SOME DEGENERATIONS OF G_2 AND CALABI-YAU VARIETIES

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Abstract. We introduce a variety $\hat{G}_2$ parameterizing isotropic five-spaces of a general degenerate four-form in a seven dimensional vector space. It is in a natural way a degeneration of the variety G_2, the adjoint variety of the simple Lie group $\mathbb{G}_2$. It occurs that it is also the image of $\mathbb{P}^5$ by a system of quadrics containing a twisted cubic. Degenerations of this twisted cubic to three lines give rise to degenerations of G_2 which are toric Gorenstein Fano fivefolds. We use these two degenerations to construct geometric transitions between Calabi–Yau threefolds. We prove moreover that every polarized K3 surface of Picard number 2, genus 10, and admitting a g^1_5 appears as linear sections of the variety $\hat{G}_2$.

1. Introduction

We shall denote by G_2 the adjoint variety of the complex simple Lie group $\mathbb{G}_2$. In geometric terms this is a subvariety of the Grassmannian $G(5, V)$ consisting of 5-spaces isotropic with respect to a chosen non-degenerate 4-form ω on a 7-dimensional vector space V. In this context the word non-degenerate stands for 4-forms contained in the open orbit in $\bigwedge^4 V$ of the natural action of Gl(7). It is known (see [1]) that this open orbit is the complement of a hypersurface of degree 7. The hypersurface is the closure of the set of 4-forms which can be decomposed into the sum of 3 simple forms. The expected number of simple forms needed to decompose a general 4-form is also 3, meaning that our case is defective. In fact this is the only known example (together with the dual $(k, n) = (3, 7)$) with $3 \leq k \leq n-3$ in which a general k-form in an n-dimensional space cannot be decomposed into the sum of an expected number of simple forms. A natural question comes to mind. What is the variety $\hat{G}_2$ of 5-spaces isotropic with respect to a generic 4-form from the hypersurface of degree 7? From the above point of view it is a variety which is not expected to exist. We prove that the Plücker embedding of $\hat{G}_2$ is linearly isomorphic to the closure of the image of $\mathbb{P}^5$ by the map defined by quadrics containing a fixed twisted cubic. We check also that $\hat{G}_2$ is singular along a plane and appears as a flat deformation of G_2.

Next, we study varieties obtained by degenerating the twisted cubic to a reducible cubic. All of them appear to be flat deformations of G_2. However only one of them appears to be a linear section of $G(5, V)$. It corresponds to the variety of 5-spaces isotropic with respect to a 4-form from the tangential variety to the Grassmannian $G(4, 7)$. The two other degenerations corresponding to configurations of lines give rise to toric degenerations of G_2. The variety G_2 as a spherical variety is proved in [7] to have such degenerations, but for G_2 it is not clear whether the constructed degeneration is Fano. In the context of applications, mainly for purposes of mirror symmetry of Calabi–Yau manifolds, it is important that these degenerations lead to Gorenstein toric Fano varieties. Our two toric varieties are both Gorenstein and

Fano, they admit respectively 3 and 4 singular strata of codimension 3 and degree 1. Hence the varieties obtained by intersecting these toric 5-folds with a quadric and a hyperplane have 6 and 8 nodes respectively. The small resolutions of these nodes are Calabi–Yau threefolds which are complete intersections in smooth toric Fano 5-folds and are connected by conifold transition to the Borcea Calabi–Yau threefolds of degree 36, which are sections of G_2 by a hyperplane and a quadric, and will be denoted X_{36}. This is the setting for the methods developed in [3] to work and provide a partially conjectural construction of mirror. Note that in [5] the authors found a Gorenstein toric Fano fourfold whose hyperplane section is a nodal Calabi–Yau threefold admitting a smoothing which has the same hodge numbers, degree, and degree of the second Chern class as X_{36}. It follows by a theorem of Wall that it is diffeomorphic to it and by connectedness of the Hilbert scheme is also a flat deformation of it. However, a priori the two varieties can be in different components of the Hilbert scheme, hence do not give rise to a properly understood conifold transition. In this case it is not clear what is the connection between the mirrors of these varieties.

The geometric properties of $\hat{G}_2$ are also used in the paper for the construction of another type of geometric transitions. A pair of geometric transitions joining X_{36} and the complete intersection of a quadric and a quartic in $\mathbb{P}^5$. The first is a conifold transition involving a small contraction of two nodes the second a geometric transition involving a primitive contraction of type III.

In the last section we consider a different application of the considered constructions. We apply it to the study of polarized K3 surfaces of genus 10. By the Mukai linear section theorem (see [14]) we know that a generic polarized K3 surface of genus 10 appears as a complete linear section of G_2. A classification of the non-general cases has been presented in [10]. The classification is however made using descriptions in scrolls, which is not completely precise in a few special cases. We use our construction to clarify one special case in this classification. This is the case of polarized K3 surfaces (S, L) of genus 10 having a g^1_5 (i.e. a smooth representative of L admits a g^1_5). In particular we prove that a smooth linear section of G_2 does not admit a g^1_5. Then, we prove that each smooth two dimensional linear section of $\hat{G}_2$ has a g^1_5 and that K3 surfaces appearing in this way form a component of the moduli space of such surfaces. More precisely we get the following.

