Notes on Kebekus lectures on differential forms on singular spaces

by Mateusz Michałek

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The present paper contains the notes taken by Mateusz Michałek from a series of lectures by Stefan Kebekus during IMPANGA Summer School 2010.

1 Lecture one

The aim of this series of lectures is to give an exposition on the extension of well known results concerning differential forms on manifolds to the case of normal varieties.

1.1 Classical theory

Let us first recall basic facts that are the motivation for further constructions.

Fact 1.1. *Let X be a smooth, projective variety of dimension n over $\mathbb{C}$. There exist the sheaf of differential forms Ω^1_X and the dualizing sheaf $\omega_X = \Omega^n_X$. The following theorems hold:*

- *For any locally free sheaf F on X we have $H^i(X, F) \cong H^{n-i}(X, F^* \otimes \omega_X)^\vee$ (Serre duality)*
- *For any ample invertible sheaf L on X we have $H^i(X, L \otimes \omega_X) = 0$ for all $i > 0$ (Kodaira vanishing)*

1.2 Three constructions for singular varieties

From now on we assume that X is a normal, possibly singular, variety. There exist several constructions of sheaves of differential forms that in general give different results.

Construction 1.1 (Kähler differentials)**.** Let Ω_X^1 be the sheaf of Kähler differentials. On $U = \operatorname{Spec} A$ the sections $\Omega_X^1(U)$ form an $\mathcal{O}_X(U)$–module generated by formal symbols df for $f \in A$ that satisfy the relations $d(f+g) = df + dg$, $d(fg) = g(df) + f(dg)$ and $dc = 0$ for c a constant.

This first construction has got many advantages. First of all it is very natural. Moreover, given a morphism of two varieties $f : X \to Y$ one can pull-back differential forms from Y to X.

It also has a few disadvantages: Ω_X^1 does not have to be locally free. Moreover, it does not have to be reflexive and may even have torsion. For a precise criterion when the sheaf of Kähler differentials on a cone over a smooth projective variety has torsion see [GRo]. One does not have a Harder-Narasimhan theory. For $\Omega_X^n := \bigwedge^n \Omega_X^1$ neither Serre duality nor Kodaira vanishing holds.

The motivation for the second construction is Serre duality. For a projective scheme over a field one always has the Grothendieck dualizing sheaf ω_X. This means that for any coherent sheaf F we have an isomorphism $\operatorname{Hom}(F, \omega_X) \cong H^n(X, F)^\vee$. Moreover, if X is Cohen-Macaulay then Serre duality applies. Unfortunately, in case of a singular variety, the sheaves ω_X and Ω_X^n can be different.

Let us now describe the sections of ω_X. The intuition is that these are differential forms defined away from the singularities. More precisely let Z be the singular locus of X and let $U = X \setminus Z$. We have got $i : U \to X$ the natural inclusion. One has $\omega_X = i_*(\Omega_U^n)$. As the sections of ω_U do not depend on a subset of codimension 2, neither do the sections of ω_X. We call this property *the second Riemann extension property*, or using the notation of [B, p.128] we say that ω_X is *normal*. The sheaf ω_X is also torsion free so using the characterization [Har80, Prop. 1.6] it is reflexive.

Construction 1.2 (Reflexive differentials)**.** We define $\Omega_X^{[p]}$ as $i_*(\Omega_U^p)$.

One of the advantages of this construction is that we get a reflexive and in particular torsion free sheaf. As a consequence we have got a Harder-Narasimhan filtration. As already mentioned Serre duality applies if X is

Cohen-Macaulay. One can also have results on the positivity on moduli spaces - for precise results see Theorem 3.13 and the discussion after it. Let us note that $\Omega_X^{[p]}$ and $(\Omega_X^p)^{**}$ are both reflexive sheaves that agree on U. As the codimension of the singular locus it at least 2, they must be equal.

However for the sheaf $\Omega_X^{[p]}$ the vanishing theorems do not hold and in general one cannot define the pull-back.

The following theorem will be the motivation for the last construction.

Theorem 1.2 (Grauert-Riemenschneider [GR])**.** *Let $\pi : \tilde{X} \to X$ be a resolution of singularities of the variety X. Let $\tilde{\omega}_X := \pi_*(\omega_{\tilde{X}})$. Then the following holds:*

1. *The sheaf $\tilde{\omega}_X$ is a subsheaf of the Grothendieck dualizing sheaf ω_X and the inclusion does not depend on the resolution.*

2. *For any ample invertible sheaf L and any integer $q > 0$ we have $H^q(X, \tilde{\omega}_X \otimes L) = 0$.* □

Construction 1.3. Let us choose any resolution of singularities $\pi : \tilde{X} \to X$. We define $\tilde{\Omega}_X^p := \pi_* \Omega_{\tilde{X}}^p$.

The push-forward of a torsion free sheaf has no torsion, so $\tilde{\Omega}_X^p$ is also torsion free. Moreover, by the above theorem one obtains results on vanishing. In this case one can also define the pull-back properly. Unfortunately the results on Serre duality do not apply.

