Special multi-flags at the crossroads of algebraic geometry and differential geometry, P. Mormul (University of Warsaw)

The spaces of jets $J^r(1,1)$ and $J^r(1,m)$, $m>1$ and the canonical contact systems on them ← so-called Cartan distributions on jets

coords $t, x, x_1, x_2, \dots, x_r$,

Pfaffian description:

$$\begin{aligned} & dx - x_1\,dt && D^1 \\ & dx_1 - x_2\,dt && D^2 \\ & dx_2 - x_3\,dt && D^3 \\ & \dots\dots \\ & dx_{r-1} - x_r\,dt && D^r \end{aligned}$$

For instance $m=2$,

coords t, x, y;
x_1, y_1;
x_2, y_2;
$\dots$
x_r, y_r.

$$\underset{\substack{rk=r+2,\\ \text{the full tangent bundle}}}{TJ^r(1,1)} \supset \underset{rk\, r+1}{D^1} \supset \underset{rk\, r}{D^2} \supset \underset{rk\, r-1}{D^3} \supset \dots \supset \underset{rk\, 3}{D^{r-1}} \supset \underset{rk\, 2}{D^r}$$

$$D^{j-1} = D^j + [D^j, D^j] = [D^j, D^j]$$

The tower of consecutive Lie squares forms a 1-flag, the linear dimensions at each point $[2, 3, 4, \dots, r, r+1, r+2]$.

Pfaffian description

$$\begin{aligned} dx - x_1\,dt &= 0 = dy - y_1\,dt && D^1 \\ dx_1 - x_2\,dt &= 0 = dy_1 - y_2\,dt && D^2 \\ dx_2 - x_3\,dt &= 0 = dy_2 - y_3\,dt && D^3 \\ & \dots\dots \\ dx_{r-1} - x_r\,dt &= 0 = dy_{r-1} - y_r\,dt && D^r \end{aligned}$$

$$\underset{\substack{rk=2r+3,\\ \text{the full tangent bundle}}}{TJ^r(1,2)} \supset \underset{rk\, 2r+1}{D^1} \supset \underset{rk\, 2r-1}{D^2} \supset \underset{rk\, 2r-3}{D^3} \supset \dots \supset \underset{rk\, 5}{D^{r-1}} \supset \underset{rk\, 3}{D^r}$$

$$D^{j-1} = D^j + [D^j, D^j] = [D^j, D^j], \quad j = r, r-1, \dots, 1,$$

$$D^0 = TJ^r(1,2)$$

The tower of consecutive Lie squares forms a 2-flag, the linear dimensions at each point $[3, 5, 7, \dots, 2r-1, 2r+1, 2r+3]$.

The key property of the Cartan distribution on $J^r(1,1)$ is now being raised to the level of **Definition**:

$D \subset TM$, a rank 2 subbundle, generates a Goursat flag on M when $\underset{\text{rk } 2}{D} \subset \underset{\text{rk } 3}{[D,D]} \subset \underset{\text{rk } 4}{[[D,D],[D,D]]} \subset \ldots\ldots \subset \underset{\text{rk } r+2}{TM}$

the tower of its consecutive Lie squares grows regularly and very slowly in ranks: $2,\ 2+1,\ (2+1)+1, \ldots,\ r+2 = \dim M$ at each point of M.

→ a Goursat flag of length $r \geq 2$ ←
in what follows

Q. What are (locally only!) all objects D encompassed by this definition?

A. [until 1978:] only jet-like, that is – only like the Cartan distribution on $J^r(1,1)$ ‖ Engel 1889, von Weber 1896, E. Cartan 1914, E. Goursat 1922

[from 1978 on:] $r = 2 \Rightarrow$ only like in $J^2(1,1)$
Engel's theorem 1889

$r \geqslant 3 \Rightarrow$ not only like in $J^r(1,1)$
Giaro – Kumpera – Ruiz, CRAS (1978)

a rich tree of singularities has emerged:

$r = 3$: jet-like & le modèle exceptionnel de GKR

$r = 4$: — " — & 4 other local geometries

$r = 5$: — " — & 12 other local geometries

. . . .

$r = 8$: a real numerical modulus appears (∞ many local geom's)

For $m>1$ the geometric essence hidden in the contact systems on $J^r(1,m)$ is subtler. Sticking for simplicity to $m=2$, for $D=D^r=$ the Cartan object on $J^r(1,2)$:

$$D^1 \underset{\mathrm{cod}\,1}{\supset} F = (\partial_{x_1}, \partial_{y_1}, \partial_{x_2}, \partial_{y_2}, \dots, \partial_{x_r}, \partial_{y_r})$$

– the one before last term of the derived flag of D possesses a codim 1 involutive subdistribution.

Attn. Only seemingly similar was the situation on $J^r(1,1)$. Then an analogous codim 1 involutive subdistribution of D^1 was not unique. For $m\geq 2$ it is unique – it is the so-called Covariant subdistribution of D^1 in the E. Cartan's terminology. And the existence of such an F for D^1 we insert into the abstract Definition: $m>1$

$D\subset TM$, a rank-$(m+1)$ subbundle, generates a Special m-Flag on M when $\underset{\mathrm{rk}\, m+1}{D} \subset \underset{\mathrm{rk}\, 2m+1}{[D,D]} \subset \underset{\mathrm{rk}\, 3m+1}{[[D,D],[D,D]]} \subset \dots \subset \underset{\mathrm{rk}\,(r+1)m+1}{TM}$

the tower of its consecutive Lie squares grows regularly in ranks: $m+1, (m+1)+m, (m+1)+2m, \dots, (m+1)+rm = \dim M$ at each point of M $\boxed{\text{AND}}$ the one before last term in this tower of subbundles possesses a codim 1 involutive subdistribution ($\leftarrow$ unique in fact).

The list of the authors of $\uparrow$, not exhaustive: Kumpera–Rubin 1999–2002, Shibuya–Yamaguchi 2009, Adachi 2010

Not mentioned Élie Cartan $\to$ to be explained....