Proposition 1.1. *Let (S, L) be a polarized K3 surface of genus 10 such that L admits exactly one g^1_5, then (S, L) is a proper linear section of one of the four considered degenerations of G_2.*

Proposition 1.2. *If (S, L) is a polarized K3 surface of genus 10 such that L admits a g^1_5 induced by an elliptic curve, and S has Picard number 2, then (S, L) is a proper linear section of $\hat{G}_2$.*

The methods used throughout the paper are elementary and rely highly on direct computations in chosen coordinates including the use of Macaulay2 and Magma.

2. The variety G_2

In this section we recall a basic description of the variety G_2 using equations.

Lemma 2.1. *The variety G_2 appears as a five dimensional section of the Grassmannian $G(2, 7)$ with seven hyperplanes (non complete intersection). It parametrizes* *the set of 2-forms* $\{[v_1 \wedge v_2] \in G(2,V) \mid v_1 \wedge v_2 \wedge \omega = 0 \in \bigwedge^6 V\}$*, where* V *is a seven dimensional vector space and* ω *a non-degenerate four-form on it.*

By [1] we can choose $\omega = x_1 \wedge x_2 \wedge x_3 \wedge x_7 + x_4 \wedge x_5 \wedge x_6 \wedge x_7 + x_2 \wedge x_3 \wedge x_5 \wedge x_6 + x_1 \wedge x_3 \wedge x_4 \wedge x_6 + x_1 \wedge x_2 \wedge x_4 \wedge x_5$. The variety G_2 is then described in its linear span $W \subset \mathbb{P}(\bigwedge^2 V)$ with coordinates $(a \dots n)$ by 4×4 Pfaffians of the matrix:

$$\begin{pmatrix} 0 & -f & e & g & h & i & a \\ f & 0 & -d & j & k & l & b \\ -e & d & 0 & m & n & -g-k & c \\ -g & -j & -m & 0 & c & -b & d \\ -h & -k & -n & -c & 0 & a & e \\ -i & -l & g+k & b & -a & 0 & f \\ -a & -b & -c & -d & -e & -f & 0 \end{pmatrix}.$$

3. The variety $\hat{G}_2$

From [1] there is a hypersurface of degree 7 in $\bigwedge^4 V$ parameterizing four-forms which may be written as a sum of three pure forms. The generic element of this hypersurface corresponds to a generic degenerate four-form ω_0. After a suitable change of coordinates we may assume (see [1]) that $\omega_0 = x_1 \wedge x_2 \wedge x_3 \wedge x_7 + x_4 \wedge x_5 \wedge x_6 \wedge x_7 + x_2 \wedge x_3 \wedge x_5 \wedge x_6 + x_1 \wedge x_3 \wedge x_4 \wedge x_6$. Let us consider the variety $\hat{G}_2 = \{[v_1 \wedge v_2] \in G(2,V) \mid v_1 \wedge v_2 \wedge \omega_0 = 0 \in \bigwedge^6 V\}$. Analogously as in the non-degenerate case it is described in it's linear span by 4×4 Pfaffians of a matrix of the form

$$\begin{pmatrix} 0 & 0 & e & g & h & i & a \\ 0 & 0 & -d & j & -g & l & b \\ -e & d & 0 & m & n & k & c \\ -g & -j & -m & 0 & 0 & -b & d \\ -h & g & -n & 0 & 0 & a & e \\ -i & -l & k & b & -a & 0 & f \\ -a & -b & -c & -d & -e & -f & 0 \end{pmatrix}.$$

Directly from the equations we observe that $\hat{G}_2$ contains a smooth Fano fourfold F described in the space (b, c, d, f, j, l, m, k) by the 4×4 Pfaffians of the matrix

$$\begin{pmatrix} 0 & -d & j & l & b \\ d & 0 & m & k & c \\ -j & -m & 0 & -b & d \\ -l & -k & b & 0 & f \\ -b & -c & -d & -f & 0 \end{pmatrix}$$

Remark 3.1. In fact we see directly a second such Fano fourfold F' isomorphic to F and meeting F in a plane. It is analogously the intersection of $\hat{G}_2$ with the space (e, h, i, a, n, c, k, f). We shall see that there is in fact a one parameter family of such Fano fourfolds any two intersecting in the plane (c, k, f).

Observation 3.2. *The image of the projection of* $\hat{G}_2$ *from the plane spanned by* (c, k, f) *is a hyperplane section of* $\mathbb{P}^1 \times \mathbb{P}^5$*.*

Proof. The projection maps $\hat{G}_2$ to $\mathbb{P}^{10}$ with coordinates $(a, b, d, e, g, h, i, j, l, m, n)$ Observe that the equations of the projection involve 2×2 minors of the matrix

$$\begin{pmatrix} e & g & h & i & a & -n \\ -d & j & -g & l & b & m \end{pmatrix},$$

as these equations appear in the description of $\hat{G}_2$ and do not involve c, k, f. It follows that the image is contained in a hyperplane section P of a $\mathbb{P}^1 \times \mathbb{P}^5$. Next we check that the map is an isomorphism over the open subset given by $g = 1$ of P. □

Proposition 3.3. *The Hilbert scheme of projective 3-spaces contained in $\hat{G}_2$ is a conic. Moreover the union of these 3-spaces is a divisor D of degree 8 in $\hat{G}_2$.*

Proof. We start by proving the following lemmas.