1.3 Comparison of the three constructions

The advantages and disadvantages of each construction are summed up in the following table.

Type	Reflexive	Pull-back	No Torsion	Duality	Vanishing
Ω_X^p	×	✓	×	×	×
$\Omega_X^{[p]}$	✓	×	✓	✓	×
$\tilde{\Omega}_X^p$	×	✓	✓	×	✓

To sum up one might paraphrase Grauert: "We need to make a choice when defining differential forms on singular spaces." The reason for this, is that depending on what property we are interested in, we might have to consider different sheaves. If we need the results on vanishing the last construction is the most appropriate, but if we want to make advantage of Serre duality we should consider the second construction. Of course one would be interested in having just one sheaf that would satisfy both. This motivates an interesting question: when the last two constructions partially coincide, that is for which X we have $\omega_X = \tilde{\omega}_X$.

To answer it let us consider a resolution of singularities $\pi : \tilde{X} \to X$. We assume that the π-exceptional locus E is an snc[1] divisor that is mapped onto the singular locus of X. From the definition, we know that the sections of $\tilde{\omega}_X$ are differential forms on $\tilde{X}$. We also know that $\tilde{X} \setminus E$ is isomorphic to $X \setminus Z$, where Z is the singular locus of X. This means that sections of ω_X on an open set U are differential forms on $\pi^{-1}(U) \setminus E$. We see that definitions of ω_X and $\tilde{\omega}_X$ agree if and only if each differential form defined on $\pi^{-1}(U) \setminus E$ extends to a differential form on $\pi^{-1}(U)$.

Under the assumption that X is Gorenstein both definitions agree iff X is canonical[2].

Main aim: Our aim will be to address the same problem for p-forms, that is when $\tilde{\Omega}^p_X$ and $\Omega^{[p]}_X$ coincide. Under mild assumptions on singularities that are always satisfied when dealing with the minimal model program, we will see that any p-form defined away from the exceptional divisor, extends on it. Let us now state precisely the theorem - all necessary definitions are given in lecture 2.

Theorem 1.3 (Greb, Kebekus, Kovács, Peternell, Theorem 1.5 [GKKP])**.** *Let (X, D) be a log canonical pair. Let $P \subset X$ be the non-klt locus and let $\pi : \tilde{X} \to X$ be a log resolution of singularities. Let $\tilde{D}$ be the largest reduced divisor contained in $\pi^{-1}(P)$. Then for any integer p the sheaf $\pi_* \Omega^p_{\tilde{X}}(\log \tilde{D})$ is reflexive and equal to $\Omega^{[p]}_X(\log D)$.*

Equivalently any p-form defined on the smooth locus of an open set $U \subset X$ can be extended to a p-form on any resolution of singularities. □

For values of p that are small with respect to the dimension of singular locus stronger results are known. The reader is advised to consult [SvS] and [F].

[1]simple normal crossing

[2]The formal definition of canonical singularities will appear in the next section. As we will see it follows from definition that X is canonical iff on the resolution of singularities any n-form defined outside the exceptional locus extends on it.

2 Lecture 2

We will start by recalling well-known facts on the Minimal Model Program and logarithmic sheaves.

2.1 Minimal Model Program

First we will make a short review of the classical results about the Minimal Models for surfaces.

Given a smooth algebraic surface X we may blow-down all -1-curves. We obtain a map $\lambda : X \to X_\lambda$ such that depending on the Kodaira dimension $\kappa(X)$ of X either:

1. If $\kappa(X) \geq 0$:

   The canonical divisor K_{X_λ} is nef and defines a fibration $K_{X_\lambda} : X_\lambda \to Z$ such that $\dim Z = \kappa(X) = \kappa(X_\lambda)$. This is a situation of the Kodaira fiber space.

2. If $\kappa(X) < 0$:

   There exists a fibration $m : X_\lambda \to Z$ such that $-K_{X_\lambda}$ is ample on fibers and $\rho(Z) + 1 = \rho(X_\lambda)$, where ρ is the Picard number. This is the Mori fiber space.

The aim of the Minimal Model Program would be to extend this result to higher dimensions in the following way:

Dream 2.1. *Let X be a projective manifold. Then we have a birational map $\lambda : X \dashrightarrow X_\lambda$ such that X_λ is normal, $\mathbb{Q}$-factorial, with sufficiently mild singularities. Moreover, λ^{-1} does not contract any divisor and one of the following holds:*

1. *If $\kappa(X) \geq 0$:*

   The divisor K_{X_λ} is nef and defines a map $X_\lambda \to Z$, such that $\dim Z = \kappa(K_X) = \kappa(K_{X_\lambda})$.

2. *If $\kappa(X) < 0$:*

   There exists a morphism $m : X_\lambda \to Z$ such that $\rho(Z) + 1 = \rho(X_\lambda)$ and $-K_{X_\lambda}$ is ample on fibers.