WARNING. That extra condition on $\exists\, F \underset{\mathrm{cod}\,1}{\subset} D^1$, F involutive is central. Already for $m=2$, $r=1$, $\dim M = 5$,

$$TM = \underset{\mathrm{rk}\,5}{[D,D]} \supset \underset{\mathrm{rk}\,3}{D}$$

there always exists $\underset{\mathrm{rk}\,2}{F} \subset \underset{\mathrm{rk}\,3}{D}$ s.t. $[F,F] \subset D$. We are interested in $\underbrace{[F,F] = F}_{\text{Special!}}$, while the situation $[F,F] = D$ is wild,

with a functional modulus of the local classification of such D's $\leftarrow$ Cartan's 1910 cinq variables paper....

Q. What are (locally) all objects encompassed by this definition?

A. $r = 1 \Rightarrow$ only jet-like (equivalent to the Cartan on $J^1(1,m)$)

$r \geq 2 \Rightarrow$ not only jet-like.

$r = 2 \Rightarrow$ jet-like or else one singular local geometry

. . . .

$r = 4$, $m = 2 \Rightarrow$ jet like & 33 other local geometries.

Can one have a grasp of all possible local behaviours of Goursat distributions and/or all distributions generating special m-flags $(m > 1)$ at a time?

YES, thanks to Cartan prolongations and generalized Cartan prolongations (gCp's). **Definition** of gCp (becomes Cp when $m = 1$):

$$\begin{array}{c} TM \supset D \\ \downarrow \\ M \end{array} \quad \text{a vector bundle in itself, } \mathrm{rk}\, D = m+1 \quad \rightsquigarrow \quad \begin{array}{c} PD \\ \downarrow \pi \\ M \end{array},$$

$D^{(1)}$ living on PD — a bigger mfd of $\dim = m + \dim M$, A new rank-$(m+1)$ distribution / subbundle in $T\mathbb{P}D$:

$$D^{(1)}(\xi) = \left(d\pi|_{\xi}\right)^{-1}(\xi), \quad \xi \subset D(\pi(\xi))$$

point upstairs — line downstairs

$$\dim \ker d\pi|_{\xi} = m, \quad \dim D^{(1)}(\xi) = 1+m = \mathrm{rk}\, D$$

new distributions have constant rank $m+1$ $\longrightarrow$ $\mathrm{rk}\, D^{(1)} = \mathrm{rk}\, D$

new manifolds grow in dimensions: $+m, +m, +m, \dots$ $\longrightarrow$ $\dim \mathbb{P}D = \dim M + m$

$$\begin{array}{l} \vdots \\ \downarrow \\ \mathbb{P}D^{(1)}, \ (D^{(1)})^{(1)} \subset T\mathbb{P}D^{(1)} \\ \downarrow \\ \mathbb{P}D, \ D^{(1)} \subset T\mathbb{P}D \\ \downarrow \\ M \end{array} \Bigg\} \text{ the tower of consecutive gCp's — a prototype of monster towers}$$

Theorem (Bryant–Hsu, earlier E. Cartan, also Kumpera–Rubin, ...)

$(m=1)$ Performing series of Cp's starting from $M = \mathbb{R}^2$ (or S^2, or...) and $D = T\mathbb{R}^2$ (or TS^2, or...), one gets a tower of manifolds, each of them hosting a locally universal Goursat structure of the relevant corank.

$(m>1)$ Performing series of gCp's starting from $M = \mathbb{R}^{m+1}$ (or $S^{m+1}, \dots$) and $D = T\mathbb{R}^{m+1}$ (or $TS^{m+1}, \dots$), one gets a tower of mfds, each of them hosting a locally universal distribution generating a special m-flag, of the relevant length.

This fantastic thm allows to visualise Goursats and special multi-flags. One locally Cartan-prolongs in the vicinity of a given horizontal direction $\xi \subset D$. Key distinction: ξ – „ordinary" (non-vertical) or ξ – vertical.

The exceptional model is the outcome of 3 Cartan prolongations:

$$\left(\begin{pmatrix} x_3\begin{pmatrix}1\\x_1\\x_2\end{pmatrix} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \right) = \left(x_3\begin{pmatrix}1\\x_1\\x_2\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \frac{\partial}{\partial x_3} \right) \xleftarrow[\text{thru vertical direction}]{\text{Cartan prolongation}} \left(\begin{pmatrix}1\\x_1\\x_2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right)$$

$$\left(\begin{pmatrix}1\\x_1\\x_2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right) = \left(\begin{pmatrix}1\\x_1\\0\end{pmatrix} + x_2\begin{pmatrix}0\\0\\1\end{pmatrix}, \frac{\partial}{\partial x_2} \right)$$

$$\begin{array}{cc} \frac{\partial}{\partial t} & \frac{\partial}{\partial x} \\ \parallel & \parallel \end{array}$$

$$\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right) \xrightarrow[\text{thru horizontal}]{\text{Cartan prol}} \left(\begin{pmatrix}1\\0\end{pmatrix} + x_1\begin{pmatrix}0\\1\end{pmatrix}, \frac{\partial}{\partial x_1} \right) = \left(\begin{pmatrix}1\\x_1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right) \xrightarrow[\text{thru horizontal direction}]{\text{Cartan prol.}} \left(\begin{pmatrix}1\\x_1\\0\end{pmatrix} + x_2\begin{pmatrix}0\\0\\1\end{pmatrix}, \frac{\partial}{\partial x_2} \right)$$

$$\begin{array}{cc} \parallel & \parallel \\ Z_1 & Z_2 \end{array}$$

One always starts from a pair of generators (Z_1, Z_2)

1, thru non-vertical $(Z_1 + cZ_2)(0)$:

$$\left(Z_1 + \overbrace{(c + x_{new})}^{x_{new}} Z_2, \frac{\partial}{\partial x_{new}} \right)$$

2, thru vertical dir. span $Z_2(0)$:

$$\left(x_{new} Z_1 + Z_2, \frac{\partial}{\partial x_{new}} \right)$$

Attn. One can start from a triple of generators

$$(Z_1, Z_2, Z_3) \xrightarrow{1} \left(Z_1 + X_{new} Z_2 + Y_{new} Z_3, \frac{\partial}{\partial x_{new}}, \frac{\partial}{\partial y_{new}} \right)$$

$$(Z_1, Z_2, Z_3) \xrightarrow{2} \left(x_{new} Z_1 + Z_2 + Y_{new} Z_3, \frac{\partial}{\partial x_{new}}, \frac{\partial}{\partial y_{new}} \right)$$

$$(Z_1, Z_2, Z_3) \xrightarrow{3} \left(x_{new} Z_1 + y_{new} Z_2 + Z_3, \frac{\partial}{\partial x_{new}}, \frac{\partial}{\partial y_{new}} \right)$$

→ [this will be] a $\overbrace{\text{multi-dimensional}}^{\text{or: generalized}}$ Cartan prolongation.

