalized in the case of higher genus curves provided one updates the notion of $g^r_d(L)$ to that of *a section of a Grassmann bundle*, a path which was first indicated in [20] and then further developed in [39] and [8]. Go to the next two sections for a sketch of the construction.

4 Wrońskians of Sections of Grassmann Bundles (in general)

This section is a survey of the contruction appeared in [39], partly published in [8], with some applications in [20].

4.1 Let $\varrho_d : F \to X$ be a vector bundle of rank $d+1$ over a smooth complex projective variety X of dimension $m \geq 0$. For each $0 \leq r \leq d$, let $\varrho_{r,d} : G(r+1, F) \to X$ be the Grassmann bundle of $(r+1)$-dimensional subspaces of the fibers of F. For $r = 0$ we shall write $\varrho_{0,d} : \mathbb{P}(F) \longrightarrow X$, where $\mathbb{P}(F) := G(1, F)$ is the *projective bundle* associated to F. The bundle $G(r+1, F)$ carries universal exact sequence (cf. [15, Appendix B.5.7]):

$$0 \longrightarrow \mathcal{S}_r \xrightarrow{\iota_r} \varrho^*_{r,d}F \longrightarrow \mathcal{Q}_r \longrightarrow 0, \tag{23}$$

where $\mathcal{S}_r$ is the *universal subbundle* of $\varrho^*_{r,d}F$ and $\mathcal{Q}_r$ is the *universal quotient bundle*.

Let

$$\Gamma(\varrho_{r,d}) := \{\text{holomorphic } \gamma : X \to G(r+1, F) \mid \varrho_{r,d} \circ \gamma = id_X\}$$

be the set of holomorphic sections of $\varrho_{r,d}$. The choice of $\gamma \in \Gamma(\varrho_{r,d})$ amounts to specify a vector sub-bundle of F of rank $r+1$. In fact the pull-back $\gamma^*\mathcal{S}_r$ via $\gamma \in \Gamma(\varrho_{r,d})$ is a rank $r+1$ subbundle of F. Conversely, given a rank $r+1$ subbundle $\mathcal{V}$ of F, one may define the section $\gamma_{\mathcal{V}} \in \Gamma(\varrho_{r,d})$ by $\gamma_{\mathcal{V}}(P) = \mathcal{V}_P \in G(r+1, F_P)$. The set $\Gamma(\varrho_{r,d})$ is huge and may have a very nasty behavior: even the case when $X = \mathbb{P}^1$ and $F = J^d O_{\mathbb{P}^1}(d)$, is far from being trivial. In fact it is related with the small quantum cohomology of Grassmannians, see [2]. A first simplification is to fix $\xi \in Pic(X)$ to study the space

$$\Gamma_\xi(\varrho_{r,d}) = \{\gamma \in \Gamma(\varrho_{r,d}) \mid \bigwedge^{r+1} \gamma^*\mathcal{S}_r = \xi\}.$$

Again, if $\xi = O_{\mathbb{P}^1}(n)$ and $F = J^d O_{\mathbb{P}^1}(d)$, then $\Gamma_n(\varrho_{r,d}) := \Gamma_{O_{\mathbb{P}^1}(n)}(\varrho_{r,d})$ can be identified with the space of the holomorphic maps $\mathbb{P}^1 \to G(r+1, d+1)$ of degree n, compactified in [2] via a Quot-scheme construction. We shall see the easiest case ($n = 0$) in Section 6. In the following, for our limited purposes, we shall restrict the attention to the definitely simpler set

$$\Gamma_{\mathtt{triv}}(\varrho_{r,d}) := \{\gamma \in \Gamma(\varrho_{r,d}) \mid \gamma^*\mathcal{S}_r \text{ is a trivial rank } (r+1) \text{ subbundle of } F \to X\}.$$

4.2 Proposition. *The set $\Gamma_{\mathtt{triv}}(\varrho_{r,d})$, if non empty, can be identified with an open set of the Grassmannian $G(r+1, H^0(F))$.*

Proof. If $\gamma \in \Gamma_{\mathtt{triv}}(\varrho_{r,d})$, there is an isomorphism $\phi : X \times \mathbb{C}^{r+1} \to \gamma^*\mathcal{S}_r$. Then $\psi := \gamma^*(\iota_r) \circ \phi : X \times \mathbb{C}^{r+1} \to F$ is a bundle monomorphism. Let $\sigma_i : X \to F$ defined by $\sigma_i(P) = \psi(P, \mathbf{e}_i)$. It is clearly a holomorphic section of F. Furthermore $\sigma_0, \sigma_1, \ldots, \sigma_r$ span an $(r+1)$-dimensional subspace U_γ of $H^0(F)$ which does not depend on the choice of the isomorphism ϕ. Thus $\gamma^*\mathcal{S}_r$ is isomorphic to $X \times U_\gamma$ and $\gamma(P) = \{u(P) \mid u \in U_\gamma\} \in G(r+1, F_P)$. Conversely, if $U \in G(r+1, H^0(F))$, one constructs a vector bundle morphism $\phi : X \times U \to F$ via $(P, u) \mapsto u(P)$. This morphism drops rank if $\bigwedge^{r+1} \phi = 0$, this is a closed condition and so there is an open set $\mathcal{U} \subseteq G(r+1, F)$ such that for $U \in \mathcal{U}$, the map ϕ_U makes $X \times U$ into a vector subbundle of F. One so obtains a section γ_U by setting $\gamma_U(P) = U_P \in G(r+1, F_P)$. The easy check that $\gamma_{U_\gamma} = \gamma$ and that $U_{\gamma_U} = U$ is left to the reader. ∎

4.3 Assume now that F comes equipped with a system $F_\bullet$ of bundle epimorphisms $q_{ij} : F_i \longrightarrow F_j$, for each $-1 \le j \le i \le d$, such that $F_d = F$, where F_i has rank $i+1$, $q_{ii} = id_{F_i}$ and $q_{ij}q_{jk} = q_{ik}$ for each triple $d \ge i \ge j \ge k \ge -1$. We set $F_{-1} = 0$ by convention. The map $q_{dj} : F \to F_j$ will be simply denoted by q_j and $\{\ker(q_i)\}$ gives a filtration of F by subbundles of rank $d-i$. Let

$$\partial_i : \mathcal{S}_r \longrightarrow \varrho_{r,d}^* F_i$$

be the composition of the universal monomorphism $\mathcal{S}_r \to \varrho_{r,d}^* F$ with the map q_i. The universal morphism ι_r can be so identified with ∂_d.

4.4 For each $\boldsymbol{\lambda} \in \mathcal{P}^{(r+1)\times(d-r)}$ the subscheme:

$$\Omega_{\boldsymbol{\lambda}}(\varrho_{r,d}^* F_\bullet) = \{\Lambda \in G(r+1, F) \,|\, \mathrm{rk}_\Lambda \partial_{j+\lambda_{r-j}-1} \le j, \qquad 0 \le j \le r\}, \tag{24}$$

of $G(r+1, F)$, is the $\boldsymbol{\lambda}$-*Schubert variety* associated to the system $F_\bullet$ and to the partition $\boldsymbol{\lambda}$. The Chow classes modulo rational equivalence $\{[\Omega_{\boldsymbol{\lambda}}(\varrho_{r,d}^* F_\bullet)] \,|\, \boldsymbol{\lambda} \in \mathcal{P}^{(r+1)\times(d-r)}\}$ freely generate $A_*(G(r+1, F))$ as a module over $A^*(X)$ through the structural map $\varrho_{r,d}^*$.

4.5 For each $0 \le h \le d+1$, let $N_h(F) := \ker(F \xrightarrow{q_{d-h}} F_{d-h})$. It is a vector bundle of rank h. One can define Schubert varieties according to such a *kernel* flag $N_\bullet(F)$ by setting, for each partition $\boldsymbol{\lambda}$ of length at most $r+1$:

$$\Omega_{\boldsymbol{\lambda}}(\varrho_{r,d}^* N_\bullet(F)) = \{\Lambda \in G(r+1, F) \,|\, \Lambda \cap N_{d+1-(j+\lambda_{r-j})}(F) \ge r+1-j\}\,.$$

It is a simple exercise of linear algebra to show that

$$\Omega_{\boldsymbol{\lambda}}(\varrho_{r,d}^* F_\bullet) = \Omega_{\boldsymbol{\lambda}}(\varrho_{r,d}^* N_\bullet(F)).$$

Both descriptions are useful according to the purposes. The first description is more suited to describe Weierstrass points as in Section 3 (it gives an algebraic generalization of the rank sequence in a Brill-Nöther matrix, see [1, p. 154]), while the second is useful when dealing with linear systems on the projective line (see Section 6 below).