It turns out that in case of higher dimensions one should work with pairs (X, D), where D is a divisor. Such a setting often allows to make inductive arguments, by passing to the divisor and hence decreasing the dimension.

Let us now remind definitions concerning types of singularities. All the definitions and much more information can be found in [KM] and [KMM]. Let X be an algebraic variety with an effective $\mathbb{Q}$-divisor D. Let $f : Y \to X$ be a resolution of singularities and let $\tilde{D}$ be the strict transform of D.

Definition 2.2 (log resolution)**.** *We assume that the exceptional locus E of f is a divisor that is mapped onto the singular locus of X. We say that f is a log resolution of the pair (X, D) if $E + \tilde{D}$ is an snc divisor.*

Let us fix canonical divisors K_Y and K_X respectively on Y and X, such that $f_*(K_Y) = K_X$. We assume that $K_X + D$ is $\mathbb{Q}$-Cartier so that its pull-back is well defined. For a given resolution of singularities f we define a_i such that

$$K_Y + \tilde{D} = f^*(K_X + D) + \sum a_i E_i,$$

where E_i are the irreducible exceptional divisors of f. As D is a $\mathbb{Q}$-divisor we may write it as $D = \sum b_i D_i$, where D_i are irreducible and $b_i \in \mathbb{Q}$. We call b_i the coefficients of D.

The following definitions are crucial for the Minimal Model Program [KM, Definition 2.34, Theorem 2.44].

Definition 2.3 (canonical, lc, dlt, klt)**.** *We say that the pair (X, D) is*

- *canonical if there exists a log resolution f such that $a_i \geq 0$ for all i,*
- *lc (log canonical) if all the coefficients of D are less or equal to 1 and there exists a log resolution f such that $a_i \geq -1$ for all i,*
- *dlt (divisorial log terminal) if all the coefficients of D are less or equal to 1 and there exists a log resolution f such that $a_i > -1$ for all i,*
- *klt (Kawamata log terminal) if all the coefficients of D are strictly smaller then 1 and there exists a log resolution f such that $a_i > -1$.*

Moreover, in the cases lc and klt one can write "for any log resolution" instead of "there exists a log resolution" [KMM, Lemma 0.2.12].

For the dimension equal to three the Dream 2.1 comes true in the following setting.

Theorem 2.4. *Let* (X, D) *be a dlt pair where* $\dim X = 3$*. Then we have a birational map* $\lambda : X \dashrightarrow X_\lambda$ *such that* (X_λ, D_λ) *is a dlt pair. Moreover, one of the following holds:*

1. *The divisor* $K_{X_\lambda} + D_\lambda$ *is nef and defines a map* $X_\lambda \to Z$*, such that* $\dim Z = \kappa(K_X + D) = \kappa(K_{X_\lambda} + D_\lambda)$*.*

2. *There exists a morphism* $m : X_\lambda \to Z$ *such that* $\rho(Z) + 1 = \rho(X_\lambda)$ *and* $-K_{X_\lambda} + D_\lambda$ *is ample on fibers.* □

More information on the Minimal Model Program can be found in [KM] and [KMM]. For the recent developments in this area the reader is advised to consult [BCHM], where the case of a klt pair and a big divisor is treated. One also hopes that results similar to Theorem 2.4 can be established in higher dimensions.

2.2 Logarithmic sheaves

Now, we will review basic facts about logarithmic sheaves. First let us consider the following motivation. Let X be a compact manifold and D a smooth divisor. Let $U = X \setminus D$. By $\mathcal{T}$ we denote the tangent sheaf. Given a vector field $s \in H^0(U, \mathcal{T}_U)$ one may ask when it extends to X and stabilizes D. If both of these conditions are satisfied then we obtain a flow on X that stabilizes D. This will be a motivation to define $\mathcal{T}_X(-\log D)$. Its sections will correspond to vector fields whose flow stabilizes D. The formal definition is as follows.

Definition 2.5 ($\mathcal{T}_X(-\log D)$)**.** *Let* X *be a projective manifold,* D *an snc divisor. Let* U *be any open set on which* D *is defined by an equation* ϕ_U*. We set:*

$$\mathcal{T}_X(-\log D)(U) = \{\partial \in Der(\mathcal{O}_X(U)) : \partial\phi_U \in \mathcal{I}_D(U)\},$$

where $\mathcal{I}_D$ *is the ideal sheaf of* D*.*

One can see that the above definition on affine pieces gives a locally free sheaf on X that is a subsheaf of the tangent bundle. By dualizing the inclusion $\mathcal{T}_X(-\log D) \subseteq \mathcal{T}_X$ we get $\Omega^1_X \subseteq \mathcal{T}_X(-\log D)^* =: \Omega^1_X(\log D)$.