At the very beginning there will be

$$\begin{array}{ccc} \begin{pmatrix}1\\0\\0\end{pmatrix}, & \begin{pmatrix}0\\1\\0\end{pmatrix}, & \begin{pmatrix}0\\0\\1\end{pmatrix}. \\ \parallel & \parallel & \parallel \\ \frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{array}$$

Think about the outcome of $3 \circ 2 \circ 1$ performed over,

or about the outcome of $3 \circ 1 \circ 2 \circ 1 \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$

→ will show up in a couple of mins

In the variables $t, x, y, x_1, y_1, x_2, y_2, x_3, y_3$, that outcome $3\circ 2\circ 1\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$:

$$\left(x_3\begin{pmatrix} x_2\begin{pmatrix} 1 \\ x_1 \\ y_1 \end{pmatrix} \\ 1 \\ c_2+y_2 \end{pmatrix} \\ \begin{matrix} y_3 \\ 1 \\ 0 \\ 0 \end{matrix}\ ,\ \frac{\partial}{\partial x_3},\ \frac{\partial}{\partial y_3} \right) = 1.2.3$$

the same;

the middle gen. Cartan prolong. 2 is:

$$\left(\begin{bmatrix} 1 \\ x_1 \\ y_1 \\ 0 \\ 0 \end{bmatrix}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1} \right) \xrightarrow{2} \left(\begin{bmatrix} x_2\begin{pmatrix} 1 \\ x_1 \\ y_1 \end{pmatrix} \\ 1 \\ y_2 \end{bmatrix}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} \right)$$

The outcome $3\circ 1\circ 2\circ 1\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$,

written simply as $\underrightarrow{1.2.1.3}$:

$$\left(x_4\begin{pmatrix} x_2\begin{pmatrix} 1 \\ x_1 \\ y_1 \end{pmatrix} \\ 1 \\ c_2+y_2 \\ b_3+x_3 \\ c_3+y_3 \end{pmatrix} \\ \begin{matrix} y_4 \\ 1 \\ 0 \\ 0 \end{matrix}\ ,\ \frac{\partial}{\partial x_4},\ \frac{\partial}{\partial y_4} \right)$$

The outcome $2\circ 1\circ 2\circ 1\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$,

written as $\underrightarrow{1.2.1.2}$:

$$\left(x_4\begin{pmatrix} x_2\begin{pmatrix} 1 \\ x_1 \\ y_1 \end{pmatrix} \\ 1 \\ c_2+y_2 \\ b_3+x_3 \\ c_3+y_3 \end{pmatrix} \\ \begin{matrix} 1 \\ c_4+y_4 \\ 0 \\ 0 \end{matrix}\ ,\ \frac{\partial}{\partial x_4},\ \frac{\partial}{\partial y_4} \right)$$

both have local geometry 1 S 1 S; how to distinguish them geometrically?

→ To what a distribution tangent is the fourth prolongation of the curve (s^3, s^5, s^7) ? ← also simple in [Gibson-Hobbs]

$$t = s^3$$
$$x = s^5$$
$$y = s^7$$

$$x_1 = \frac{dx}{dt} = \frac{5}{3} s^2$$
$$y_1 = \frac{dy}{dt} = \frac{7}{3} s^4$$

$$x_2 = \frac{dt}{dx_1} = \frac{9}{10} s$$
$$y_2 = \frac{dy_1}{dx_1} = \frac{14}{5} s^2$$

$$x_3 = \frac{dx_1}{dx_2} = \frac{100}{27} s$$
$$y_3 = \frac{dy_2}{dx_2} = \frac{56}{9} s$$

$$x_4 = \frac{dx_3}{dx_2} = \frac{1000}{243}$$
$$y_4 = \frac{dy_3}{dx_2} = \frac{560}{81}$$

$$1 \,.\, 2 \,.\, 2 \,.\, 1$$

$$dx - x_1 dt \qquad dy - y_1 dt$$
$$dt - x_2 dx_1 \qquad dy_1 - y_2 dx_1$$
$$dx_1 - x_3 dx_2 \qquad dy_2 - y_3 dx_2$$
$$dx_3 - \frac{1000}{243} dx_2 \qquad dy_3 - \frac{560}{81} dx_2$$
$$dx_3 - \left(\frac{1000}{243} + x_4\right) dx_2 \qquad dy_3 - \left(\frac{560}{81} + y_4\right) dx_2$$

$$\rightarrow \left(\begin{array}{l} x_3\left(\begin{array}{l} x_2\left(\begin{bmatrix} 1 \\ x_1 \\ y_1 \end{bmatrix}\right) \\ \begin{bmatrix} 1 \\ y_2 \end{bmatrix} \end{array} \right) \\ \begin{bmatrix} 1 \\ y_3 \end{bmatrix} \\ \begin{bmatrix} \frac{1000}{243} + x_4 \\ \frac{560}{81} + y_4 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array} , \quad \begin{array}{l} \begin{bmatrix} \\ \\ \\ \end{bmatrix} \\ \begin{bmatrix} \\ \\ \end{bmatrix} \\ \begin{bmatrix} \\ \\ \end{bmatrix} \\ \begin{bmatrix} \\ \\ \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array} , \quad \begin{array}{l} \begin{bmatrix} \\ \\ \\ \end{bmatrix} \\ \begin{bmatrix} \\ \\ \end{bmatrix} \\ \begin{bmatrix} \\ \\ \end{bmatrix} \\ \begin{bmatrix} \\ \\ \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array} \right) \begin{array}{l} \leftarrow \frac{\partial}{\partial t} \\ \leftarrow \frac{\partial}{\partial x} \\ \leftarrow \frac{\partial}{\partial y} \\ \\ \\ \leftarrow \frac{\partial}{\partial x_4} \\ \leftarrow \frac{\partial}{\partial y_4} \end{array}$$

Two examples first...

→ To what a distribution tangent is the fourth prolongation of the curve (s^4, s^6, s^7) ?