Lemma 3.4. *Let a plane P have four points of intersection with a $G(5, V)$, such that they span this plane. Then $P \cap G(5, V)$ is a conic parameterizing all five-spaces containing a three-space W.*

Proof. The proof follows from [15], as three points in $G(2, V)$ always lie in a $G(2, A)$ for some subspace A of dimension 6. □

Lemma 3.5. *A projective three-space $\Pi \subset G(2, 7)$ is contained in $\hat{G}$ if and only if there exists a vector u in V and a four-space $v_1 \wedge v_2 \wedge v_3 \wedge v_4 \in G(4, 7)$ such that $u \wedge \omega_0 = u \wedge v_1 \wedge v_2 \wedge v_3 \wedge v_4$ and Π is generated by $u \wedge v_1$, $u \wedge v_2$, $u \wedge v_3$, $u \wedge v_4$.*

Proof. To prove the if part we observe that our conditions imply $u \wedge v_i \wedge \omega_0 = 0$ for $i = 1, \ldots, 4$. Let us pass to the proof of the only if part. Observe first that any projective three-space contained in $G(2, V)$ is spanned by four points of the form $u \wedge v_1$,$u \wedge v_2$,$u \wedge v_3$,$u \wedge v_4$. By our assumption on ω_0 the form $u \wedge \omega_0 \neq 0$, and it is killed by the vectors $u, v_1, \ldots, v_4$, hence equals $u \wedge v_1 \wedge v_2 \wedge v_3 \wedge v_4$. □

Now, it follows from Lemma 3.5 that the set of projective three-spaces contained in $\hat{G}$ is parametrized by those $[v] \in \mathbb{P}(V)$ for which $v \wedge \omega_0 \in G(5, 7)$. The form ω_0 may be written as the sum of three simple forms corresponding to three subspaces P_1, P_2, P_3 of dimension 4 in V, each two meeting in a line and no three having a nontrivial intersection. Hence the form $v \wedge \omega_0$ may be written as the sum of three simple forms corresponding to three subspaces of dimension 5 each spanned by v and one of the spaces P_i. By lemma 3.4 the sum of these three 5-forms may be a simple form only if they all contain a common 3-space. But this may happen only if v lies in the space spanned by the lines $P_i \cap P_j$. Now it is enough to see that the condition $v \wedge \omega_0$ is simple, corresponds for the chosen coordinate space to $((v \wedge \omega_0)^*)^2 = 0$ and perform a straightforward computation to see that it induces a quadratic equation on the coefficients of $v \in \mathrm{span}\{P_1 \cap P_2,\ P_1 \cap P_3,\ P_2 \cap P_3\}$.

In coordinates the constructed divisor is the intersection of $\hat{G}_2$ with $\{g = h = j = 0\}$. The latter defines on $G(5, 7)$ the set of lines intersecting the distinguished plane. We compute in Macaulay2 its degree. □

Remark 3.6. From the above proof it follows that the form ω_0 defines a conic Q in $\mathbb{P}(V)$ by $Q = \{[v] \in \mathbb{P}(V) \mid v \wedge \omega_0 \in G(5, 7)\}$. Observe that any secant line of this conic is an element of $\hat{G}_2$. Indeed let $v_1, v_2 \in V$ be two vectors such that $[v_1]$, $[v_2] \in Q$. Then $v_i \wedge \omega_0$ defines a 5 space $\Pi_i \subset V$ for $i = 1, 2$. Consider now the product $v_1 \wedge v_2 \wedge \omega_0$. If it is not zero it defines a hyperplane in V. It follows that $\dim(\Pi_1 \cap \Pi_2) = 4$ and ω_0 can then be written in the form $\omega_0 = u_1 \wedge u_2 \wedge u_3 \wedge u_4 + v_1 \wedge v_2 \wedge \alpha$. According to [1] this decomposition corresponds to a non general degenerate form ω_0 giving us a contradiction.

The proof implies also that each $\mathbb{P}^3$ contained in $\hat{G}_2$ is a $\mathbb{P}^3$ of lines passing through a chosen v in the conic and contained in the projective four-space corresponding to $v \wedge \omega_0$.

Remark 3.7. For any three points v_1, v_2, v_3 lying on the distinguished conic Q, there exists a decomposition of ω_0 into the sum of 3 simple forms $\alpha_1, \alpha_2, \alpha_3$ such that $v_1 \wedge (\omega_0 - \alpha_1) = v_3 \wedge (\omega_0 - \alpha_2) = v_3 \wedge (\omega_0 - \alpha_3) = 0$. In other words for any triple of points on the conic there is a decomposition with corresponding 4-spaces P_1,P_2,P_3 such that $(v_1, v_2, v_3) = (P_1 \cap P_2, P_1 \cap P_3, P_2 \cap P_3)$.