Fact 2.6. *For a projective manifold* X *and an snc divisor* D *one has the following description of* $\Omega^1_X(\log D)$*:*

$$\Omega^1_X(\log D) = \{\delta \in \Omega^1_X \otimes \mathcal{O}_X(D) : d\delta \in \Omega^2_X \otimes \mathcal{O}_X(D)\}.$$

Fact 2.7. *The following equality holds:*

$$\bigwedge^n \Omega^1_X(\log D) = \mathcal{O}_X(K_X + D).$$

For X_0 non compact manifold, we may compactify it to a variety X in such a way that $D = X \setminus X_0$ is an snc divisor.

Fact 2.8 (Kodaira-Itaka dimension for non compact varieties)**.** *The Kodaira-Itaka dimension $\kappa(\bigwedge^n \Omega^1_X(\log D))$ does not depend on X and hence it is a birational invariant of X_0. We denote it $\kappa(X_0)$.*

Proof. This follows from the fact stated in [I, p.326]. For an integer $m > 0$ one can consider birational invariant $\overline{P}_m(X_0)$ of X_0 called logarithmic m-genus. There are positive real numbers α, $\beta > 0$ such that for m sufficiently large $\alpha m^{\kappa(X_0)} \leq \overline{P}_m(X_0) \leq \beta m^{\kappa(X_0)}$. Hence the Kodaira-Itaka dimension is indeed a birational invariant. $\square$

Let us now consider a manifold X and a smooth codimension one subvariety D.

Fact 2.9. *One has got the following exact sequence called "residue sequence":*

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \to \mathcal{O}_D \to 0.$$

It can be generalized for higher differentials.

Fact 2.10 (Property 2.3 b, [EV92])**.** *Let X be a smooth variety and D a reduced, irreducible divisor. There is the following exact sequence:*

$$0 \to \Omega^p_X \to \Omega^p_X(\log D) \to \Omega^{p-1}_D \to 0.$$

Let us assume that (X, D) is a dlt pair. In this case reflexive differentials $\Omega^{[p]}(\log D) = (\Omega^p(\log D))^{**}$ have got a lot of nice properties, similar with logarithmic Kähler differentials on smooth spaces with an snc divisor. For example we get the following exact sequence.

Fact 2.11. *Let (X, D) be a dlt pair. There exists an exact sequence:*

$$0 \to \Omega^{[p]}_X \to \Omega^{[p]}_X(\log D) \to \Omega^{p-1}_D \to 0.$$

For a much more general analogue of the sequence for manifolds the reader is advised to consult [GKKP, Theorem 11.7]. There and in [KK08a] one can find more results on reflexive differentials on dlt pairs.

3 Lecture 3

In this lecture we will focus on applications of theorems on reflexive differentials.

3.1 Pull-back of reflexive differentials

First let us note that the pull-back of logarithmic differential sheaves is closely related to our main aim: the extension of the differential form from the smooth locus onto the resolution of singularities.

Let $\pi : \tilde{X} \to X$ be a resolution of singularities. Suppose that we have a pull-back map $d\pi : \pi^*(\Omega_X^{[p]}) \to \Omega_{\tilde{X}}^{[p]} = \Omega_{\tilde{X}}^p$. The last equality holds, because $\tilde{X}$ is smooth. A section of $\Omega_X^{[p]}$ is a differential form on the smooth locus. Hence the existence of $d\pi$ precisely guarantees that all such sections extend to differential forms on $\tilde{X}$. This is one of the reasons why the following theorem is crucial.

Theorem 3.1 (Theorem 4.3 [GKKP])**.** *Let $f : X \to Y$ be a morphism of normal varieties. Suppose that Y is klt and the image of X is not contained in the singular locus. Then there exists the pull-back morphism $df : f^*(\Omega_Y^{[p]}) \to \Omega_X^{[p]}$.* □

Remark 3.2. *The statement of [GKKP, Theorem 4.3] is much stronger. The proof presented there uses the main theorem of that paper [GKKP, Theorem 1.5] on the extension of reflexive differentials. Hence the three conditions:*

1. *existence of the pull-back map for reflexive differentials,*
2. *extension of differential forms from the smooth locus onto the resolution of singularities,*
3. *reflexivity of the sheaf $\pi_*\Omega_{\tilde{X}}^p(\log \tilde{D})$ from the Construction 1.3*

are closely related.

We will now present some applications of Theorem 3.1.

3.2 Bogomolov-Sommese vanishing

Using Theorem 3.1 one can obtain an extension of Bogomolov-Sommese vanishing. Let us first recall the theorem.

Theorem 3.3 (Bogomolov-Sommese vanishing, cf. Corollary 6.9 [EV92])**.** *Let X be a projective manifold, D an snc divisor and $\mathcal{A} \subseteq \Omega_X^p(\log D)$ an invertible subsheaf. Then the Kodaira-Itaka dimension $\kappa(\mathcal{A})$ is not greater then p.* □

Here the definition of the Kodaira-Itaka dimension for a reflexive sheaf is as follows.