← one of simple curves in [Gibson–Hobbs]

$t = s^4$

$x = s^6$

$y = s^7$

$x_1 = \frac{dx}{dt} = \frac{3}{2}s^2$

$y_1 = \frac{dy}{dt} = \frac{7}{4}s^3$

$x_2 = \frac{dt}{dx_1} = \frac{4}{3}s^2$

$y_2 = \frac{dy_1}{dx_1} = \frac{7}{4}s$

$x_3 = \frac{dx_1}{dy_2} = \frac{12}{7}s$

$y_3 = \frac{dx_2}{dy_2} = \frac{32}{21}s$

$x_4 = \frac{dx_3}{dy_2} = \frac{48}{49}$

$y_4 = \frac{dy_3}{dy_2} = \frac{128}{147}$

1 . 2 . 3 . 1

$dx - x_1 dt \qquad dy - y_1 dt$

$dt - x_2 dx_1 \qquad dy_1 - y_2 dx_1$

$dx_1 - x_3 dy_2 \qquad dx_2 - y_3 dy_2$

$dx_3 - \frac{48}{49}dy_2 \qquad dy_3 - \frac{128}{147}dy_2$

$dx_3 - \left(\frac{48}{49} + x_4\right)dy_2 \qquad dy_3 - \left(\frac{128}{147} + y_4\right)dy_2$

$$\longrightarrow \left(\begin{bmatrix} x_3\left(\begin{array}{c} x_2\begin{pmatrix} 1 \\ x_1 \\ y_1 \end{pmatrix} \\ 1 \\ y_2 \end{array}\right) \\ y_3 \\ 1 \\ \frac{48}{49}+x_4 \\ \frac{128}{147}+y_4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ 0 \\ 1 \end{bmatrix} \right)$$

← $\frac{\partial}{\partial t}$

← $\frac{\partial}{\partial x}$

← $\frac{\partial}{\partial y}$

← $\frac{\partial}{\partial x_4}$

← $\frac{\partial}{\partial y_4}$

Such **polynomial normal forms** are called EKR's, (E – Extended, K – Kumpera, R – Ruiz)

for $m=1$ – just KR's. Are encoded by words of length r over $\{1,2,3,\dots,m,m+1\}$, but NOT in the # $\underbrace{(m+1)(m+1)\cdots(m+1)}_{r \text{ times}}$, NOT!

m=1 (classical case of Goursat): $1.1.\underbrace{\begin{Bmatrix}1\\2\end{Bmatrix}.\begin{Bmatrix}1\\2\end{Bmatrix}\cdots\begin{Bmatrix}1\\2\end{Bmatrix}}_{r-2}$,

$\# = 2^{r-2}$. $r=3$, $\# = 2^{3-2} = 2$: $\begin{cases} 1.1.1 & \text{jet-like (Cartan)} \\ 1.1.2 & \text{modèle exceptionnel (Kumpera–Ruiz)} \end{cases}$

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m>1 (EKR's of special m-flags); words $j_1 j_2 j_3 \cdots j_r$ over $\{1,2,\dots,m,m+1\}$ starting with $j_1=1$ and subject to the so-called

→ **least upward jumps rule** ←

$$j_{l+1} > \max(j_1,j_2,\dots,j_l) \Rightarrow j_{l+1} = 1+\max(j_1,j_2,\dots,j_l),\quad l \le r-1$$

For instance: $1.\underline{3}.2$ – not allowed, $1.2.3.\underline{5}.1$ – not allowed...

Important. Sets of germs equivalent to EKR's subject to the least upward jumps rule are invariant wrt the automorphisms of the special m-flags' structures living on the stages of SmFMT (Special m-Flags Monster Tower). These are the singularity classes in the stages of SmFMT. Their # = ?

m=2 $\Rightarrow \# = \frac{1}{3!}(3^r+3)$, $r \ge 3$; for inst. $\frac{1}{3!}(3^4+3) = 14$ when $r=4$

m=3 $\Rightarrow \# = \frac{1}{4!}(4^r+6\cdot 2^r+8)$, $r \ge 4$; for inst. $\frac{1}{4!}(4^4+6\cdot 2^4+8) = 15$ when $r=4$

Those, just outlined, singularity classes, in any given stage of the SmFMT, form a natural stratification of that stage,

$$\mathrm{codim}(j_1 \cdot j_2 \cdots j_r) = \#(j_i=2) + 2\#(j_i=3) + \ldots + m\#(j_i=m+1).$$

For instance, $\mathrm{codim}(1.1\ldots 1) = 0$, the only open (and dense) stratum – jet-like, Cartan geometry. two natural stratifications !?! YES!

At the same time, here is the abstract in a recent (2017) paper by Castro – Colley – Kennedy – Shanbrom:

> The monster tower is a tower of spaces over a specified base, each space in the tower is a parameter space for curvilinear data up to a specified order. We describe and analyze a natural stratification of these spaces.

That „other" monster tower is the same Special-Multi-Flags Tower, up to the letters being used. From Lejeune-Jalabert' 2006 survey type paper: „Chains of points in the Semple tower":

> What Semple realizes (...) is the following. At each point P on the n^{th} stage $\mathcal{M}(n)$ of the tower over a nonsingular variety $\mathcal{M}$ of dimension r, there is a linear subspace F_P of dimension r of the tangent space to $\mathcal{M}(n)$ at P, which contains the tangents to every n^{th} derivate of a curve C in $\mathcal{M}$ which happens to pass through P. This F_P is called focal.
> The next stage $\mathcal{M}(n+1)$ is thus defined to be (...) pairs (P, L) where L is a line in the focal space F_P at P on $\mathcal{M}(n)$.

see Proc. London Math. Soc. 4 (1954), 24 – 49.

So the tower is one and the same:

Semple tower = SrFMT

but the stratifications in its stages are much different. First of all, the authors C–C–K–S use the same glasses – the EKR coordinates on tower's stages, but label them differently!

Recalling Mormul & Pelletier's labelling:

$(Z_1, Z_2, \dots, Z_{m+1})$ – local polynomial normal form in the sing. class $j_1 j_2 \cdots j_k$ $(k < r)$, and one performs, still locally, the gC_p in the vicinity of a direction $\mathrm{span}(Z_l + a_{l+1}Z_{l+1} + \dots + a_{m+1}Z_{m+1})$, $l \in \{1, 2, \dots, m, m+1\}$ determined univocally (!).