Remark 3.8. A three form defining $\hat{G}_2$ has a five dimensional family of presentations into the sum of three simple forms corresponding to three subspaces P_1, P_2, P_3, however all these presentations induce the same space $\operatorname{span}\{P_1 \cap P_2, P_1 \cap P_3, P_2 \cap P_3\}$. This space corresponds to the only projective plane in $\hat{G}$ consisting of lines contained in a projective plane. All other planes contained in $\hat{G}$ consist of lines passing through a point and contained in a three-space.

Proposition 3.9. *The projection of $\hat{G}_2$ from F is a birational map onto $\mathbb{P}^5$ whose inverse is the map φ defined by the system of quadrics in $\mathbb{P}^5$ containing a common twisted cubic.*

Proof. Observe that the considered projection from F decomposes into a projection from the plane spanned by c, k, f and the canonical projection from $\mathbb{P}^1 \times \mathbb{P}^5$ onto $\mathbb{P}^5$. The latter restricted to P is the blow down of $\mathbb{P}^1 \times \mathbb{P}^3$. It follows that the map is an isomorphism between the open set given by $g = 1$ and its image in $\mathbb{P}^5$. Let us write down explicitly the inverse map. Let (x, y, z, t, u, v) be a coordinate system in $\mathbb{P}^5$. Consider a twisted cubic curve given by $u = 0$, $v = 0$ and the minors of the matrix

$$\begin{pmatrix} x & y & z \\ t & x & y \end{pmatrix}.$$

Let L be the system of quadrics containing the twisted cubic. Choose the coordinates $(a, \ldots, n)$ of $H^0(L)$ in the following way: $(a, \ldots, n) = (uy, vy, yt - x^2, -vx, ux, y^2 - xz, uv, -u^2, uz, v^2, -xy + zt, vz, vt, ut)$. We easily check that the corresponding map is well defined and inverse to the projection by writing down the matrix defining $\hat{G}_2$ with substituted coordinates.

$$\begin{pmatrix}
0 & 0 & ux & uv & -u^2 & uz & uy \\
0 & 0 & vx & v^2 & -uv & vz & vy \\
-ux & -vx & 0 & vt & -ut & -xy + zt & yt - x^2 \\
-uv & -v^2 & -vt & 0 & 0 & -vy & -vx \\
u^2 & uv & ut & 0 & 0 & uy & ux \\
-uz & -vz & xy - zt & vy & -uy & 0 & y^2 - xz \\
-uy & -vy & -yt + x^2 & vx & -ux & -y^2 + xz & 0
\end{pmatrix}$$

□

Remark 3.10. The images of the 4-dimensional projective spaces containing the twisted cubic form a pencil of smooth Fano fourfolds each two meeting in the plane which is the image of the $\mathbb{P}^3$ spanned by the twisted cubic. The statement follows from the fact that we can change coordinates in $\mathbb{P}^5$ and hence we can assume that any two chosen Fano fourfolds obtained in this way are F and F' in Remark 3.1.

Lemma 3.11. *The singular locus of $\hat{G}_2$ is a plane.*

Proof. To see that the distinguished plane is singular it is enough to observe that each line secant to the distinguished conic C is the common element of two projective three-spaces contained in $\hat{G}_2$. These are the spaces of lines corresponding to the points of intersections of the secant line with C. By the same argument it follows also that the divisor D' is singular in the plane. To check smoothness outside let us perform the following argument. Clearly the system $|2H - E|$ on the blow up of $\mathbb{P}^5$ in the twisted cubic separates points and tangent directions outside the pre-image transform of the $\mathbb{P}^3$ spanned by the twisted cubic. It remains to study the image of the exceptional divisor, which is D'. Now observe that for any F_1 and F_2 in the pencil of Fano fourfolds described in Remark 3.10 there is a hyperplane in $\mathbb{P}^{13}$ whose intersection with $\hat{G}_2$ decomposes in F_1, F_2 and D'. It follows that the singularities of $\hat{G}_2$ may occur only in the singularities of D' and in the base points of the pencil. We hence need only to prove that D' is smooth outside the distinguished plane. This follows directly from the description of the complement of the plane in D' as a vector bundle over the product of the twisted cubic with $\mathbb{P}^1$. □

Remark 3.12. Observe that the map induced on $\mathbb{P}^5$ contracts only the secant lines of the twisted cubic to distinct points of the distinguished $\mathbb{P}^2$.

Remark 3.13. A generic codimension 2 section of $\hat{G}_2$ by 2 hypersurfaces is nodal. We check this by taking a codimension 2 linear section and looking at its singularity.