4.6 Definition. *The* $F_\bullet$-Wrońskian *subvariety of* $G(r+1, F)$ *is*

$$\mathfrak{W}_0(\varrho_{r,d}^* F_\bullet) := \Omega_{(1)}(\varrho_{r,d}^* F_\bullet).$$

By (24), the $F_\bullet$-Wroński variety $\mathfrak{W}_0(\varrho_{r,d}^* F_\bullet)$ of $G(r+1, F)$ is the degeneracy scheme of the natural map $\partial_r : \mathcal{S}_r \longrightarrow \varrho_{r,d}^* F_r$, i.e. the zero scheme of the map

$$\bigwedge^{r+1} \partial_r : \bigwedge^{r+1} \mathcal{S}_r \longrightarrow \bigwedge^{r+1} \varrho_{r,d}^* F_r.$$

The map

$$W_0(\varrho_{r,d}^* F_\bullet) := \bigwedge^{r+1} \partial_r \in Hom(\bigwedge^{r+1} \varrho_{r,d}^* \mathcal{S}_r, \bigwedge^{r+1} \varrho_{r,d}^* F_r) = H^0(X, \bigwedge^{r+1} \varrho_{r,d}^* F_r \otimes \bigwedge^{r+1} \varrho_{r,d}^* \mathcal{S}_r^\vee), \tag{25}$$

is the *Wroński section* (of the line bundle $\bigwedge^{r+1} \varrho_{r,d}^* F_r \otimes \bigwedge^{r+1} \varrho_{r,d}^* \mathcal{S}_r^\vee$). The $F_\bullet$-Wroński variety is then a Cartier divisor, because it is the zero scheme of the Wroński section (25). In this setting, the Schubert subvariety $\Omega_{\boldsymbol{\lambda}}(\varrho_{r,d}^* F_\bullet)$ of $G(r+1, F)$, associated to the partition $\boldsymbol{\lambda} \in \mathcal{P}^{(r+1)\times(d-r)}$, plays the role of a generalized Wroński subvariety associated to the system $F_\bullet$.

4.7 Among all such Schubert varieties associated to $F_\bullet$ one can recognize some distinguished ones. It is natural to define the $F_\bullet$-*base locus* subvariety of $G(r+1, F)$ as

$$\mathcal{B}(\varrho_{r,d}^* F_\bullet) = \Omega_{(1^{r+1})}(\varrho_{r,d}^* F_\bullet);$$

and the $F_\bullet$-*cuspidal locus* subvariety as

$$\mathcal{C}(\varrho_{r,d}^* F_\bullet) = \Omega_{(1^r)}(\varrho_{r,d}^* F_\bullet).$$

Each Schubert subvariety $\Omega_{\boldsymbol{\lambda}}(\varrho_{r,d}^* F_\bullet)$ has codimension $|\boldsymbol{\lambda}|$ in $G(r+1, F)$. In particular, the base locus variety $\mathcal{B}(\varrho_{r,d}^* F_\bullet)$ has codimension $r+1$.

4.8 Let $\gamma \in \Gamma(\varrho_{r,d})$. The $F_\bullet$-*ramification locus* of γ is the subscheme $\gamma^{-1}(\mathfrak{W}_0(\varrho_{r,d}^* F_\bullet))$ of X, its $F_\bullet$-*base locus* is $\gamma^{-1}(\mathcal{B}(\varrho_{r,d}^* F_\bullet))$ and its $F_\bullet$-*cuspidal locus* is $\gamma^{-1}(\mathcal{C}(\varrho_{r,d}^* F_\bullet))$. The definition of *Wroński map* defined on sections of Grassmann bundles equipped with filtrations, as in Section 4.3, is very natural too.

4.9 Definition. *For $\gamma \in \Gamma(\varrho_{r,d})$, the section*

$$W_0(\gamma) := \gamma^*(W_0(\varrho_{r,d}^* F_\bullet)) \in H^0(X, \bigwedge^{r+1} F_r \otimes \bigwedge^{r+1} \gamma^* \mathcal{S}_r^\vee)$$

*will be called the $F_\bullet$-*Wrońskian *of γ.*

The class in $A^*(X)$ of the ramification locus of γ is:

$$\begin{aligned} [Z(W_0(\gamma))] &= [\gamma^{-1}(\mathfrak{W}_0(\varrho_{r,d}^* F_\bullet))] = \gamma^*[\mathfrak{W}_0(\varrho_{r,d}^* F_\bullet)] = \\ &= c_1(\bigwedge^{r+1} F_r \otimes \bigwedge^{r+1} \gamma^* \mathcal{S}_r^\vee) \cap [X] = (c_1(F_r) - \gamma^* c_1(\mathcal{S}_r)) \cap [X]. \end{aligned} \tag{26}$$

If X is a curve, the expected dimension of the ramification locus is 0 and so, when γ is not entirely contained in the Wroński variety, the *total weight* w_γ of the ramification points of γ is by definition the degree of the cycle $[\gamma^{-1}(\mathfrak{W}_0(\varrho_{r,d}^* F_\bullet))]$:

$$w_\gamma = \int_X (c_1(F_r) - \gamma^* c_1(\mathcal{S})) \cap [X].$$

According to the definitions above, a point $P \in X$ is a ramification point of $\gamma \in \Gamma(\varrho_{r,d})$ if $W_0(\gamma)(P) = 0$, which amounts to say that the map $\gamma^* \partial_r : \gamma^* \mathcal{S}_r \to F_r$ drops rank at P.

4.10 Definition. *Fix $\xi \in Pic(X)$. The holomorphic map:*

$$\begin{cases} \Gamma_\xi(\varrho_{r,d}) & \longrightarrow & \mathbb{P}H^0(\bigwedge^{r+1} F_r \otimes \xi^\vee) \\ \gamma & \longmapsto & W_0(\gamma) \pmod{\mathbb{C}^*} \end{cases}$$

is the Wroński map *defined on $\Gamma_\xi(\varrho_{r,d})$.*

Indeed $W_0(\gamma)$ is a section of $\gamma^*(\bigwedge^{r+1}\varrho_{r,d}^*F_r\otimes\bigwedge^{r+1}\mathcal{S}_r^\vee)=\bigwedge^{r+1}F_r\otimes\xi^\vee$. The class of the *ramification locus* of γ, as in (26), can be now expressed as:

$$[Z(W_0(\gamma))]=(c_1(F_r)-\xi))\cap[X]\in A^*(X).$$

4.11 The Extended Wroński Map. It is particularly easy to express the Wrońskian of a section $\gamma\in\Gamma_{\mathtt{triv}}(\varrho_{r,d})$. Let $U\in G(r+1,F)$ such that $\gamma=\gamma_U$. The pull-back of the map $\partial_r:\mathcal{S}_r\longrightarrow\varrho_{r,d}^*F_r$ is

$$\gamma^*\partial_r:X\times U\longrightarrow F_r. \tag{27}$$

The Wrońskian is the determinant of the map (27):

$$\bigwedge^{r+1}\gamma^*\partial_r:\bigwedge^{r+1}(X\times U)\to\bigwedge^{r+1}F_r.$$

Once a basis $(u_0,u_1,\ldots,u_r)$ of U is chosen, the Wrońskian

$$\bigwedge^{r+1}\gamma^*\partial_r\in H^0(X,\bigwedge^{r+1}F_r)$$

is represented by the holomorphic section $X\to\bigwedge^{r+1}F_r$ given by:

$$P\mapsto q_r(u_0)(P)\wedge q_r(u_1)(P)\wedge\ldots\wedge q_r(u_r)(P)\in\bigwedge^{r+1}F_P,$$

where q_r is the epimorphism introduced in 4.3. Changing basis the section gets multiplied by a non-zero constant, and so the Wroński map

$$\Gamma_{\mathtt{triv}}(\varrho_{r,d})\longrightarrow\mathbb{P}H^0(X,\bigwedge^{r+1}F_r)$$

defined by $\gamma\mapsto W_0(\gamma)\bmod\mathbb{C}^*\in\mathbb{P}H^0(X,\bigwedge^{r+1}F_r)$ coincides with the map

$$\begin{cases} G(r+1,H^0(F)) & \longrightarrow & \mathbb{P}H^0(X,\bigwedge^{r+1}F_r)\\ \\ U & \longmapsto & q_r(u_0)\wedge q_r(u_1)\wedge\ldots\wedge q_r(u_r)\bmod\mathbb{C}^*\end{cases}$$

where $\mathbf{u}=(u_0,u_1,\ldots,u_r)$ is any basis of U.