In the outcome one gets a longer normal form $j_1 j_2 \cdots j_k j_{k+1}$, where $j_{k+1} = \min(l, 1 + \max(j_1, j_2, \dots, j_k))$ ← the least upward jumps rule!

Eventually one gets the singularity class $j_1 j_2 \cdots j_{r-1} j_r$, the considered germ [of a special m-flag of length r] belongs to.

Recapitulating CCKS' labelling: they operate with vastly redundant set of $(m+1)^r$ charts $C(p_1 p_2 \cdots p_r)$ on $M(r)$, $p_1, p_2, \dots, p_r \in \{1, 2, \dots, m+1\}$.

How to choose (always locally!) step by step these indices $p_1, p_2, \dots, p_r$?

$x_1, \dots, x_{m+1}$

$x_1(p_1), \dots, x_{m+1}(p_1)$

$x_1(p_1 p_2), \dots, x_{m+1}(p_1 p_2)$

– – – – – –

$x_1(p_1 p_2 \cdots p_r), \dots, x_{m+1}(p_1 p_2 \cdots p_r)$

Suppose the coordinates $C(p_1 p_2 \cdots p_k)$, $k < r$, have already been chosen. And, in the next step, one locally Cartan-prolongs in the vicinity of a focal direction span(v) : $v X_l(p_1 p_2 \cdots p_k) \neq 0$ ($1 \leq l \leq m+1$, such l is, generally speaking, not univocally determined). Then it is legitimate to take $p_{k+1} = l$ and

$$x_j\,(p_1 p_2 \cdots p_k\, p_{k+1}) = \begin{cases} x_l(p_1 p_2 \cdots p_k), & j = l, \\ \dfrac{dx_j\,(p_1 p_2 \ldots p_k)}{dx_l(p_1 p_2 \cdots p_k)}, & j \neq l. \end{cases}$$

Eventually a chart $C(p_1 p_2 \cdots p_{r-1} p_r)$ is got that serves well as local glasses for watching the focal structure living on the stage $M(r)$.

The 4th prolongation of the curve (s^3, s^5, s^7) hitting the singularity class 1.2.2.1 (slide № 8) viewed in the CCKS chart $C(1211)$:

$$\begin{array}{ccc}
t & x & y \\
\| & \| & \| \\
x_1 & x_2 & x_3 \\
\| & & \\
\boxed{x_1(1)} \leftarrow p_1 = 1 & x_2(1) = \dfrac{dx_2}{dx_1} & x_3(1) = \dfrac{dx_3}{dx_1} \\
 & \| & \\
x_1(12) = \dfrac{dx_1(1)}{dx_2(1)} & \boxed{x_2(12)} \leftarrow p_2 = 2 & x_3(12) = \dfrac{dx_3(1)}{dx_2(1)} \\
\| & & \\
\boxed{x_1(121)} \leftarrow p_3 = 1 & x_2(121) = \dfrac{dx_2(12)}{dx_1(12)} & x_3(121) = \dfrac{dx_3(12)}{dx_1(12)} \\
\| & & \\
\boxed{x_1(1211)} \leftarrow p_4 = 1 & x_2(1211) = \dfrac{dx_2(121)}{dx_1(121)} & x_3(1211) = \dfrac{dx_3(121)}{dx_1(121)}
\end{array}$$

The 4^{th} prolongation of the curve (s^4, s^6, s^7) hitting the sing. class 1.2.3.1 (slide № 9) viewed in the CCKS chart

$C(1233)$:

coords in the initial $\mathbb{R}^3(t,x,y)$

$$t = x_1 = \boxed{x_1(1)} \quad (p_1 = 1)$$

$$x = x_2, \quad x_2(1) = \frac{dx_2}{dx_1} = \boxed{x_2(12)} \quad (p_2 = 2)$$

$$y = x_3, \quad x_3(1) = \frac{dx_3}{dx_1}$$

$$x_1(12) = \frac{dx_1(1)}{dx_2(1)} \qquad x_3(12) = \frac{dx_3(1)}{dx_2(1)} = \boxed{x_3(123)} \quad (p_3 = 3)$$

$$x_1(123) = \frac{dx_1(12)}{dx_3(12)} \qquad x_2(123) = \frac{dx_2(12)}{dx_3(12)}$$

$$x_1(1233) = \frac{dx_1(123)}{dx_3(123)} \qquad x_2(1233) = \frac{dx_2(123)}{dx_3(123)} \qquad x_3(123) = \boxed{x_3(1233)} \quad (p_4 = 3)$$

Q. How to translate, in general, an EKR code $j_1.j_2.\cdots.j_r$ into a CCKS word $p_1 p_2 \cdots p_r$? (Algorithmically!)

Code Words of CCKS – to be used later for labelling the strata (A – a finite subset of integers greater than 1):

(1) The first symbol is R.

(2) Immediately following the symbol V_A, one may put any V_B, $B \subset A \cup \{j\}$, j being the position of the symbol ($V_\emptyset = R$).

(3) $\#(A) \leq m$.

Example & the auxiliary integers n_j $(j = 2, 3, \ldots, r)$:

$$W = R\, V_2\, V_{23}\, V_{23}\, V_{25}\, V_5\, V_5\, V_5 \quad \left.\begin{matrix} \\ n_2 = 4, \quad n_3 = 2, \quad n_4 = 0, \quad n_5 = 4, \quad n_6 = n_7 = n_8 = 0 \end{matrix}\right\}\ m = 2 \text{ or more}$$

Such code words W give rise to intersection loci I_W....

Much like a naked EKR code $j_1.j_2.\cdots.j_r$ means a stratum in the r^{th} stage of SmFMT, also a code word $W = RV_{A_2}V_{A_3}\cdots V_{A_r}$ represents a concrete geometric locus in the r^{th} stage of Semple tower. But – a small difference – not yet a stratum. The authors, for clarity, write I_W for the geom. locus $\longleftrightarrow$ code word W.

For inst., $I_{\underbrace{RR\dots R}_{r}} = M(r)$, the entire r^{th} stage.

Attn. Note the difference: in the special flags' language the class $\underbrace{1.1.\dots.1}_{r}$ is <u>not</u> the entire r^{th} stage – it is just the jet-like Cartan stratum.