Lemma 3.14. *The variety $\hat{G}$ is a flat deformation of G.*

Proof. We observe that both varieties arise as linear sections of $G(2, V)$ by some $\mathbb{P}^{10}$. Moreover we easily find an algebraic family with those as fibers. Indeed consider the family parameterized by $t \in \mathbb{C}$ of varieties given in $\mathbb{P}^{13}$ by the 4×4 Pfaffians of the matrices:

$$\begin{pmatrix} 0 & -tf & e & g & h & i & a \\ f & 0 & -d & j & -g-tk & l & b \\ -e & d & 0 & m & n & k & c \\ -g & -j & -m & 0 & tc & -b & d \\ -h & -k & -n & -c & 0 & a & e \\ -i & -l & g+k & b & -a & 0 & f \\ -a & -b & -c & -d & -e & -f & 0 \end{pmatrix}.$$

For each $t \in \mathbb{C}$ the equations describe the variety of isotropic five-spaces with respect to the form $\omega_t = x_1 \wedge x_2 \wedge x_3 \wedge x_7 + x_4 \wedge x_5 \wedge x_6 \wedge x_7 + x_2 \wedge x_3 \wedge x_5 \wedge x_6 + x_1 \wedge x_3 \wedge x_4 \wedge x_6 + t x_1 \wedge x_2 \wedge x_4 \wedge x_5$. The latter is a nondegenerate fourform for $t \neq 0$. It follows that for $t \neq 0$ the corresponding fiber of the family is isomorphic to G_2 and for $t = 0$ it is equal to $\hat{G}_2$.

The assertion then follows from the equality of their Hilbert polynomials, which we compute using MACAULAY 2. □

4. Further degenerations

Observe that one can further degenerate $\hat{G}_2$ by considering degenerations of the twisted cubic C in $\mathbb{P}^5$. In particular the twisted cubic can degenerate to one of the following:

- the curve C_0 which is the sum of a smooth conic and a line intersecting it in a point
- a chain C_1 of three lines spanning a $\mathbb{P}^3$
- a curve C_2 consisting of three lines passing through a common point and spanning a $\mathbb{P}^3$

Let us consider the three cases separately.

Let us start with the conic and the line. In this case we can assume that the ideal of C_0 is given in $\mathbb{P}^5$ by $\{u = 0, v = 0\}$ and the minors of the matrix

$$\begin{pmatrix} x, y, z \\ t, x, 0 \end{pmatrix}.$$

Then the image of $\mathbb{P}^5$ by the system of quadrics containing C_0 can also be written as a section of $G(2, 7)$ consisting of two-forms killed by the four-form $\omega_1 = x_1 \wedge x_2 \wedge x_3 \wedge x_7 + x_2 \wedge x_3 \wedge x_5 \wedge x_6 + x_1 \wedge x_3 \wedge x_4 \wedge x_6$. To find the deformation family we consider the family of varieties isomorphic to $\hat{G}_2$ corresponding to twisted cubics given by $\{u = 0, v = 0\}$ and the minors of the matrix

$$\begin{pmatrix} x, y, z \\ t, x, \lambda y \end{pmatrix}.$$

We conclude comparing Hilbert polynomials.

Remark 4.1. The forms ω_0 and ω_1 represent the only two orbits of forms in $\bigwedge^3(V)$ whose corresponding isotropic varieties are flat degenerations of G_2. To prove it we use the representatives of all 9 orbits contained in [1] and check one by one the invariants of varieties they define using Macaulay2. In all other cases the dimension of the isotropic variety is higher.

In the case of a chain of lines the situation is a bit different.

Proposition 4.2. *The variety G_2 admits a degeneration over a disc to a Gorenstein toric Fano 5-fold whose only singularities are 3 conifold singularities in codimension 3 toric strata of degree 1.*

Proof. As $\hat{G}_2$ is a degeneration of G_2 over a disc it is enough to prove that $\hat{G}_2$ admits such a degeneration. We know that the latter is the image of $\mathbb{P}^5$ by the map defined by the system of quadrics containing a twisted cubic C. Let us choose a coordinate system (x, y, z, t, u, v) such that C is given in $\mathbb{P}^5$ by $\{u = 0, v = 0\}$ and the minors of the matrix

$$\begin{pmatrix} x, y, z \\ t, x, y \end{pmatrix},$$

then choose the chain of lines C_1 to be defined by $\{u = 0, v = 0\}$ and the minors of the matrix

$$\begin{pmatrix} 0, y, z \\ t, x, 0 \end{pmatrix}.$$

Let T be the variety in $\mathbb{P}^{13}$ defined as the closure of the image of $\mathbb{P}^5$ by the system of quadrics containing C_0. It is an anti-canonically embedded toric variety with corresponding dual reflexive polytope:

$$
\begin{array}{ccccccc}
(& 0 & 0 & 1 & 0 & 0 &) \\
(& 0 & 0 & 0 & 1 & 0 &) \\
(& -1 & -1 & -1 & -1 & -1 &) \\
(& 0 & 0 & 0 & 0 & 1 &) \\
(& 1 & 0 & 0 & 0 & 0 &) \\
(& 0 & 1 & 0 & 0 & 0 &) \\
(& 1 & 1 & 1 & 0 & 1 &) \\
(& 0 & 0 & -1 & 0 & -1 &) \\
(& 1 & 1 & 0 & 1 & 1 &)
\end{array}
$$

We check using Magma that the singular locus of this polytope has three conifold singularities along codimension 3 toric strata of degree 1.