4.12 Here is a quick review of intersection theory on $G(r+1,F)$ which is necessary for enumerative geometry purposes. First recall some basic terminology and notation. Let $a=a(t)=\sum_{n\geq 0}a_nt^n$ be a formal power series with coefficients in some ring A and $\boldsymbol{\lambda}$ be a partition as in (4). Set $a_n=0$ for $n<0$. The $\boldsymbol{\lambda}$-*Schur polynomial* associated to a is, by definition:

$$\Delta_{\boldsymbol{\lambda}}(a)=\det(a_{i+\lambda_{r-i}-j})_{0\leq i,j\leq r}=\begin{vmatrix} a_{\lambda_r} & a_{\lambda_{r-1}+1} & \cdots & a_{\lambda_0+r}\\ a_{\lambda_r-1} & a_{\lambda_{r-1}} & \cdots & a_{\lambda_0+r-1}\\ \vdots & \vdots & \ddots & \vdots\\ a_{\lambda_r-r} & a_{\lambda_{r-1}-(r-1)} & \cdots & a_{\lambda_0}\end{vmatrix}. \tag{28}$$

The Chern polynomial of a bundle $\mathcal{E}$ is denoted by $c_t(\mathcal{E})$. Write $c_t(\mathcal{Q}_r - \varrho_{r,d}^* F))$ for the ratio $c_t(\mathcal{Q}_r)/c_t(\varrho_{r,d}^* F)$ of Chern polynomials. According to the Basis Theorem [15, p. 268], the Chow group $A^*(G(r+1, F))$ is a free $A^*(X)$-module (via the structural morphism $\varrho_{r,d}^* : A^*(X) \to A^*(G(r+1, F)))$ generated by

$$\{ \Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q}_r - \varrho_{r,d}^* F)) \cap [G(r+1, F)] \,|\, \boldsymbol{\lambda} \in \mathcal{P}^{(r+1)\times(d-r)} \},$$

If $r = 0$, let

$$\mu^i := (-1)^i c_1(\mathcal{S}_0)^i \cap [\mathbb{P}(F)]$$

for each $i \geq 0$. Then, by [15, Ch. 14], $(\mu^0, \mu^1, \ldots, \mu^d)$ is an $A^*(X)$-basis of $A^*(\mathbb{P}(F))$ and for each $j \geq 0$ the following relation, defining the Chern classes of F, holds:

$$\mu^{d+1+j} + \varrho_{0,d}^* c_1(F)\mu^{d+j} + \ldots + \varrho_{0,d}^* c_{d+1}(F)\mu^j = 0\,. \tag{29}$$

A main result of [21] says that $\bigwedge^{r+1} A^*(\mathbb{P}(F))$ can be equipped with a structure of $A^*(G(r+1, F))$-module of rank 1. It is generated by $\mu^0 \wedge \mu^1 \wedge \ldots \wedge \mu^r$ in such a way that, for each $\boldsymbol{\lambda} \in \mathcal{P}^{(r+1)\times(d-r)}$,

$$\Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q}_r - \varrho_{r,d}^* F)) \cdot \mu^0 \wedge \mu^1 \wedge \ldots \wedge \mu^r = \mu^{\lambda_r} \wedge \mu^{1+\lambda_{r-1}} \wedge \ldots \wedge \mu^{r+\lambda_0}. \tag{30}$$

We shall see in the last section that $\Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q}_r - \varrho_{r,d}^* F))$ are related to Wrońskians associated to a fundamental system of solutions of a suitable differential equation. Define now:

$$\epsilon^i := [\Omega_{(i)}(\varrho_{0,d}^* F_\bullet)] \in A^*(\mathbb{P}(F)), \quad 0 \leq i \leq d,$$

where $\Omega_{(i)}(\varrho_{0,d}^* F_\bullet)$ is nothing but the zero locus in codimension i of the map $\partial_{i-1} : \mathcal{S}_0 \to F_{i-1}$. Because of the relation:

$$\epsilon^i = \sum_{j=0}^{i} \varrho_{0,d}^* c_j(F_{i-1})\mu^{i-j}, \tag{31}$$

it follows that $(\epsilon^0, \epsilon^1, \ldots, \epsilon^d)$ is an $A^*(X)$-basis of $A^*(\mathbb{P}(F))$ as well. For $\boldsymbol{\lambda} \in \mathcal{P}^{(d+1)\times(d-r)}$ let $\epsilon^{\boldsymbol{\lambda}} := \epsilon^{\lambda_r} \wedge \epsilon^{1+\lambda_{r-1}} \wedge \ldots \wedge \epsilon^{r+\lambda_0} \in \bigwedge^{r+1} A^*(\mathbb{P}(F))$. Again by [21], the set $\{\epsilon^{\boldsymbol{\lambda}} \,|\, \boldsymbol{\lambda} \in \mathcal{P}^{(r+1)\times(d-r)}\}$ is an $A^*(X)$-basis of $A^*(G(r+1, F))$. Denote by $[\Omega_{\boldsymbol{\lambda}}(\varrho_{r,d}^* F_\bullet)]$ the class in $A^*(G(r+1, F))$ of the $F_\bullet$-Schubert variety $\Omega_{\boldsymbol{\lambda}}(\varrho_{r,d}^* F_\bullet)$.

4.13 Theorem. *The following equality holds:*

$$[\Omega_{\boldsymbol{\lambda}}(\varrho_{r,d}^* F_\bullet)] = [\Omega_{(\lambda_r)}(\varrho_0^* F_\bullet)] \wedge [\Omega_{(1+\lambda_{r-1})}(\varrho_1^* F_\bullet)] \wedge \ldots \wedge [\Omega_{(r+\lambda_0)}(\varrho_r^* F_\bullet)] = \epsilon^{\boldsymbol{\lambda}} \tag{32}$$

modulo the identification of $A^*(G(r+1, F))$ with $\bigwedge^{r+1} A^*(\mathbb{P}(F))$. ∎

Equality (32) is an elegant and compact re-interpretation of the determinantal formula of Schubert Calculus proven by Kempf and Laksov in [31] to compute classes of degeneracy loci of maps of vector bundles. This formula was first generalized in [43, (8.3)], see also [44, Example 3.5 and Appendix 4]. Then a far reaching generalization was obtained in [16] with help of the correspondences in flag bundles. In fact in [43], the $\mathcal{P}$-ideals of polynomials supported on degeneracy loci were studied, giving a deeper insight in enumerative geometry of these loci. Formula (32) was basically discovered in [19] for trivial bundles. The present formulation is as in [39].

Let us sketch the proof of Theorem 4.13. Set $\mu_j := d - r - \lambda_{r-j}$, then

$$\boldsymbol{\mu} := (\mu_0, \mu_1, \ldots, \mu_r) \in \mathcal{P}^{(r+1)\times(d-r)}.$$

Denote $A_j := N_{\mu_{r-j}+j+1}$ (see Section 4.5), i.e A_j fits into the exact sequence

$$0 \to A_j \to F \to F_{d-(j+\mu_{r-j})-1} \to 0.$$

Then $0 \subsetneq A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_r$ is a flag of subbundles of F_d. The Schubert variety

$$\Omega(A_0, A_1, \ldots, A_r) = \{\Lambda \in G(r+1, F) \,|\, \Lambda \cap A_i \geq i\}$$

coincides with $\Omega_{\boldsymbol{\lambda}}(\varrho^*_{r,d}F_\bullet)$ defined by (24), as a simple check shows. Formula 7.9 in [33], which translates the determinantal formula proven in [31], implies

$$[\Omega(A_0, A_1, \ldots, A_r)] = [\Omega(A_0)] \wedge [\Omega(A_1)] \wedge \ldots \wedge [\Omega(A_r)],$$

which is thence equivalent to (32). ∎

5 Wrońskians of Sections of Grassmann Bundles of Jets

5.1 The general framework of Section 4 shows that the notion of linear system can be generalized into that of pairs $(\gamma, F_\bullet)$, where $F_\bullet$ is a vector bundle on X equipped with a filtration and γ a section of the Grassmann bundle $G(r+1, F)$. This picture can be fruitfully applied in the case of (families of) smooth complex projective curves of genus $g \geq 0$. For the time being let C be any one such, and let $L \in Pic^d(C)$. In this section we shall denote by $\varrho_d : J^dL \to C$ the bundle of jets of $L \to C$ up to the order d. Accordingly, for each $0 \leq r \leq d$, we shall denote $\varrho_{r,d} : G(r+1, J^dL) \to C$ the Grassmann bundle of $(r+1)$-dimensional subspaces of fibers of ϱ. The natural filtration of J^dL given by the quotients $J^dL \to J^iL \to 0$, for $-1 \leq i \leq d$, will be denoted $J^\bullet L$ (setting $J^{-1}L = 0$).

5.2 The *kernel filtration* of J^dL

$$N_\bullet(L) : 0 \subset N_1(L) \subset \ldots \subset N_d(L) \subset N_{d+1}(L) = J^dL \tag{33}$$

is defined through the exact sequence of vector bundles $0 \to N_h(L) \to J^d L \to J^{d-h}L \to 0$, where $N_h(L)$ is a vector bundle of rank h. It will be also called the *osculating flag* – see below and Section 6. The fiber of $N_h(L)$ at $P \in C$ will be denoted by $N_{h,P}(L)$.

As in the previous section, the $\boldsymbol{\lambda}$-generalized *Wrońskian* subvariety of $G(r+1, J^d L)$ is $\Omega_{\boldsymbol{\lambda}}(\varrho_{r,d}^* J^\bullet L)$, which has codimension $|\boldsymbol{\lambda}|$ in $G(r+1, J^d L)$. By virtue of Proposition 4.2, the space $\Gamma_{\mathtt{triv}}(\varrho_{r,d})$ of sections γ of $\varrho_{r,d}$ such that $\gamma^*\mathcal{S}_r$ is a trivial subbundle of $J^d L$, can be identified with an open subset of $G(r+1, H^0(J^d L))$. Hence $\gamma^*\mathcal{S}_r$ is of the form $C \times U$ for some $U \in G(r+1, H^0(J^d L))$. As in section 4 we gain a Wroński map:

$$\Gamma_{\mathtt{triv}}(\varrho_{r,d}) \longrightarrow \mathbb{P}H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}), \tag{34}$$

defined by $\gamma \mapsto W_0(\gamma) \pmod{\mathbb{C}^*}$. As we said, this map is the restriction to the open subset $\Gamma_{\mathtt{triv}}(\varrho_{r,d}) \subseteq G(r+1, H^0(J^d L))$ of the determinant map

$$G(r+1, H^0(J^d L)) \to \mathbb{P}H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}),$$

sending U to $t_r(u_0) \wedge t_r(u_1) \wedge \ldots \wedge t_r(u_r) \pmod{\mathbb{C}^*}$, where $(u_0, u_1, \ldots, u_r)$ is a basis of U and t_r denotes the epimorphism $J^d L \to J^r L$.