Definition (of a CCKS stratum). $S_W := I_W \setminus \bigcup_{I_{W'} \subsetneqq I_W} I_{W'}$.

$\text{codim}(S_W) = n_2 + n_3 + \dots + n_r$. For inst. $\underset{\displaystyle n_2 + n_3 + n_5 = 4+2+4 = 10.}{\underset{\parallel}{\text{codim}(S_{RV_2V_{23}V_{23}V_{25}V_5V_5V_5})}}$

Let us stick to $m=2$ (special 2-flags):

$r=2$: $1.1 = S_{RR}$ $\longleftarrow$ an open dense stratum

$1.2 = S_{RV_2} \left(= I_{RV_2}\right)$ $\longleftarrow$ a codim 1 stratum

sing classes

RV strata of CCKS

$r=3$: $1.1.1 = S_{RRR}$ $\longleftarrow$ an open dense stratum

$1.1.2 = S_{RRV_3}$ $\longleftarrow$ codim 1

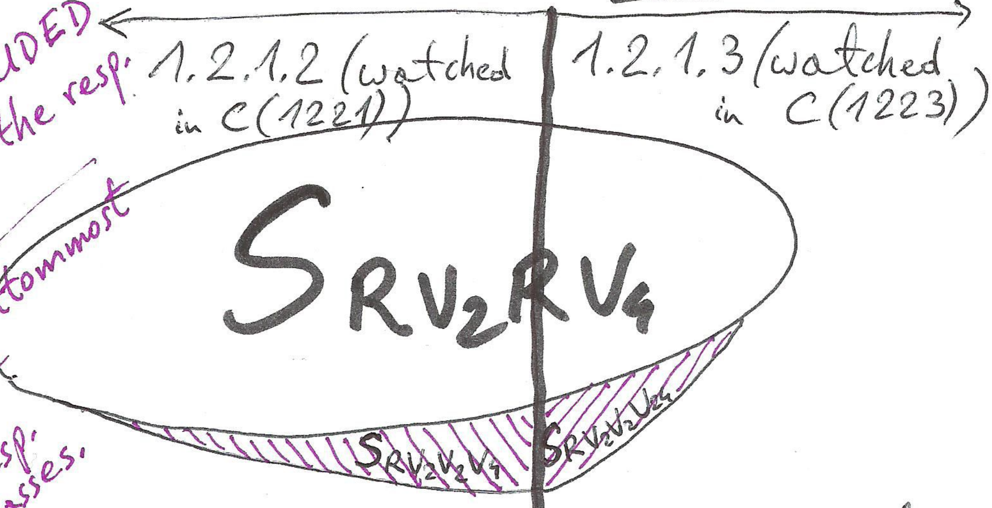
$1.2.1 = S_{RV_2R} \cup S_{RV_2V_2}$ $\longleftarrow$ codim 1 $\cup$ codim 2

$1.2.2 = S_{RV_2V_3}$ $\longleftarrow$ codim 2

$1.2.3 = S_{RV_2V_{23}} \left(= I_{RV_2V_{23}}\right)$ $\longleftarrow$ codim $= \#(2) + 2\#(3) = 1 + 2\cdot 1$
$= n_2 + n_3 = 2 + 1 = 3$

$r = 4, 5, \dots$ two last slides [not completed]

RV strata vis à vis 1.2.1.2 and 1.2.1.3 ...

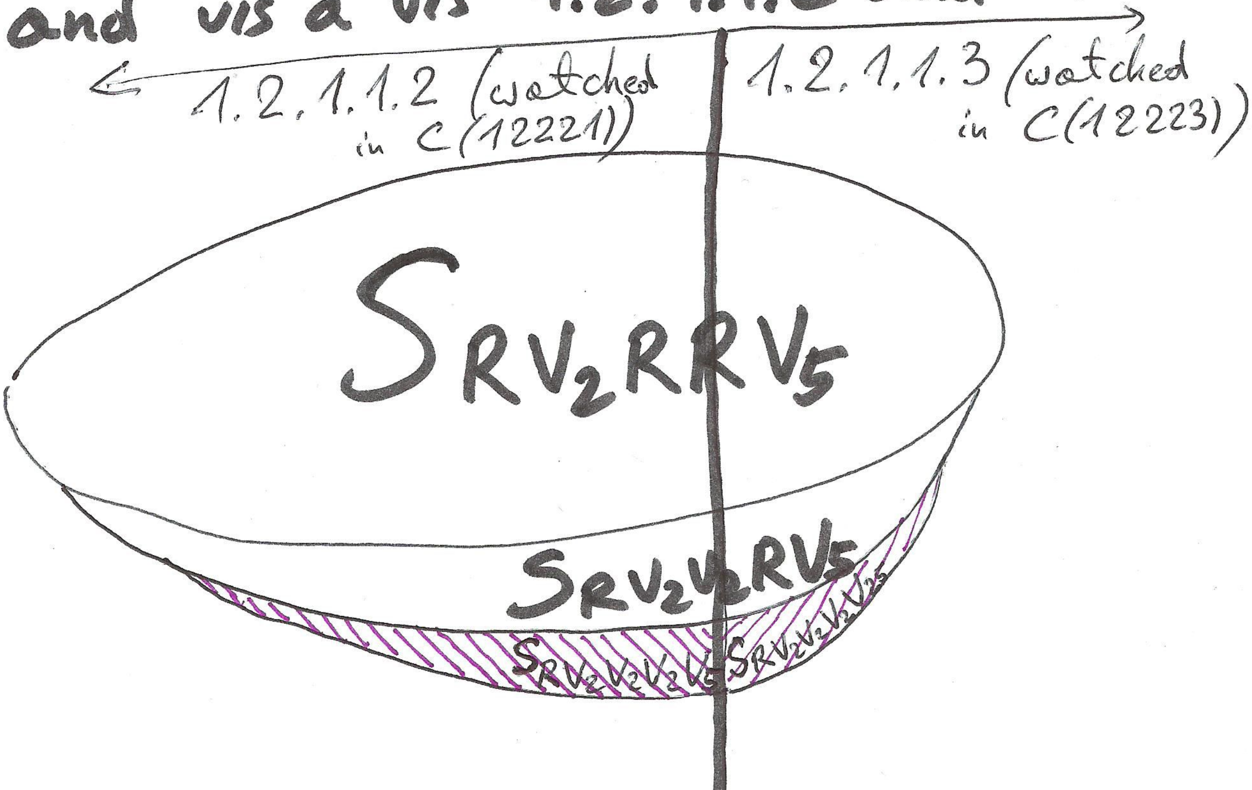


• These singularity classes are INCLUDED in the unions of the resp. S_{RV} strata.