Definition 3.4 (Kodaira-Iitaka dimension of a sheaf)**.** *Let X be a normal projective variety and $\mathcal{A}$ a reflexive sheaf of rank one. If $h^0(X, (\mathcal{A}^{\otimes n})^{**}) = 0$ for all $n \in \mathbb{N}$, then we say that $\mathcal{A}$ has Kodaira-Iitaka dimension $\kappa(\mathcal{A}) := -\infty$. Otherwise, set $M := \{n \in \mathbb{N} : h^0(X, (\mathcal{A}^{\otimes n})^{**}) > 0\}$, recall that the restriction of $\mathcal{A}$ to the smooth locus of X is locally free and consider the natural rational mapping*

$$\phi_n : X \dashrightarrow \mathbb{P}(H^0(X, (\mathcal{A}^{\otimes n})^{**})^\vee) \text{ for each } n \in M.$$

The Kodaira-Iitaka dimension of A is then defined as

$$\kappa(\mathcal{A}) := \max_{n \in M} \dim \overline{\phi_n(X)}.$$

The Bogomolov-Sommese vanishing can be generalized to log canonical pairs as follows.

Theorem 3.5 (Bogomolov-Sommese vanishing for lc pairs, Theorem 7.2 [GKKP])**.** *Let (X, D) be an lc pair, where X is projective. If $\mathcal{A} \subseteq \Omega_X^{[p]}(\log D)$ is a $\mathbb{Q}$-Cartier reflexive subsheaf of rank one, then $\kappa(\mathcal{A}) \leq p$.*

Proof in a simple case. We assume that $D = 0$ and X is klt. Let us consider a Cartier divisor $\mathcal{A} \subseteq \Omega_X^{[p]}$. Let $\pi : \tilde{X} \to X$ be a resolution of singularities with an exceptional divisor E. The pull-back $\pi^*(\mathcal{A})$ is a Cartier divisor on $\tilde{X}$. By Theorem 3.1 it is a subsheaf of $\Omega_{\tilde{X}}^{[p]} = \Omega_{\tilde{X}}^p$. Using the standard Bogomolov-Sommese vanishing theorem we obtain $\kappa(\pi^*(\mathcal{A})) \leq p$. As π is surjective, we have $\kappa(\mathcal{A}) = \kappa(\pi^*(\mathcal{A}))$, so $\kappa(\mathcal{A}) \leq p$. □

3.3 Lipman-Zariski conjecture

Here we will present the application of reflexive differentials to prove a special case of the Lipman-Zariski conjecture.

Conjecture 3.6 (Lipman-Zariski conjecture)**.** *Let X be a variety such that the tangent sheaf $\mathcal{T}_X$ is locally free. Then X is smooth.*

We give a proof of an interesting special case.

Theorem 3.7 (Lipman-Zariski Conjecture for klt spaces, Thm. 6.1 [GKKP])**.** *Let X be a klt space such that the tangent sheaf $\mathcal{T}_X$ is locally free. Then X is smooth.*

Proof. Suppose that X is not smooth. As the question is local, we may assume that $\mathcal{T}_X$ is free of rank n. Let $\theta_1, \ldots, \theta_n$ be global sections of $\mathcal{T}_X$ that generate the tangent sheaf. We consider a resolution of singularities $\pi : \tilde{X} \to X$ called the *functorial* resolution [Kol07, Theorems 3.35 and 3.45]. Let E be the exceptional divisor. As the singular locus of X is invariant with respect to any automorphism and due to the fact that we have chosen a functorial resolution, we may apply [GKK10, Corollary 4.7]. We see that $\pi_*(\mathcal{T}_{\tilde{X}}(-\log E))$ is reflexive. Hence we can lift the sections θ_i to

$$\theta_i' \in H^0(\tilde{X}, \mathcal{T}_{\tilde{X}}(-\log E)) \subseteq H^0(\tilde{X}, \mathcal{T}_{\tilde{X}}).$$

The smooth locus of X is isomorphic to $\tilde{X} \setminus E$. As the sections θ_i were independent, also the sections θ_i' must be independent on $\tilde{X} \setminus E$. This means that we can find differential forms $\rho_1, \ldots, \rho_n \in H^0(\tilde{X} \setminus E, \Omega^1_{\tilde{X}})$ that are a dual basis to θ_i' on $\tilde{X} \setminus E$. On this set we have $\rho_i(\theta_j') = \delta_{ij} \mathbf{1}_{\tilde{\mathbf{X}} \setminus \mathbf{E}}$, where δ_{ij} is the Kronecker delta and $\mathbf{1}_{\tilde{\mathbf{X}} \setminus \mathbf{E}}$ is a constant function equal to 1. Using the extension theorem 1.3 we can extend each ρ_i to $\rho_i' \in H^0(\tilde{X}, \Omega^1_{\tilde{X}})$. Of course the equality $\rho_i'(\theta_j') = \delta_{ij} \mathbf{1}_{\tilde{\mathbf{X}}}$ holds on $\tilde{X}$. Let q be a smooth point of the divisor E. As θ_i' were in $H^0(\tilde{X}, \mathcal{T}_{\tilde{X}}(-\log E))$ the tangent vectors $\theta_i'(q)$ must in fact be tangent to E[3]. In particular they have to be linearly dependent. This contradicts the equality $\rho_i(\theta_j'(q)) = \delta_{ij}$. □

[3] The intuition that the flow corresponding to a section of $\mathcal{T}(-\log D)$ stabilizes D might be helpful.