5.3 We notice now that each $g^r_d(L)$, i.e. $V \in G(r+1, H^0(L))$, can be seen in fact as an element of $\Gamma_{\mathtt{triv}}(\varrho_{r,d})$, because $\mathcal{D}_d : C \times V \to J^d L$ realizes $C \times V$ as a (trivial) vector subbundle of $J^d L$. Indeed $D_d V := \{D_d v \,|\, v \in V\}$ is an $(r+1)$-dimensional subspace of $H^0(J^d L)$ because the map $J^d L \to L \to 0$ induces the surjection $H^0(J^d L) \to H^0(L) \to 0$, see e.g. [8], and then $\mathcal{D}_d v = 0$ implies $v = 0$.

We have thus an injective map $G(r+1, H^0(L)) \hookrightarrow \Gamma_{\mathtt{triv}}(\varrho_{r,d}) \subseteq G(r+1, H^0(J^d L))$, sending V to $\gamma_{D_d V}$, and

$$W_0(\gamma_{\mathcal{D}_d V}) := D_r u_0 \wedge D_r u_1 \wedge \ldots \wedge D_r u_r \mod \mathbb{C}^* = W_0(V)$$

which proves that our Wroński map defined on $\Gamma_{\mathtt{triv}}(\varrho_{r,d})$, which is in general strictly larger than $G(r+1, H^0(L))$, coincides with the Wrońskian $W_0(V)$ defined in section 3. We are so in condition of defining *generalized Wrońskian subloci*. Recall the natural evaluation map

$$\mathrm{ev} : C \times \Gamma_{\mathtt{triv}}(\varrho_{r,d}) \longrightarrow G(r+1, J^d L)$$

sending $(P, \gamma) \mapsto \gamma(P)$. If $\Omega_{\boldsymbol{\lambda}}(\varrho_{r,d}^* J^\bullet L)$ is a generalized Wroński variety of $G(r+1, J^d L)$, then $\mathrm{ev}^{-1}(\Omega_{\boldsymbol{\lambda}}(\varrho_{r,d}^* J^\bullet L))$ cuts the locus of pairs (P, γ) such that $\gamma(P) \in \Omega_{\boldsymbol{\lambda}}(J^\bullet L)$. We also set $\mathrm{ev}_P(\gamma) = \gamma(P)$, for each $P \in C$. It follows that the general section of any irreducible component of $\mathrm{ev}_P^{-1}(\Omega_{\boldsymbol{\lambda}}(J^\bullet L))$ is a section having $\boldsymbol{\lambda}$ as a ramification partition.

5.4 The map $D_{d,P} : H^0(L) \to J^d_P L$ sending $v \mapsto D_d v(P)$ is a vector space monomorphism. If $V \in G(r+1, H^0(L))$, then $v \in V \cap D_{d,p}^{-1} N_{h,P}(L)$ if and only if $D^h v(P) = 0$, i.e. if and only if v vanishes at P with multiplicity at least h. This explains the terminology *osculating flag* used in Section 5.2.

5.5 Example. More details about the present example are in [20]. Let $\pi : \mathfrak{X} \longrightarrow S$ be a proper flat family of smooth projective curves of genus $g \geq 2$. The *Hodge bundle* of the family is $\mathbb{E}_\pi := \pi_* K_\pi$. The vector bundle map over $\mathfrak{X}$

$$\pi^*\mathbb{E}_\pi \to J^{2g-2}K_\pi$$

is injective and then it induces a section $\gamma_K : \mathfrak{X} \to G(g, J^{2g-2}K_\pi)$. In this case the *cuspidal locus* of γ_K, which is by definition $\gamma_K^{-1}(\Omega_{1^{g-1}}(J^\bullet K))$, coincides with the locus in $\mathfrak{X}$ of the Weierstrass points of the hyperelliptic fibers of π. With the notation as in 4.12 and 4.13, its class in $A^{g-1}(\mathfrak{X})$ is given by

$$[\gamma_K^{-1}(\Omega_{1^{g-1}}(J^\bullet K)] = \gamma_K^*[\Omega_{1^{g-1}}(J^\bullet K)] = \gamma_K^*(\epsilon^0 \wedge \epsilon^2 \wedge \ldots \wedge \epsilon^g)$$

and can be easily computed through a straightforward computation (see [20, Section 3], where the computation was performed for $g = 4$). Since on each hyperelliptic fiber there are precisely $2g+2$ Weierstrass points, the class of the hyperelliptic locus in $A^{g-2}(S)$ is given by

$$[H] = \frac{1}{2g+2} \cdot \pi_*\gamma_K^*(\epsilon^0 \wedge \epsilon^2 \wedge \ldots \wedge \epsilon^g)\,,$$

which yields precisely the formula displayed in [38, p. 314].

5.6 If $C = \mathbb{P}^1$ and $L = O_{\mathbb{P}^1}(d)$, then $\Gamma_{\mathtt{triv}}(\varrho_{r,d})$ coincides in this case with $G(r+1, H^0(L))$ and our picture allows to rephrase in an elegant way the situation exposed in the first part of [9]. The *Wroński map* $\Gamma_{\mathtt{triv}}(\varrho_{r,d}) \to \mathbb{P}H^0(O_{\mathbb{P}^1}((r+1)(d-r)))$ coincides with (20), modulo the identification of $\Gamma_{\mathtt{triv}}(\varrho_{r,d})$ with $G(r+1, H^0(O_{\mathbb{P}^1}(d)))$. In other words, when C is not rational, the theory exposed up to now is a generalization of the theory of linear systems on the projective line, for which we want to spend some additional words in a separate section.

6 Linear Systems on $\mathbb{P}^1$ and the Intermediate Wrońskians

In the case of linear systems g^r_d defined on the projective line, the picture outlined in Section 5 gets simpler. However, even this case is particularly rich of nice geometry interacting with other parts of mathematics.

For the sake of brevity, denote by L_d the invertible sheaf $O_{\mathbb{P}^1}(d)$, i.e. the unique line bundle on $\mathbb{P}^1$ of a fixed degree d. The elements of a basis $\mathbf{x} := (x_0, x_1)$ of $H^0(L_1)$ can be regarded as homogeneous coordinates $(x_0 : x_1)$ on $\mathbb{P}^1$. Furthermore $H^0(L_d) = \mathrm{Sym}^d H^0(L_1)$, i.e. $H^0(L_d)$ can be identified with the $\mathbb{C}$-vector space generated by the monomials $\{x_0^i x_1^{d-i}\}_{0 \leq i \leq d}$, and a g^r_d on $\mathbb{P}^1$ is a point of $G(r+1, H^0(L_d))$. Any basis $\mathbf{v} := (v_0, v_1, \ldots, v_r)$ of $V \in G(r+1, H^0(L_d))$ defines a rational map

$$\varphi_V : \mathbb{P}^1 \to \mathbb{P}^r\,, \quad P \mapsto (v_0(P) : v_1(P) : \ldots : v_r(P))\,. \tag{35}$$

If V has no base points (that is, if $\dim V(-P) = \dim V - 1$ for each $P \in \mathbb{P}^1$), then the image of (35) is a non-degenerated (that is, not contained in any hyperplane) rational curve of degree d in $\mathbb{P}^r$. In particular, if $r+1 = \dim H^0(L_d)$, then $V = H^0(L_d)$ and $\varphi_V(\mathbb{P}^1)$ is nothing else than the rational normal curve of degree d. Each curve of degree d in $\mathbb{P}^r$ can be seen as the rational normal curve in $\mathbb{P}H^0(L_d)$ composed with a projection $\mathbb{P}H^0(L_d) \to \mathbb{P}^r$ whose center is a complementary linear subvariety of $V \in G(r+1, H^0(L_d))$ (see e.g. [9], [30]).