Consider the family of quadrics parameterized by λ containing the curves C_λ defined by $\{u = 0, v = 0\}$ and the minors of the matrix

$$
\begin{pmatrix} \lambda x, y, z \\ t, x, \lambda y \end{pmatrix}.
$$

For each $\lambda \neq 0$ the equations of the image of $\mathbb{P}^5$ by the corresponding system of quadrics agree with the minors of the matrix:

$$
\begin{pmatrix}
0 & 0 & \lambda e & g & h & i & a \\
0 & 0 & -\lambda d & j & -g & l & b \\
-\lambda e & \lambda d & 0 & m & n & k & c \\
-g & -j & -m & 0 & 0 & -\lambda b & d \\
-h & g & -n & 0 & 0 & \lambda a & e \\
-i & -l & k & \lambda b & -\lambda a & 0 & f \\
-a & -b & -c & -d & -e & -f & 0
\end{pmatrix},
$$

in the coordinates $(a, \dots, n) = (uy, vy, yt - x^2, -vx, ux, y^2 - xz, uv, -u^2, uz, v^2, -xy + zt, vz, vt, ut)$. The latter define a variety isomorphic to $\hat{G}_2$ for each λ. It is easy to check that this family degenerates to T when λ tends to 0. By comparing Hilbert polynomials we obtain that it is a flat degeneration of $\hat{G}_2$, hence of G_2. $\square$

In the case of the twisted cubic degenerating to three lines meeting in a point we obtain a Gorenstein toric Fano 5-fold with 4 singular strata of codimension 3 and degree 1 which is a flat deformation of G_2. The corresponding dual reflexive polytope is:

$$
\begin{array}{ccccccc}
(& -1 & -1 & -1 & -1 & -1 &) \\
(& 0 & 0 & 1 & 0 & 0 &) \\
(& 0 & 0 & 0 & 1 & 0 &) \\
(& 0 & 0 & 0 & 0 & 1 &) \\
(& 1 & 0 & 0 & 0 & 0 &) \\
(& 0 & 1 & 0 & 0 & 0 &) \\
(& 1 & 1 & 1 & 1 & 0 &) \\
(& 1 & 1 & 1 & 0 & 1 &) \\
(& 1 & 1 & 0 & 1 & 1 &) \\
(& 2 & 2 & 1 & 1 & 1 &)
\end{array}
$$

4.1. **Application to mirror symmetry.** One of the methods of computing mirrors to Calabi-Yau threefolds is to find their degenerations to complete intersections in Gorenstein toric Fano varieties. Let us present the method, contained in [3], in our context. We aim to use the constructed toric degeneration to compute the mirror of the Calabi-Yau threefold X_{36}. As the construction is still partially conjectural we omit details in what follows.

Consider the degeneration of G_2 to T. We have, X_{36} is a generic intersection of G_2 with a hyperplane and a quadric. On the other hand when we intersect T with a generic hyperplane and a generic quadric we get a Calabi-Yau threefold $\hat{Y}$ with 6 nodes. It follows that $\hat{Y}$ is a flat degeneration of X_{36}. Moreover $\hat{Y}$ admits a small resolution of singularities, which is also a complete intersection in a toric variety. We shall denote it by Y. The variety Y is a smooth Calabi-Yau threefold connected to X by a conifold transition. Due to results of [2] the variety Y has a mirror family $\mathcal{Y}^*$ with generic element denoted by Y^*. The latter is found explicitly as a complete intersection in a toric variety obtained from the description of Y by the method of nef partitions. The authors in [2] prove that there is in fact a canonical isomorphism between $H^{1,1}(Y)$ and $H^{1,2}(Y^*)$. Let us consider the one parameter subfamily $\mathcal{X}^*$ of the family $\mathcal{Y}^*$ corresponding to the subspace of $H^{1,2}$ consisting of elements associated by the above isomorphism to the pullbacks of Cartier divisors from $\hat{Y}$.

The delicate part of this mirror construction of X_{36} is to prove that a generic Calabi–Yau threefold from the subfamily $\mathcal{X}^*$ has 6 nodes satisfying an appropriate number of relations. This is only a conjecture ([3, Conj. 6.1.2]) which we are still unable to solve, also in this case. Assume that the conjecture is true. We then obtain a construction of the mirror family of X_{36} as a family of small resolutions of the elements of the considered subfamily.

5. A geometric bi-transition

In this section we construct two geometric transitions between Calabi–Yau threefolds based on the map from Proposition 3.9.

Let us consider a generic section X of $\hat{G}_2$ by a hyperplane and a quadric. Observe that X has exactly two nodes and admits a smoothing to a Borcea Calabi-Yau threefold X_{36} of degree 36. Observe moreover that X contains a system of smooth K3 surfaces each two intersecting in exactly the two nodes. Namely these are the intersections of the pencil of F with the quadric and the hyperplane. Blowing up any of them is a resolution of singularities of X. Let us consider the second resolution i.e. the one with the exceptional lines flopped. It is a Calabi-Yau threefold Z with a fibration by K3 surfaces of genus 6 and generic Picard number 1. Observe moreover that according to proposition 3.9 the map φ^{-1} factors through the blow up $\tilde{\mathbb{P}}^5$ of $\mathbb{P}^5$ in the twisted cubic C. Let E be the exceptional divisor of the blow up and H the pullback of the hyperplane from $\mathbb{P}^5$. In this context Z is the intersection of two generic divisors of type $|2H - E|$ and $|4H - 2E|$ respectively.