3.4 Hyperbolicity of moduli

We will now present applications to moduli problems - for more details the reader is advised to consult [K]. First let us state the part of Shavarevich conjecture dealing with hyperbolicity. It was proved by Parshin and Arakelov [Par68], [Ara71].

Theorem 3.8 (Shavarevich hyperbolicity, [Par68], [Ara71])**.** *Let $f^o : X^o \to Y^o$ be a smooth, complex, projective family of curves of genus $g > 1$ over a smooth quasi-projective base curve Y^o. If Y^o is isomorphic to one of the following varieties:*

- *the projective line $\mathbb{P}^1$,*
- *the affine line $\mathbb{A}^1$,*
- *the affine line minus one point $\mathbb{C}^*$, or*
- *an elliptic curve,*

then any two fibers of f^o are necessarily isomorphic. □

In other words any map form Y^o as above to the moduli stack of algebraic curves $\mathcal{M}_g$ is constant. If Y^o is higher dimensional, then the map to the moduli stack must contract all curves isomorphic to those mentioned in the theorem.

The straightforward generalization of Theorem 3.8 to families of higher dimension does not hold. A counterexample for surfaces over a curve is presented in [K, 2.1]. However families of minimal surfaces [Mig95], [OV01] and families of canonically polarized manifolds[4] of any dimension are well studied [Kov00], [VZ03]. In particular the following theorem holds.

Theorem 3.9 (Hyperbolicity for families of canonically polarized varieties)**.** *Let $f : X^o \to Y^o$ be a smooth, complex, projective family of canonically polarized varieties of arbitrary dimension, over a smooth quasi-projective base curve Y^o. Then the conclusion of Shafarevich hyperbolicity, Theorem 3.8 holds.* □

Now, we will focus on generalizations of Theorem 3.8. First we define the variation of a family of canonically polarized complex manifolds.

[4] A variety is canonically polarized iff the canonical bundle is ample.

Definition 3.10 (Variation of the family, isotriviality)**.** *Let $X^o \to Y^o$ be a projective family of canonically polarized complex manifolds over an irreducible base Y^o. This defines a map from Y^o to the moduli scheme*[5]*. The variation of f^o, denoted by $\mathrm{Var}(f^o)$, is defined as the dimension of the image of Y^o in the moduli space.*

We have $\mathrm{Var}(f^o) = 0$ iff all fibers of f^o are isomorphic; in this case, the family f^o is called "isotrivial".

A general definition of the variation of a family can be found in [K, 2.5]. In [Vie01] Viehweg proposed the following generalization of the Shafarevich Hyperbolicity Theorem 3.8.

Conjecture 3.11 (Viehweg's conjecture)**.** *Let $f^o : X^o \to Y^o$ be a smooth projective family of canonically polarized varieties, over a quasi-projective manifold Y^o. If $\mathrm{Var}(f^o) = \dim Y^o$ then $\kappa(Y^o) = \dim Y^o$, i.e. Y^o is of log general type.*

Remark 3.12. *In fact this conjecture was formulated by Viehweg in bigger generality - the details can be found in [Vie01].*

The following result can be found in [KK08b] and implies Viehweg's conjecture for $\dim Y^o \leq 3$.

Theorem 3.13 (Relation between the moduli map and the MMP, Theorem 1.2 [KK08b])**.** *Let $f^o : X^o \to Y^o$ be a smooth projective family of canonically polarized varieties, over a quasi-projective manifold Y^o of dimension $\dim Y^o \leq 3$. Let Y be a smooth compactification of Y^o such that $D := Y \setminus Y^o$ is an snc divisor.*

Then any run of the minimal model program on the pair (Y, D) will terminate in a Kodaira or Mori fiber space whose fibration factors the moduli map birationally.

We will give an idea of a proof in a very simple case, highlighting the relation with extension of differential forms. However first let us present the statement of the theorem in detail.

[5]For the existence of coarse moduli scheme in this case see [Vie95].