Keeping the notation of Section 5, let $\varrho_d : J^d L_d \to \mathbb{P}^1$ be the bundle of d-jets of L_d. Then $\mathcal{D}_d : \mathbb{P}^1 \times H^0(L_d) \to J^d L$ (cf. (22) is an injective morphism between vector bundles of the same rank, that is, an isomorphism. In particular, the map

$$\begin{cases} \mathcal{D}_{d,P} & : & H^0(L_d) & \longrightarrow & J^d_P L_d \\ & & P & \longmapsto & D_d v(P) \end{cases} \tag{36}$$

is an isomorphism of vector spaces, for each $P \in \mathbb{P}^1$. We define the *osculating flag* at P of $H^0(L_d)$,

$$\mathcal{F}_{\bullet,P} : 0 \subset \mathcal{F}_{1,P} \subset \ldots \subset \mathcal{F}_{d,P} \subset \mathcal{F}_{d+1,P} = J^d_P L,$$

by setting (cf. 5.2)

$$\mathcal{F}_{h,P} = \mathcal{D}^{-1}_{d,p}(N_{h,P}(L)) \subseteq H^0(L_d).$$

In other words, $v \in V \cap \mathcal{F}_{h,P}$ if and only if v vanishes at P with multiplicity at least h, that is, $\mathcal{D}_h v(P) = 0$. In fact, $\mathcal{F}_{h,P}$ may be identified with the vector subspace of the homogeneous polynomials of $H^0(L_d)$ that vanish at P with multiplicity at least h. Yet another interpretation of $\mathcal{F}_{h,P}$ is the set of all hyperplanes of $\mathbb{P}H^0(L_d)$ intersecting the rational normal curve in $\mathbb{P}H^0(L_d))$ at P with multiplicity at least $d - h$.

6.1 The Riemann-Roch formula shows that $h^0(L_d) = h^0(J^d L_d)$; thus the injective "derivative map" $D_d : H^0(L_d) \to H^0(J^d L_d)$ is an isomorphism which itself induces a biholomorphism:

$$G(r+1, H^0(L_d)) \to G(r+1, H^0(J^d L_d)).$$

So one concludes that $\Gamma_{\mathtt{triv}}(\varrho_{r,d}) = G(r+1, H^0(J^d L_d)) \cong G(r+1, H^0(L_d))$ parameterizes all the g^r_d's on $\mathbb{P}^1$ (with base points or not). In particular it is compact.

For $V \in G(r+1, H^0(L_d))$, denote by γ_V the corresponding element of $\Gamma_{\mathtt{triv}}(\varrho_{r,d})$. The evaluation morphism $\mathbb{P}^1 \times G(r+1, H^0(L_d)) \to G(r+1, J^d L_d)$ maps (P, V) to $\gamma_V(P) \in G(r+1, J^d_P L_d)$.

6.2 By 6.1, the *Wroński map* $\gamma \mapsto W_0(\gamma)$ (see (34)) coincides with the Wroński map (20) of Section 3.9:

$$G(r+1, H^0(L_d)) \to \mathbb{P}H^0(L_{(r+1)(d-r)}), \quad V \mapsto W_0(V). \tag{37}$$

It is a finite surjective morphism (see e.g. [9], [30], [47]). Its degree $N_{r,d}$ is precisely the Plücker degree of the Grassmannian $G(r+1, d+1)$:

$$N_{r,d} = \int \sigma_{(1)}^{(r+1)(d-r)} \cap [G(r+1, d+1)] = \frac{1!2!\ldots r! \cdot (r+1)(d-r)!}{(d-r)!(d-r+1)! \cdot \ldots \cdot d!}.$$

Thus, given a homogeneous polynomial W of degree $(d-r)(r+1)$ in two indeterminates (x_0, x_1), there are at most $N_{r,d}$ distinct g^r_d's having W as a Wrońskian. The number $N_{r,d}$ was calculated by Schubert himself in 1886, cf. [51] and [15, p. 274]. In the case of real rational curves, the degree of the Wroński map was obtained by L. Goldberg for $r = 1$ ([25]), and for any $r \geq 1$ by A. Eremenko and A. Gabrielov ([10]). For more considerations on real Wroński map see also [30].

6.3 For any partition $\boldsymbol{\lambda} \in \mathcal{P}^{(r+1)\times(d-r)}$ define

$$\Omega_{\boldsymbol{\lambda}}(P) := \Omega_{\boldsymbol{\lambda}}(\mathcal{F}_{\bullet,P}) \subseteq G(r+1, H^0(L_d)).$$

It is a Schubert variety of codimension $|\boldsymbol{\lambda}|$ in $G(r+1, H^0(L_d))$. If $\boldsymbol{\lambda}(V,P)$ is the order partition of V at P (see Section 3.12) then

$$V \in \Omega^{\circ}_{\boldsymbol{\lambda}(V,P)}(P) \subseteq \Omega_{\boldsymbol{\lambda}(V,P)}(P)\,,$$

and P is a V-ramification point if and only if $|\boldsymbol{\lambda}(V,P)| > 0$. The Wrońskian $W_0(V)$ of V vanishes exactly at the V-ramification points. The total weight of the V-ramification points equals the dimension of $G(r+1, H^0(L_d))$ (one can see that by putting $g = 0$ in (21)).

Let $\{(\underline{P}, \underline{w})\} := \{(P_0, w_0), (P_1, w_1), \ldots, (P_k, w_k)\}$ be a $k+1$-tuple of pairs where $P_i \in \mathbb{P}^1$ and w_i's are positive integers such that

$$\sum_{i=1}^{k} w_i = (r+1)(d-r). \tag{38}$$

Thus, in notation of Section 3.14, if

$$D_{w_i-1}W_0(V) \in H^0(J^{w_i-1}L_{(r+1)(d-r)})$$

vanishes at P_i, for every $0 \leq i \leq k$, then $P_0, P_1, \ldots, P_k$ are exactly the ramification points of V, each one of weight $wt_V(P_i) = w_i = |\lambda(P_i, V)|$. We have

$$\begin{aligned} V &\in \Omega^{\circ}_{\boldsymbol{\lambda}(V,P_0)}(P_0) \cap \Omega^{\circ}_{\boldsymbol{\lambda}(V,P_1)}(P_1) \cap \ldots \cap \Omega^{\circ}_{\boldsymbol{\lambda}(V,P_k)}(P_k) = \\ &= \Omega_{\boldsymbol{\lambda}(V,P_0)}(P_0) \cap \Omega_{\boldsymbol{\lambda}(V,P_1)}(P_1) \cap \ldots \cap \Omega_{\boldsymbol{\lambda}(V,P_k)}(P_k)\,. \end{aligned} \tag{39}$$

Condition (38) means that the "expected dimension" of the intersection (39) is zero. Intersections of Schubert varieties associated with the osculating flags of the normal rational curve were first studied by D. Eisenbud and J. Harris in the eighties, [9]. In particular, they showed that the intersection (39) is zero-dimensional indeed, and hence the number of distinct elements in the intersection is at most

$$\int_{G(r+1,H^0(L_d))} \sigma_{\boldsymbol{\lambda}(P_0,V)} \cdot \sigma_{\boldsymbol{\lambda}(P_1,V)} \cdot \ldots \cdot \sigma_{\boldsymbol{\lambda}(P_k,V)} \cap [G(r+1, H^0(L_d))]\,,$$

where $\sigma_{\boldsymbol{\lambda}}$ is the Schubert cycle defined by the equality $\sigma_{\boldsymbol{\lambda}} \cap [G(r+1, H^0(L_d))] = [\Omega_{\boldsymbol{\lambda}}]$. This fact was used in [7] to deduce explicit formulas (and a list up to $n = 40$) for the number of space rational curves of degree $n-3$ having $2n$ *hyperstalls* at $2n$ prescribed points.

6.4 Preimages of the Wroński Map. Notice that if $P \in \mathbb{P}^1$ is a base point of V, it occurs in the V-ramification locus as well, and the Wrońskian vanishes at it with weight $(r+1)$. The set B_P of linear systems having P as base point is a closed subset of $G(r+1, H^0(L_d))$ of codimension $(r+1)$. In fact $B_P := \mathrm{ev}_P^{-1}(\mathcal{B}(\varrho_{r,d}^* J^\bullet L_d))$, which is a closed subset of codimension $(r+1)$ (cf. Section 4.7).

Let $\{(\underline{P}, \underline{w})\}$ be as in 6.3. Denote by $\mathcal{G}_{r,d}(\underline{P})$ the set of all $V \in G(r+1, H^0(L_d))$ whose base locus contains no P_i, $0 \leq i \leq r$. It is an open dense subset of codimension $(r+1)$,

$$\mathcal{G}_{r,d}(\underline{P}) = G(r+1, H^0(L_d)) \setminus (B_{P_0} \cup B_{P_1} \cup \ldots \cup B_{P_k}) .$$

Consider now a $(k+1)$-tuple of partitions

$$\vec{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k) , \quad |\boldsymbol{\lambda}_j| = w_j, \quad 0 \leq j \leq k.$$

We shall write:

$$\boldsymbol{\lambda}_j := \lambda_{j,0} \geq \lambda_{j,1} \geq \ldots \geq \lambda_{j,r} .$$

The elements of

$$I(\vec{\boldsymbol{\lambda}}, \underline{P}) = \Omega_{\boldsymbol{\lambda}_0}(P_0) \cap \Omega_{\boldsymbol{\lambda}_1}(P_1) \cap \ldots \cap \Omega_{\boldsymbol{\lambda}_k}(P_k) \cap \mathcal{G}_{r,d}(\underline{P}) \subset G(r+1, H^0(L_d)) \tag{40}$$

correspond to the base point free linear systems ramifying at $\underline{P}$ according to $\vec{\boldsymbol{\lambda}}$.