•• Moreover, the bottommost rose S_{RV} strata are INCLUDED in the resp. sing. classes.

[Now RV strata are being discussed, not RV intersection loci]

... and vis à vis 1.2.1.1.2 and 1.2.1.1.3

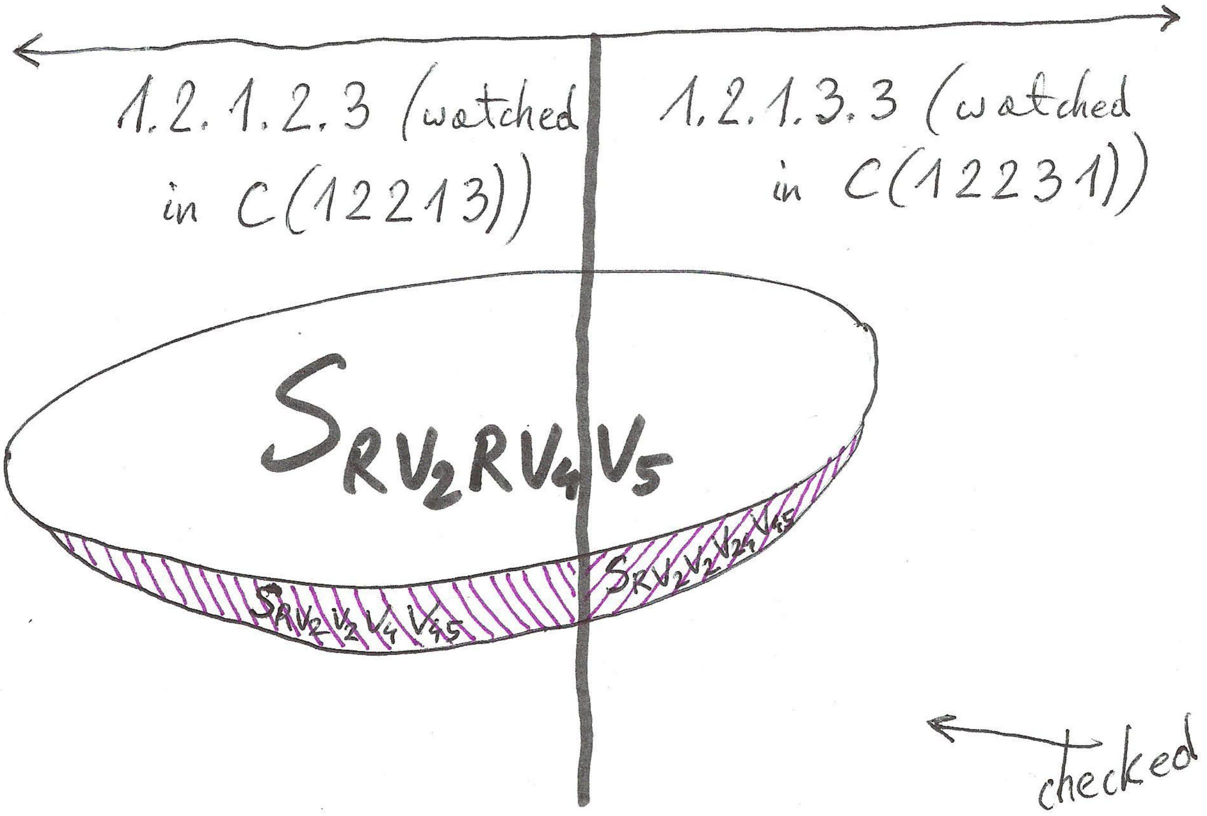


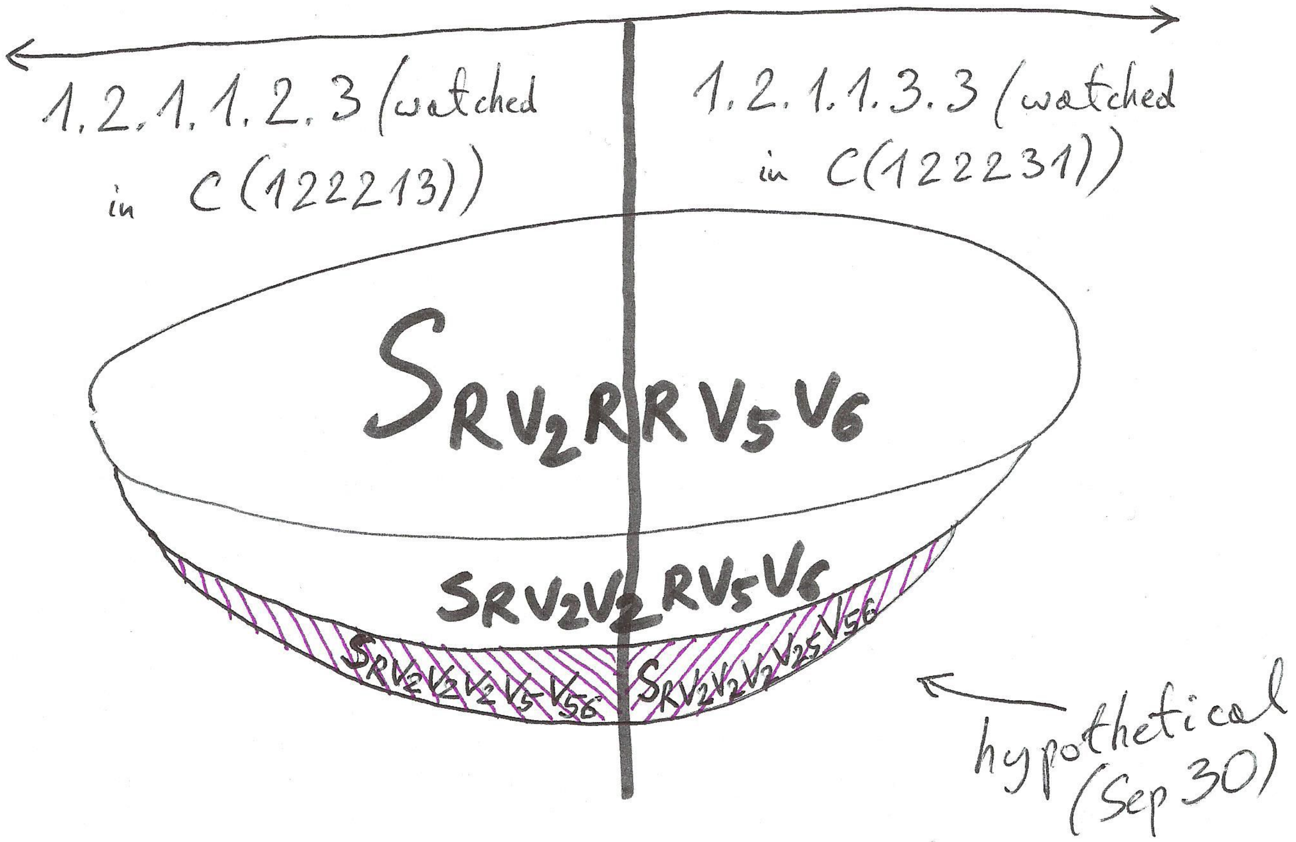
... And so it goes on, with $S_{RV_2RRRV_6}$,