3.5 Discussion of Theorem 3.13

Let $\mathcal{M}$ be the coarse moduli space for polarized manifolds and let $\mu^o : Y^o \to \mathcal{M}$ be the moduli map associated with the family f^o. This defines a rational map $\mu : Y \dashrightarrow \mathcal{M}$. Let $\lambda : Y \dashrightarrow Y_\lambda$ be the map obtained by running the minimal model program 2.4 and let $Y_\lambda \to Z_\lambda$ be the associated Kodaira or Mori fiber space. Theorem 3.13 asserts the existence of a map $Z_\lambda \dashrightarrow \mathcal{M}$, such that the following diagram commutes:

$$\begin{array}{ccccc} X^o & & & & \\ \downarrow & & & & \\ Y^o & \subseteq & Y & \overset{\lambda}{\dashrightarrow} & Y_\lambda \\ & & \mu \downarrow & & \downarrow \\ & & \mathcal{M} & \dashleftarrow & Z_\lambda \end{array}$$

If $\kappa(Y^o) \geq 0$, then the minimal model program terminates in a Kodaira fiber space. In this case $\dim Z_\lambda = \kappa(Y^o)$. In the special case when $\kappa(Y^o) = 0$, then f^o is isotrivial.

If $\kappa(Y^o) = -\infty$ then the minimal model program terminates in a Mori fiber space. In this case $\dim Z < \dim Y^o$ and the moduli map μ is not generically finite.

It follows that if Y^o is not of log general type, then the map to the moduli space is not generically finite.

3.6 Idea of the proof of Theorem 3.13 in a very simple case

We will now present one of the arguments used in the proof of Theorem 3.13. We focus on the easiest case $\kappa(Y^o) = -\infty$ and $\mathrm{Var}(f^o) = \dim Y^o$. We will only show that the Picard number $\rho(Y_\lambda) \neq 1$. This would show that there is a nontrivial fiber space structure on Y_λ, which can be used for inductive arguments. Assume to the contrary that $\rho(Y_\lambda) = 1$.

The main ingredient is the following theorem of Viehweg and Zuo [VZ02, 1.4(i)]:

Theorem 3.14 (Existence of pluri-differentials)**.** *Let $f^o : X^o \to Y^o$ be a smooth projective family of canonically polarized complex manifolds, over a* *smooth complex quasi-projective base. Assume that the family is not isotrivial and fix a smooth projective compactification Y of Y^o such that $D := Y \setminus Y^o$ is an snc divisor.*

Then there exist a number $m > 0$ and an invertible sheaf $\mathcal{A} \subseteq \mathrm{Sym}^m \Omega^1_Y(\log D)$ whose Kodaira-Itaka dimension is at least the variation of the family, i.e. $\kappa(\mathcal{A}) \geq \mathrm{Var}(f^o)$. □

Remark 3.15. *It turns out that the use of reflexive differentials allows to extend this result to the singular case. This way we obtain a reflexive, rank one sheaf $\mathcal{A}' \subseteq ((\Omega^{[1]}_{Y_\lambda}(\log D_\lambda))^{\otimes m})^{**}$ such that $\kappa(\mathcal{A}') \geq \mathrm{Var}(f^o)$.*

Let us remind that we are working under the hypothesis: $\kappa(Y^o) = -\infty$, $\mathrm{Var}(f^o) = \dim Y^o$, $\rho(Y_\lambda) = 1$ and we want to obtain a contradiction. Let $C \subset Y_\lambda$ be a general complete intersection curve. As we have supposed that $\rho(Y_\lambda) = 1$ and $\kappa(Y^o) = -\infty$, we know that $-(K_{Y_\lambda} + D_\lambda)$ is ample. As C avoids the singular locus of Y_λ the sheaf $\Omega^{[1]}_{Y_\lambda}(\log D_\lambda)_{|C}$ equals $\Omega^1_{Y_\lambda}(\log D_\lambda)_{|C}$ and is locally free on C. We obtain:

$$c_1(\Omega^{[1]}_{Y_\lambda}(\log D_\lambda)_{|C}) < 0,$$

so

$$c_1(\Omega^1_{Y_\lambda}(\log D_\lambda)^{\otimes m}_{|C}) < 0.$$

Since the map to the moduli space induced by f^o is non constant, the variation $\mathrm{Var}(f^o)$ is positive, so the Kodaira-Itaka dimension $\kappa(\mathcal{A}') > 0$. Note that as C is smooth and $\mathcal{A}'$ is a reflexive, rank one sheaf, then $\mathcal{A}'_{|C}$ is invertible. One obtains $c_1(\mathcal{A}'_{|C}) > 0$.

Let $\mathcal{B} \subseteq \Omega^{[1]}_{Y_\lambda}(\log D_\lambda)$ be the maximal destabilizing subsheaf with respect to C. We have seen that $((\Omega^1_{Y_\lambda}(\log D_\lambda))^{\otimes m})^{**}$ contains a subsheaf that is positive with respect to C. On the other hand, since we are working in characteristic zero, $(\mathcal{B}^{\otimes m})^{**}$ is the maximal destabilizing subsheaf of $((\Omega^1_{Y_\lambda}(\log D_\lambda))^{\otimes m})^{**}$, so it also has a positive degree. This implies that $\mathcal{B}$ has a positive degree with respect to C.