The problem of determining $I(\vec{\boldsymbol{\lambda}}, \underline{P})$ leads to interesting analytic considerations related with Wrońskians. Up to a projective change of coordinates, it is not restrictive to assume that $P_0 = \infty := (0:1)$. Using the coordinate $x = x_1/x_0$, the osculating flag at ∞ shall be denoted by $\mathcal{F}_{\bullet,\infty}$. Accordingly, the partition $\boldsymbol{\lambda}_0$ will be renamed $\boldsymbol{\lambda}_\infty$. Notice that $\mathcal{F}_{j,\infty}$ coincides with the vector space Poly_j of the polynomials of degree at most j in the variable x: in fact a polynomial $P(x)$ (thought of as the affine representation of a homogeneous polynomial of degree d in two variables) vanishes at ∞ with multiplicity j if and only if it has degree $d-j$. For $V \in I(\vec{\boldsymbol{\lambda}}, \underline{P})$, let

$$W_V(x) := \frac{W_0(V)}{x_0^{(r+1)(d-r)}}$$

be the representation of the $W_0(V)$ in the affine open subset of $\mathbb{P}^1$ defined by $x_0 \neq 0$. The degree of the polynomial $W_V(x)$ is less or equal than $(r+1)(d-r)$, because of possible ramifications of V at ∞. We have

$$W_V(x) = (x-z_1)^{w_1} \cdot \ldots \cdot (x-z_k)^{w_k}, \tag{41}$$

where $z_i := x(P_i)$ are the values of the coordinate x at $P_i \in \mathbb{P}^1$; $\sum_{i=1}^k w_i = \deg W_V(x) \leq (r+1)(d-r)$.

For a basis $\mathbf{v} = (v_0, v_1, \ldots, v_r)$ of V, consider $f_i := v_i/x_0^d$ and write $\mathbf{f} = (f_0, f_1, \ldots, f_r)$. According to (3), one writes $W_V(x) = \mathbf{f} \wedge D\mathbf{f} \wedge \ldots \wedge D^r\mathbf{f}$, where

$$D^j\mathbf{f} = \left(\frac{d^j f_i}{dx^j}\right)_{0\leq i\leq r}.$$

The space V can be realized as the solution space of the following differential equation

$$E_V(g) = \begin{vmatrix} g & f_0 & f_1 & \ldots & f_r \\ Dg & Df_0 & Df_1 & \ldots & Df_r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D^r g & D^r f_0 & D^r f_1 & \ldots & D^r f_r \\ D^{r+1} g & D^{r+1} f_0 & D^{r+1} f_1 & \ldots & D^{r+1} f_r \end{vmatrix} = 0\,. \tag{42}$$

6.5 Intermediate Wrońskians. For $V \in I(\vec{\boldsymbol{\lambda}}, \underline{P})$, denote by $V_\bullet$ the flag obtained by the intersection of V and $\mathcal{F}_{\bullet,\infty}$:

$$V_\bullet = \{V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_r = V\}, \quad \dim V_j = j+1, \tag{43}$$

all the polynomials in V_j have degree $\leq d_j$, where $0 \leq d_0 < d_1 < \ldots < d_r \leq d$ is the order sequence of V at P_0 (cf. Section 3.12). Recall that V has no base point and $W_V(x)$ as in (41).

Define *the j-th intermediate Wrońskian of V* as $W_j(x) := W_{V_j}(x)$, the Wrońskian of V_j, $0 \leq j \leq r$. In particular, the r-th intermediate Wrońskian coincides with $W_V(x)$. Non-vanishing properties of intermediate Wrońskians have been recently investigated in an analytic context in [5] and [6] to study factorizations of linear differential operators with non-constant $\mathbb{C}$-valued coefficients.

Intermediate Wrońskians are important because every $V \in G(r+1, \mathrm{Poly}_d)$ is completely determined by the set of its intermediate Wrońskians $W_0(x), \ldots, W_r(x)$. Indeed, the ODE (42) can be rewritten as follows:

$$\frac{d}{dx}\frac{W_r^2(x)}{W_{r-1}(x)W_{r+1}(x)} \cdot \ldots \cdot \frac{d}{dx}\frac{W_2^2(x)}{W_3(x)W_1(x)}\,\frac{d}{dx} \cdot \frac{W_1^2(x)}{W_2(x)W_0(x)} \cdot \frac{d}{dx}\frac{g(x)}{W_1(x)} = 0\,.$$

By [41, Part VII, Section 5, Problem 62], one can take as a basis of V the following set of $r+1$ linearly independent solutions of (42):

$$\begin{aligned} g_0(x) &= W_0(x)\,, \\ g_1(x) &= W_0(x)\int^x \frac{W_0 W_2}{W_1^2}\,, \end{aligned}$$

$$
\begin{aligned}
g_2(x) &= W_0(x)\int^x \left(\frac{W_0(\xi)W_2(\xi)}{W_1^2(\xi)}\int^\xi \frac{W_1W_3}{W_2^2}\right),\\
\ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
g_r(x) &= W_0(x)\int^x \left(\frac{W_0(\xi)W_2(\xi)}{W_1^2(\xi)}\int^\xi \left(\frac{W_1(\tau)W_3(\tau)}{W_2^2(\tau)}\int^\tau \ldots \int^\eta \frac{W_{r-1}W_{r+1}}{W_r^2}\right)\ldots\right).
\end{aligned}
$$

Define now polynomials $Z_0(x), Z_1(x), \ldots, Z_r(x)$ through the following formula:

$$
Z_i(x) = \prod_{j=1}^{k}(x - z_j)^{m_j(i)}, \quad 0 \le i \le r\,, \tag{44}
$$

where

$$
m_j(i) = \lambda_{j,r} + \lambda_{j,r-1} + \ldots + \lambda_{j,r-i}, \qquad 1 \le j \le k.
$$

In particular $Z_r(x) = W_V(x)$.

6.6 Lemma ([48]) *The ratio $T_{r-i}(x) := W_i(x)/Z_i(x)$ is a polynomial of degree*

$$
(i+1)(d-i) - \sum_{l=0}^{i}\lambda_{r-l,\infty} - \sum_{j=1}^{k} m_j(i). \tag{45}
$$

In particular, $T_0(x) = 1$. Thus we have $W_{r-j}(x) = T_j(x)Z_{r-j}(x)$, $0 \le j \le r$. The roots of $T_j(x)$ are said to be *the additional roots* of the $(r-j)$-th intermediate Wrońskian. If (40) contains more than one element, then the intermediate Wrońskians of these elements all differ by the additional roots.

6.7 Non-degenerate planes. ([48]) The intersection (40) contains some distinguished elements, called *non-degenerate planes*. Denote by $\Delta(f)$ the discriminant of a polynomial $f(x)$ and by $\mathrm{Res}(f, g)$ the resultant of polynomials $f(x)$, $g(x)$.

Definition. *We call $V \in I(\vec{\lambda}, \underline{P})$ a* non-degenerate plane *if the polynomials $T_0(x), \ldots, T_{r-1}(x)$*

i) do not vanish at the ramification points $P_1, \ldots, P_k$, i.e. $T_i(z_j) \neq 0$ for all $0 \le i \le r-1$ and all $1 \le j \le k$;

ii) do not have multiple roots: $\Delta(T_i) \neq 0$, for all $0 \le i \le r$;

iii) For each $1 \le i \le r$, T_i and T_{i-1} have no common roots: $\mathrm{Res}(T_i, T_{i-1}) \neq 0$.

6.8 Relative discriminants and resultants. Non-degenerate planes correspond to critical points of a certain *generating function* which can be described in terms of *relative* discriminants and resultants. For fixed $\mathbf{z} = (z_1, \ldots, z_k)$, any monic polynomial $f(x)$ can be written in a unique way as the product of two monic polynomials $T(x)$ and $Z(x)$ satisfying

$$
f(x) = T(x)Z(x), \;\; T(z_j) \neq 0, \;\; Z(x) \neq 0 \;\text{ for any } x \neq z_j, \;\; 1 \le j \le k. \tag{46}
$$

One defines *the relative discriminant of $f(x)$ with respect to z* as being

$$\Delta_{\mathbf{z}}(f) = \frac{\Delta(f)}{\Delta(Z)} = \Delta(T)(\mathrm{Res}(Z,T))^2,$$

and *the relative resultant of $f_i(x) = T_i(x)Z_i(x)$, $i = 1, 2$, with respect to* $\mathbf{z}$ as

$$\mathrm{Res}_{\mathbf{z}}(f_1, f_2) = \frac{\mathrm{Res}(f_1, f_2)}{\mathrm{Res}(Z_1, Z_2)} = \mathrm{Res}(T_1, T_2)\mathrm{Res}(T_1, Z_2)\mathrm{Res}(T_2, Z_1).$$

If V is a non-degenerate plane in $I(\vec{\boldsymbol{\lambda}}, \underline{P})$ given by (40), then the decomposition $W_i(x) = T_{r-i}(x)Z_i(x)$ is exactly the same as displayed in (46). The *generating function* of $I(\vec{\boldsymbol{\lambda}}, \underline{P})$ is a rational function such that its critical points determine the non-degenerate elements in such an intersection. Its expression is (see [48]):

$$\Phi_{(\vec{\boldsymbol{\lambda}},\mathbf{z})}(T_1, \ldots, T_{p-1}) = \frac{\Delta_{\mathbf{z}}(W_0) \cdot \ldots \cdot \Delta_{\mathbf{z}}(W_{r-1})}{\mathrm{Res}_{\mathbf{z}}(W_1, W_2) \cdot \ldots \cdot \mathrm{Res}_{\mathbf{z}}(W_{r-1}, W_r)} \tag{47}$$

Part of the following theorem was originally obtained by A. Gabrielov (unpublished), along his investigations of the Wroński map.