Lemma 5.1. *The Picard number $\rho(Z) = 2$*

Proof. We follow the idea of [11]. Observe that both systems $|2H-E|$ and $|4H-2E|$ are base point free and big on $\tilde{\mathbb{P}}^5$. On $\tilde{\mathbb{P}}^5$ both divisors contract the proper transform of $\mathbb{P}^3$ to $\mathbb{P}^2$. It follows by [17, Thm. 6] that the Picard group of Z is isomorphic to the Picard group of $\tilde{\mathbb{P}}^5$ which is of rank 2. □

Moreover Z contains a divisor D' fibered by conics. In one hand D' is the proper transform of the divisor D from Proposition 3.3 by the considered resolution of singularities, on the other hand D' is the intersection of Z with the exceptional divisor E. It follows that D' is contracted to a twisted cubic in $\mathbb{P}^5$ by the blowing down of E and the contraction is primitive by Lemma 5.1. It follows that Z is connected by a conifold transition involving a primitive contraction of 2 lines with X, and by a geometric transition involving a type III primitive contraction with the complete intersection $Y_{2,4} \subset \mathbb{P}^5$.

Remark 5.2. We can look also from the other direction. Let C be a twisted cubic, Q_2 a generic quadric containing it, and Q_4 a generic quartic singular along it. Then the intersection $Q_2 \cap Q_4$ contains the double twisted cubic and two lines secant to it. Taking the map defined by the system of quadrics containing C the singular cubic is blown up and the two secant lines are contracted to 2 nodes.

6. Polarized K3 surfaces genus 10 with a g^1_5

In this section we investigate polarized K3 surfaces of genus 10 which appear as sections of the varieties studied in this paper.

Proposition 6.1. *A polarized K3 surface (S, L), which is a proper linear section of a G_2 does not admit a g^1_5.*

Proof. Let us first prove the following lemma:

Lemma 6.2. *Let $p_1, \ldots, p_5$ be five points on $G(2, V)$ of which no two lie on a line in $G(2, V)$ and no three lie on a conic in $G(2, V)$ and such that they span a 3-space P. Then $\{p_1, \ldots, p_5\} \subset G(2, W) \subset G(2, V)$ for some five dimensional subspace W of V.*

Proof. Let $p_1, \ldots, p_5$ correspond to planes $U_1, \ldots, U_5 \subset V$. By Lemma 3.4 we may assume that no four of these points lie on a plane. Assume without loss of generality that $p_1, \ldots, p_4$ span the 3-space. If $\dim(U_1+U_2+U_3+U_4) = 6$ the assertion follows from [15, Lemma 2.3]. We need to exclude the case $U_1 + U_2 + U_3 + U_4 = V$. In this case (possibly changing the choice of $p_1, \ldots, p_4$ from the set $\{p_1, \ldots, p_5\}$) we may choose a basis in one of the two following ways $\{v_1, \ldots, v_7\}$ such that $v_1, v_2 \in U_1$, $v_3, v_4 \in U_2$, $v_5, v_6 \in U_3$, and either $v_7, v_1 + v_3 + v_5 \in U_4$ or $v_7, v_1 + v_3 \in U_4$. Each point of P is then represented by a bi-vector

$$w = av_1 \wedge v_2 + bv_3 \wedge v_4 + cv_5 \wedge v_6 + dv_7 \wedge (v_1 + v_3 + v_5),$$

or

$$w = av_1 \wedge v_2 + bv_3 \wedge v_4 + cv_5 \wedge v_6 + dv_7 \wedge (v_1 + v_3),$$

for some $a, b, c, d \in \mathbb{C}$. By simple calculation we have $w^2 = 0$ if and only if exactly one of the a, b, c, d is nonzero, which gives a contradiction with the existence of p_5 as in the assumption. □

Now assume that L has a g^1_5. It follows from [15, (2.7)] that it is given by five points on L spanning a $\mathbb{P}^3$. By Lemma 6.2 these points are contained in a section of G_2 with a $G(2, 5)$. We conclude by [13, lem. 3.3] as five isolated points cannot be a linear section of a cubic scroll by a $\mathbb{P}^3$. □

Proposition 6.3. *Every smooth polarized surface (S, L) which appears as a complete linear section of $\hat{G}_2$ is a $K3$ surface with a g^1_5.*

Proof. As $\hat{G}_2$ is a flat deformation of G_2 it's smooth complete linear section of dimension 2 are K3 surfaces of genus 10. Moreover each of these surfaces contains an elliptic curve of degree 5 which is a section of the Fano fourfold F. $\square$

Let us consider the converse. Let (S, L) be polarized K3 surface of genus 10, such that L admits a g^1_5 induced by and elliptic curve E and do not admit a g^1_4. By the theorem of Green and Lazarsfeld [9] this is the case for instance when L admits a g^1_5 and does not admit neither a g^1_4 nor a g^2_7.