Let r be the rank of $\mathcal{B}$, that is strictly smaller then $\dim Y_\lambda$, as $c_1(\mathcal{B}_{|\mathcal{C}}) > 0$, so $\mathcal{B} \neq \Omega^{[1]}_{Y_\lambda}(\log D_\lambda)$. As Y_λ is $\mathbb{Q}$-factorial, for some $k > 0$ we have a Cartier divisor $L = (\det \mathcal{B})^{\otimes k})^{**}$. The Picard number of Y_λ is equal to one and $\mathcal{B}$ is positive with respect to C, so the divisor L has to be ample. In particular the Kodaira dimension of $\det \mathcal{B} \subseteq \Omega^{[r]}_{Y_\lambda}(\log D_\lambda)$ is equal to the dimension of Y_λ what contradicts the generalized Bogomolov-Sommese vanishing 3.5 and therefore ends the proof. □

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References

[Ara71] S. J. Arakelov, Families of algebraic curves with fixed degeneracies, *Izv. Akad. Nauk SSSR Ser. Mat.* 35 (1971), 1269-1293.

[B] W. Barth, Some properties of stable rank-2 vector bundles on $\mathbb{P}^n$, *Math. Ann.* 226 (1977), 125-150.

[BCHM] C. Birkar, P. Cascini, C. D. Hacon, J. Mckernan, Existence of minimal models for varieties of log general type, arXiv:math/0610203.

[EV92] H. Esnault, E. Viehweg, *Lectures on vanishing theorems*, DMV Seminar, vol. 20, Birkhäuser Verlag, Basel, (1992).

[F] H. Flenner, Extendability of differential forms on nonisolated singularities, *Invent. Math.* 94 (1988), no. 2, 317-326.

[GR] H. Grauert, O. Riemenschneider, Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen, *Invent. Math.* 11 (1970), 263292.

[GRo] D. Greb, S. Rollenske, Torsion and cotorsion in the sheaf of Kähler differential on some mild singularities, arXiv:1012.5940.

[GKK10] D. Greb, S. Kebekus, S. J. Kovács, Extension theorems for differential forms, and Bogomolov-Sommese vanishing on log canonical varieties, *Compos. Math.* 146 (2010), 193-219.

[GKKP] D. Greb, S. Kebekus, S. Kovács, T. Peternell, Differential forms on log canonical spaces, arXiv:1003.2913.

[Har80] R. Hartshorne, Stable reflexive sheaves, *Math. Ann.* Vol. 254, N. 2 (1980), 121-176.

[I] S. Iitaka, *Algebraic geometry. An introduction to birational geometry of algebraic varieties*, Graduate Texts in Mathematics, vol. 76, Springer-Verlag New York, Inc., (1982).

[KMM] Y. Kawamata, K. Matsuda, K. Matsuki, *Introduction to the minimal model problem*, Algebraic geometry, Sendai, 1985, 283360, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, (1987).

[K] S. Kebekus, Differential forms on singular spaces, the minimal model program, and hyperbolicity of moduli stacks, preprint.

[KK08a] S. Kebekus, S. Kovács, Families of varieties of general type over compact bases, *Adv. Math.* 218 (2008), no. 3, 649-652.

[KK08b] S. Kebekus, S. Kovács, The structure of surfaces and threefolds mapping to the moduli stack of canonically polarized varieties, arXiv:0812.2305. To appear in *Duke Math. J.*

[Kol07] J. Kollár, *Lectures on resolution of singularities*, Annals of Mathematics Studies, vol. 166, Princeton University Press, Princeton, NJ, (2007).

[KM] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, Vol. 134, (1998).

[Kov00] S. Kovács, Algebraic hyperbolicity of fine moduli spaces, *J. Alg. Geom.* 9 (2000), no. 1, 165-174.

[Mig95] L. Migliorini, A smooth family of minimal surfaces of general type over a curve of genus at most one is trivial, *J. Alg. Geom.* 4 (1995), no. 2, 353-361.

[OV01] K. Oguiso, E. Viehweg, On the isotriviality of families of elliptic surfaces, *J. Alg. Geom.* 10 (2001), no. 3, 569-598.

[Par68] A. N. Parshin, Algebraic curves over function fields, *Izv. Akad. Nauk SSSR Ser. Mat.* 32 (1968), 1191-1219.

[SvS] J. Steenbrink, D. van Straten, Extendability of holomorphic differential forms near isolated hypersurface singularities, *Abh. Math. Sem. Univ. Hamburg* 55 (1985), 97-110.

[Vie95] E. Viehweg, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 30, Springer-Verlag, Berlin, (1995).

[Vie01] E. Viehweg, *Positivity of direct image sheaves and applications to families of higher dimensional manifolds*, School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000). ICTP Lect. Notes, vol. 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, (2001), 249-284.

[VZ02] E. Viehweg, K. Zuo, *Base spaces of non-isotrivial families of smooth minimal models*, Complex geometry, (Göttingen, 2000), Springer, Berlin, (2002), 279-328.

[VZ03] E. Viehweg, K. Zuo, On the Brody hyperbolicity of moduli spaces for canonically polarized manifolds, *Duke Math. J.* 118 (2003), no.1, 103-150.