6.9 Theorem ([48]) *There is a one-to-one correspondence between the critical points with non-zero critical values of the function $\Phi_{(\vec{\boldsymbol{\lambda}},\mathbf{z})}(T_0, \ldots, T_{r-1})$ and the non-degenerate planes in the intersection $I(\vec{\boldsymbol{\lambda}}, \underline{P})$ given by* (40).

Namely, every such critical point defines the intermediate Wrońskians, and hence a non-degenerate plane, see 6.5. Conversely, for every non-degenerate plane one can calculate the intermediate Wrońskians, and the corresponding polynomials $T_i(x)$ supply a critical point with a non-zero critical value of the *generating function* (47).

6.10 Relation to Bethe vectors in the Gaudin model (see [37, 46, 48]). Once one re-writes (47) in terms of unknown roots of the polynomials T_j's, the generating function turns into the *master function* associated with the *Gaudin model* of statistical mechanics.

In the Gaudin model, the partitions $\boldsymbol{\lambda}_j$, $1 \le j \le k$, of Section 6.4 are the highest weights of $\mathfrak{sl}_{r+1}$-representations, and the j-th representation is marked by the point P_j. Recall that $\boldsymbol{\lambda}_\infty$ is the partition related to $P_0 := (0 : 1) \in \mathbb{P}^1$, after renaming $\boldsymbol{\lambda}_0$, see Section 6.4. Denote by $\boldsymbol{\lambda}_\infty^*$ the partition dual to $\boldsymbol{\lambda}_\infty$. Certain commuting linear operators, called *Gaudin Hamiltonians*, act in the subspace of singular vectors of the weight $\boldsymbol{\lambda}_\infty^*$ in the tensor product of the $\mathfrak{sl}_{r+1}$-representations of the weights $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k$, and one looks for a common eigenbasis of the Gaudin Hamiltonians.

The *Bethe Ansatz* is a method to look for common eigenvectors. It gives a family of vectors of the required weight $\boldsymbol{\lambda}_\infty^*$ meromorphically depending on a number of auxiliary complex parameters. The *Bethe system* is a system of equations on these parameters, and any member of the family that corresponds to a solution of the Bethe system is a common singular eigenvector of the Gaudin Hamiltonians called the *Bethe vector*.

It turns out that the Bethe system coincides with the system on critical points with non-zero critical value of the function $\Phi_{\vec{\lambda},\mathbf{z},}$. In other words, the auxiliary complex parameters are exactly the additional roots of the intermediate Wrońskians! Thus every non-degenerate plane of (40) defines a Bethe vector and vice versa.

This link has led to an essential progress in studies of the Gauidin model as well as in algebraic geometry (e.g., Shapiro-Shapiro conjecture), see [36] and references therein.

7 Linear ODEs and Wroński–Schubert Calculus

This last section surveys and announces the results of [22], an attempt to reconcile the first part of this survey, regarding Wrońskians of fundamental systems of solutions of linear ODEs, with the geometry described in the last four sections. The main observation is that Schubert cycles of a Grassmann bundle can be described through Wrońskians associated with a fundamental system of solutions of a linear ODE.

7.1 Let us work in the category of (not necessarily finitely generated) associative commutative $\mathbb{Q}$-algebras with unit. Let A be such a $\mathbb{Q}$-algebra. We denote by $A[T]$ and $A[[t]]$ the corresponding A-algebras of polynomials and of formal power series, respectively (here t and T are indeterminates over A). For $\phi = \sum_{n\geq 0} a_n t^n \in A[[t]]$, we write $\phi(0)$ for the "constant term" a_0. If $P(T) \in A[T]$ is a polynomial of degree $r+1$, we denote by $(-1)^i e_i(P)$ the coefficient of T^{r+1-i}, for each $0 \leq i \leq r+1$; for instance, if P is monic, $e_0(P) = 1$, we have:

$$P(T) = T^{r+1} - e_1(P)T^r + \ldots + (-1)^{r+1} e_{r+1}(P).$$

Let B be another $\mathbb{Q}$-algebra. Each $\psi \in Hom_{\mathbb{Q}}(A, B)$ induces two obvious $\mathbb{Q}$-algebra homomorphisms, $A[T] \to B[T]$ and $A[[t]] \to B[[t]]$, the both are also denoted by ψ. The former is defined by $e_i(\psi(P)) = \psi(e_i(P))$ and the latter by $\sum_{n\geq 0} a_n t^n \mapsto \sum_{n\geq 0} \psi(a_n) t^n$.

7.2 Let $E_r := \mathbb{Q}[e_1, e_2, \ldots, e_{r+1}]$ be the polynomial $\mathbb{Q}$-algebra in the set of indeterminates $(e_1, \ldots, e_{r+1})$. We call

$$U_{r+1}(T) = T^{r+1} - e_1 T^r + \ldots + (-1)^{r+1} e_{r+1}$$

the *universal monic polynomial* of degree $r+1$. Thus $e_i(U_{r+1}(T)) = e_i$ for all $0 \leq i \leq r+1$.

Let $\bar{\mathbf{h}} := (h_0, h_1, h_2, \ldots, h_r, h_{r+1}, \ldots)$ be the sequence in E_r defined by the equality of formal power series:

$$\sum_{n\geq 0} h_n t^n = \frac{1}{1 - e_1 t + \ldots + (-1)^{r+1} t^{r+1}} = 1 + \sum_{n\geq 1} (e_1 t - e_2 t^2 + \ldots + (-1)^r e_{r+1} t^{r+1})^n.$$

One gets $h_0 = 1$, $h_1 = e_1$, $h_2 = e_1^2 - e_2$, ... In general $h_n = \det(e_{j-i+1})_{1\leq i,j\leq n}$ (see [15, p. 264]).

For any $(r+1)$-tuple or a sequence $\bar{\mathbf{a}} = (a_0, a_1, \dots)$ of elements of any E_r-module, we set

$$U_0(\mathbf{a}) = a_0\,, \quad U_i(\mathbf{a}) = a_i - e_1 a_{i-1} + \dots + (-1)^i e_i a_0\,,\ 1 \le i \le r. \tag{48}$$

Although only $a_0, a_1, \dots, a_r$ appear in (48), we prefer to define U_j also for sequences. We have $U_i(\bar{\mathbf{h}}) = 0$ for all $1 \le i \le r$.

7.3 Let $\mathbf{x} := (x_0, x_1, \dots, x_r)$ and $\bar{\mathbf{f}} := (f_n)_{n\ge0}$ be two sets of indeterminates over $\mathbb{Q}$. Let

$$E_r[\mathbf{x}, \bar{\mathbf{f}}] := E_r[x_0, x_1, \dots, x_r; f_0, f_1, \dots]$$

be the $\mathbb{Q}$-polynomial algebra and $E_r[\mathbf{x}, \bar{\mathbf{f}}][[t]]$ the corresponding algebra of formal power series. Denote by $D := d/dt$ the usual formal derivative of formal power series. Its j-th iterated is:

$$D^j\left(\sum_{n\ge0} a_n \frac{t^n}{n!}\right) = \sum_{n\ge0} a_{n+j}\frac{t^n}{n!}\,, \quad a_m \in E_r[\mathbf{x}, \bar{\mathbf{f}}].$$

Evaluating the polynomial U_{r+1} at D we get the *universal differential operator:*

$$U_{r+1}(D) = D^{r+1} - e_1 D^r + \dots + (-1)^{r+1} e_{r+1}.$$

Let $f := \sum_{n\ge0} f_n \frac{t^n}{n!} \in \mathbb{Q}[\bar{\mathbf{f}}][[t]] \subseteq E_r[\mathbf{x}, \bar{\mathbf{f}}][[t]]$. Consider the universal Cauchy problem for a linear ODE with constant coefficients:

$$\begin{cases} U_{r+1}(D)y &= f, \\ D^i y(0) &= x_i, \quad 0 \le i \le r. \end{cases} \tag{49}$$

We look for solution of (49) in $E_r[\mathbf{x}, \bar{\mathbf{f}}][[t]]$.

7.4 **Theorem.** ([22]) *Let $\sum_{n\ge0} p_n \cdot t^n \in E_r[\mathbf{x}, \bar{\mathbf{f}}][[t]]$ be defined by:*

$$\sum_{n\ge0} p_n t^n = \frac{U_0(\mathbf{x}) + U_1(\mathbf{x})t + \dots + U_r(\mathbf{x})t^r + \sum_{n\ge r+1} f_{n-r-1} t^n}{1 - e_1 t + \dots + (-1)^{r+1} e_{r+1} t^{r+1}}\,, \tag{50}$$

where U_j are as in (48). Then

$$g := \sum_{n\ge0} p_n \frac{t^n}{n!} \tag{51}$$

is the unique solution of the Cauchy problem (49). ∎

The *universality* of $U_{r+1}(D)$ means the following.