We have $E.L = 5$ and $E^2 = 0$ hence $h^0(O(L)|_E) = 5$ and $h^0(O(L-E)|_E) = 5$. It follows from the standard exact sequence that $h^0(O(L-E)) \geq 6$ and $h^0(O(L-2E)) \geq 1$. We claim that $|L-E|$ is base point free: Indeed, denote by D its moving part and Δ its fixed part. Clearly $|D-E|$ is effective as $|L-2E|$ is. Observe that D cannot be of the form kE' with E' an elliptic curve, because as $D-E$ is effective we would have $E' = E$ hence $k \leq 3$ which would contradict $h^0(O(L-E)) \geq 6$. Hence we may assume that D is a smooth irreducible curve and $h^1(O(D)) = 0$. By Riemann-Roch we have:

$$4 + D^2 = 2h^0(O(D)) = 2h^0(O(D + \Delta) \geq 4 + (D + \Delta)^2$$

and analogously:

$$4 = 4 + E^2 = 2h^0(O(E)) = 2h^0(O(E + \Delta) \geq 4 + (E + \Delta)^2,$$

because $|D-E|$ being effective implies Δ is also the fixed part of $|E+\Delta|$. It follows that $L.\Delta = (D + E + \Delta).\Delta \leq 0$, which contradicts ampleness of L.

It follows from the claim that $|L-E|$ is big, nef, base point free and $h^0(O(L-E)) = 6$. Observe that $|L-E|$ is not hyper-elliptic. Indeed, first since $(L-E).E = 5$ it cannot be a double genus 2 curve. Assume now that there exists an elliptic curve E' such that $E'.(L-E) = 2$ then $L.E \leq 4$ because $|L-2E|$ being effective implies $(L-2E).E' \geq 0$. This contradicts the nonexistence of g^1_4 on L.

Hence $|L-E|$ defines a birational morphism to a surface of degree 8 in $\mathbb{P}^5$. Observe moreover that the image of an element in $|L-2E|$ is a curve of degree 3 spanning a $\mathbb{P}^3$. The latter follows from the fact that by the standard exact sequence

$$0 \longrightarrow O(E) \longrightarrow O(L-E) \longrightarrow O_\Gamma(L-E) \longrightarrow 0$$

for $\Gamma \in |L-2E|$ we have $h^0((L-E)|_\Gamma) = 4$.

Next we have two possibilities:

(1) The system $|L-E|$ is trigonal, then the image of $\varphi_{|L-E|}(S)$ is contained in a cubic threefold scroll. The latter is either the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ or a cone over a cubic rational normal scroll surface.
(2) The surface $\varphi_{|L-E|}(S)$ is a complete intersection of three quadrics.

Moreover for the image $C = \varphi_{|L-E|}(\Gamma)$ we have the following possibilities:

- Either C is a twisted cubic,
- or C is the union of a conic and a line,
- or C is the union of three lines.

Consider now the composition ψ of $\varphi_{|L-E|}$ with the birational map given by quadrics in $\mathbb{P}^5$ containing C. It is given by a subsystem of $|L| = |2(L-E)-(L-2E)|$. Moreover in every case above $\psi(S)$ spans a $\mathbb{P}^{10}$, because in each case the space of quadrics containing $\varphi_{|L-E|}(S)$ is three dimensional. It follows that ψ is defined on S by the complete linear system $|L|$. Finally (S, L) is either a proper linear section of one of the three considered degenerations of G_2 or a divisor in the blow up of a cubic scroll in a cubic curve.

In particular we have the following.

Proposition 6.4. *Let (S, L) be a polarized K3 surface of genus 10 such that L admits a g^1_5 induced by an elliptic curve E but no g^1_4. If moreover $|L - E|$ is not trigonal, then (S, L) is a proper linear section of one of the four considered degenerations of G_2.*

Remark 6.5. The system $|L - E|$ is trigonal on S if and only if there exists an elliptic curve E' on S such that one of the following holds:

(1) $L.E' = 6$ and $E.E' = 3$
(2) $L.E' = 5$ and $E.E' = 2$.

Now observe that in both cases we obtain a second g^1_5 on L. In the first case it is given by the restriction of E' and in the second we get at least a g^2_7 by restricting $|L - E - E'|$, the latter gives rise to a g^1_5 by composing the map with a projection from the singular point of the image by the g^2_7 (there is a singular point by Noether's genus formula).

We can now easily prove Proposition 1.1.

Proof. Proof of Proposition 1.1 Indeed the existence of exactly one g^1_5 excludes both the existence of a g^2_7 and of a g^1_4, hence the g^1_5 is induced by an elliptic pencil $|E|$ on S. Moreover by Remark 6.5 we see that $|L - E|$ is then trigonal. □

The Proposition 1.2 follows directly from Proposition 6.4 and the fact that in the more degenerate case we clearly get a higher Picard number due to the decomposition of C.

Remark 6.6. The K3 surfaces obtained as sections of considered varieties fit to the case $g = 10$, $c = 3$, $D^2 = 0$, and scroll of type $(2, 1, 1, 1, 1)$ from [10] (Observe that there is a misprint in the table, because $H^0(L - 2D)$ should be 1 in this case). The embedding in the scroll corresponds to the induced embedding in the projection of $\hat{G}_2$ from the distinguished plane.

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