7.5 Theorem. *Let A be a $\mathbb{Q}$-algebra, $P \in A[T]$, $\phi = \sum_{n\geq 0} \phi_n t^n/n! \in A[[t]]$ and $(b_0, b_1, \dots, b_r) \in A^{r+1}$ any $(r+1)$-tuple. Then the unique $\mathbb{Q}$-algebra homomorphism, defined by $x_i \mapsto b_i$, $e_i \mapsto e_i(P)$ and $f_i \mapsto \phi_i$, maps the universal solution g, as in (51), to the unique solution of the Cauchy problem*

$$\begin{cases} P(D)y &= \phi, \\ D^i y(0) &= b_i\,, \quad 0 \leq i \leq r. \end{cases} \tag{52}$$

For each $0 \leq i \leq r$, let $\psi_i : E_r[\mathbf{x}, \bar{\mathbf{f}}] \longrightarrow E_r$ be the unique E_r-algebra homomorphism over the identity sending $\mathbf{x} \mapsto (\underbrace{0, \dots, 0}_{i}, 1, h_1, \dots, h_{r-i})$ and $\bar{\mathbf{f}} \mapsto (0, 0, \dots)$.

7.6 Corollary. *If $u_i := \psi_i(g) \in E_r[[t]]$, where g is the unique solution of the universal Cauchy problem (49), then $\mathbf{u} = (u_0, u_1, \dots, u_r)$ is an E_r-basis of $\ker U_{r+1}(D)$.*

Proof. Using the same arguments as in Theorem 7.4, one shows that u_i is a solution of $U_{r+1}(D)y = 0$. Furthermore, if $u := a_0u_0 + a_1u_1 + \dots + a_ru_r = 0$, then u is the unique solution of $U_{r+1}(D)y = 0$, with the zero initial conditions. Then by uniqueness $u = 0$, i.e. $(u_0, \dots, u_r)$ are linearly independent. ∎

7.7 Corollary. *Let A be any $\mathbb{Q}$-algebra and $P \in A[T]$. Let $\psi : E_r \longrightarrow A$ be the unique morphism mapping $e_i \mapsto e_i(P)$. Then $(\psi(u_0), \psi(u_1), \dots, \psi(u_r))$ is an A-basis of $\ker P(D)$.* ∎

In other words, $\ker P(D) \cong \ker U_{r+1}(D) \otimes_{E_r} A$.

7.8 Let $n \geq 0$ be an integer and $\boldsymbol{\mu}$ a partition of length at most $r+1$ with weight n. Denote by $\binom{n}{\boldsymbol{\mu}}$ the coefficient of $x_0^{\mu^0} x_1^{\mu^1} \dots x_r^{\mu^r}$ in the expansion of $(x_0 + x_1 + \dots + x_r)^n$. With the usual convention $0! = 1$, one has

$$\binom{n}{\boldsymbol{\mu}} = \frac{n!}{\mu_0! \mu_1! \dots \mu_r!}.$$

In Section 4.12 the Schur polynomials $\Delta_{\boldsymbol{\mu}}(a)$ associated to partition $\boldsymbol{\mu}$ and to (the coefficients of) a formal power series $a = \sum_{n\geq 0} a_n t^n$ were defined, see (28). In our notation, the coefficients form sequence $\bar{\mathbf{a}} = (a_0, a_1, \dots)$; below we will write $\Delta_{\boldsymbol{\mu}}(\bar{\mathbf{a}})$ instead of $\Delta_{\boldsymbol{\mu}}(a)$.

7.9 Theorem. *For each partition $\boldsymbol{\lambda}$, the following equality holds:*

$$W_{\boldsymbol{\lambda}}(\mathbf{u}) = \sum_{n\geq 0} \sum_{|\boldsymbol{\mu}|=n} \binom{n}{\boldsymbol{\mu}} \Delta_{\boldsymbol{\lambda}+\boldsymbol{\mu}}(\bar{\mathbf{h}}) \frac{t^n}{n!}.$$

In particular, the "constant term" is $W_{\boldsymbol{\lambda}}(\mathbf{u})(0) = \Delta_{\boldsymbol{\lambda}}(\bar{\mathbf{h}})$. ∎

It is a straightforward combinatorial exercise made easy by the use of the basis $\mathbf{u}$ found in 7.6. See [22] for details.

7.10 Proposition. *Giambelli's formula for Wrońskians holds:*

$$W_{\boldsymbol{\lambda}}(\mathbf{u}) = \Delta_{\boldsymbol{\lambda}}(\bar{\mathbf{h}}) \cdot W_0(\mathbf{u}).$$

Proof. First of all, by Remark 2.5, $W_{\boldsymbol{\lambda}}(\mathbf{u})$ is proportional to $W_0(\mathbf{u})$, i.e. $W_{\boldsymbol{\lambda}}(\mathbf{u}) = c_{\boldsymbol{\lambda}} W_0(\mathbf{u})$ for some $c_{\boldsymbol{\lambda}} \in E_r$. Next, two formal power series are proportional if and only if the coefficients of the same powers of t are proportional, with the same factor of proportionality. Finally,

$$c_{\boldsymbol{\lambda}} = \frac{W_{\boldsymbol{\lambda}}(\mathbf{u})(0)}{W_0(\mathbf{u})(0)} = \Delta_{\boldsymbol{\lambda}}(\bar{\mathbf{h}}),$$

according to Theorem 7.9. ∎

7.11 Corollary. *Pieri's formula for generalized Wrońskians holds:*

$$h_i W_{\boldsymbol{\lambda}}(\mathbf{u}) = \sum_{\boldsymbol{\mu}} W_{\boldsymbol{\mu}}(\mathbf{u}),$$

where the sum is over the partitions $\boldsymbol{\mu} = (\mu_0, \mu_1, \ldots, \mu_r)$ such that $|\boldsymbol{\mu}| = i + |\boldsymbol{\lambda}|$ and

$$\mu_0 \geq \lambda_0 \geq \mu_1 \geq \lambda_1 \geq \ldots \geq \mu_r \geq \lambda_r\,.$$

It is well known that Giambelli's and Pieri's implies each other. See e.g. [15, Lemma A.9.4].

7.12 Let now $\varrho_{r,d} : G \to X$ be a Grassmann bundle, where $G := G(r+1, F)$ and F is a vector bundle of rank $d+1$. As recalled in Section 4.12, $A^*(G)$ is freely generated as $A^*(X)$-module (see [15, Proposition 14.6.5]) by

$$\Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q}_r - \varrho_{r,d}^* F)) \cap [G].$$

The exact sequence (23) implies that $c_t(\mathcal{S}_r)c_t(\mathcal{Q}_r) = c_t(\varrho_{r,d}^* F)$, which is equivalent to

$$1 = c_t(\mathcal{S}_r)\,\frac{c_t(\mathcal{Q}_r)}{c_t(\varrho_{r,d}^* F)} = c_t(\mathcal{S}_r)c_t(\mathcal{Q}_r - \varrho_{r,d}^* F).$$

Set $\varepsilon_i = (-1)^i c_i(\mathcal{S}_r)$ and consider the differential equation

$$D^{r+1}y - \varepsilon_1 \cdot D^r y + \ldots + (-1)^{r+1}\varepsilon_{r+1} \cdot y = 0\,. \tag{53}$$

We look for solutions in $(A^*(G) \otimes \mathbb{Q})[[t]]$. By Corollary (7.7) the unique morphism $\psi : E_r \to A^*(G) \otimes \mathbb{Q}$, sending $e_i \mapsto \varepsilon_i$, maps the universal fundamental system $(u_0, u_1, \ldots, u_r)$ to $\mathbf{v} = (v_0, v_1, \ldots, v_r)$, where $v_i = \psi(u_i)$ and, as a consequence, it maps h_i to $c_i(\mathcal{Q}_r - \varrho_{r,d}^* F)$ and $W_{\boldsymbol{\lambda}}(\mathbf{u})$ to $W_{\boldsymbol{\lambda}}(\mathbf{v})$. Then we have proven that

$$\Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q}_r - \varrho_{r,d}^* F)) = \frac{W_{\boldsymbol{\lambda}}(\mathbf{v})}{W_0(\mathbf{v})}\,.$$

In other words, the Chow group $A^*(G)$ can be identified with the $A^*(X)$-module generated by the generalized Wrońskians associated to the basis $\mathbf{v}$ of solutions of the differential equation (53). In particular we have shown that the class $[\Omega_{\boldsymbol{\lambda}}(\varrho_{r,d}^* F_\bullet)]$ of the generalized Wroński variety $\Omega_{\boldsymbol{\lambda}}(\varrho_{r,d}^* F_\bullet)$ is an $A^*(X)$-linear combination of ratios of generalized Wrońskians associated to the basis $\mathbf{v}$ of (53), by virtue of (30), (31) and (32).

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