$S_{RV_2V_2RRV_6}$,

$S_{RV_2V_2V_2RV_6}$,

$S_{RV_2V_2V_2V_2V_6}$ | $S_{RV_2V_2V_2V_2V_{26}}$ ....





...One imagines also, how it presumably is with the pair

1.2.1.2.2 | 1.2.1.3.2,

then with 1.2.1.1.2.2 | 1.2.1.1.3.2 ....

Example of comparison sing classes $\longleftrightarrow$ RV classes.

From the chart $C(3212)$ in Ex. 5.1/[CCKS] we pass to the chart $C(1232)$ $\leftarrow$ the mirror image of ↑. And alongside we write EKR symbols, and also the vector field visualisation

$$
\begin{array}{ll}
\left.\begin{array}{l} 1 \\ x_1 \\ y_1 \end{array}\right] & \leftarrow \partial_t \\
\left.\begin{array}{l} 1 \\ y_2 \end{array}\right] & \leftarrow \partial_{x_1} \\
\left.\begin{array}{l} y_3 \\ 1 \end{array}\right] & \leftarrow \partial_{y_2} \\
\left.\begin{array}{l} 1 \\ y_4 \end{array}\right] & \begin{array}{l} \leftarrow \partial_{x_3} \\ \leftarrow \partial_{y_3} \end{array} \\
\left.\begin{array}{l} 0 \\ 0 \end{array}\right] &
\end{array}
\qquad , \ \partial_{x_4}, \partial_{y_4}
$$

X_2, X_3, X_4

$x_1 = t$	$x_2 = x$	$x_3 = y$
$\boxed{x_1(1)}$	$x_2(1) = x_1$	$x_3(1) = y_1$
$x_1(12) = x_2$	$\boxed{x_2(12)}$	$x_3(12) = y_2$
$x_1(123) = y_3$	$x_2(123) = x_3$	$\boxed{x_3(123)}$
$x_1(1232) = y_4$	$\boxed{x_2(1232)}$	$x_3(1232) = x_4$

This chart $C(1232)$ allows to watch not the intersection locus $I_{RV_2V_2V_{24}}$ (as stated in [CCKS]), but rather $I_{RV_2V_{23}V_4}$.

While this is the EKR 1232.

this coincidence is purely accidental

The sing. class 1.2.3.2 has in these EKR coords the equations

$x_2 = x_3 = y_3 = x_4 = 0$. And what are, in the chart $C(1232)$, the equations of $I_{RV_2V_{23}V_4}$? $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, $p_4 = 2$

$n_2 = 2$, and: $p_1 \neq p_2 \Rightarrow x_1(p_1p_2) = x_1(12)(= x_2) = 0$
$p_1 \neq p_3 \Rightarrow x_1(p_1p_2p_3) = x_1(123)(= y_3) = 0$

$n_3 = 1$, and: $p_2 \neq p_3 \Rightarrow x_2(p_1p_2p_3) = x_2(123)(= x_3) = 0$

$n_4 = 1$, and: $p_3 \neq p_4 \Rightarrow x_3(p_1p_2p_3p_4) = x_3(1232)(= x_4) = 0$

Ok then, $1.2.3.2 = I_{RV_2V_{23}V_4}$.

$n_2 = \#(2)$ in $RV_2V_{23}V_4$ // $n_3 = \#(3)$ in $RV_2V_{23}V_4$ // $n_4 = \#(4)$ in $RV_2V_{23}V_4$

But it is known – [Mormul & Pelletier, arXiv 1011.1763] that the singularity class 1.2.3.2 is the union of TWO orbits [of the local equivalence by automorphisms of the relevant 2-flag structure]:

$$1.2.3.2 = 1.2.3.2_{-s} \cup 1.2.3.2_{+s}$$

$y_4 \neq 0$ $\qquad$ $y_4 = 0$

codim 4 $\qquad$ codim 5

[arXiv 1011.1763, p. 21]

$$\|$$

$$I_{RV_2V_{23}V_4} = S_{RV_2V_{23}V_4} \cup I_{RV_2V_{23}V_{24}}$$

$x_1(1232) \neq 0$ $\qquad$ $x_1(1232) = 0$

Codim 4 $\qquad$ codim 5

$4 = \#(2) + \#(3) + \#(4)$ $\qquad$ $\#(2) + \#(3) + \#(4) = 5$

Remark on subtleties behind the corner.

The above thinner part $1.2.3.2_{+s}$ of 1.2.3.2 has the EKR equations $x_2 = x_3 = y_3 = x_4 = y_4 = 0$ being formally identical with the EKR equations of the singularity class 1.2.3.3, itself of codimension 5. Yet 1.2.3.3 is being visualised in the chart C(1231), not C(1232) !! In the C(1231) coordinates the equations $x_2 = x_3 = y_3 = x_4 = y_4 = 0$ describe 1.2.3.3 and have nothing in common with $1.2.3.2_{+s}$. So – attention! The RV – counterpart of 1.2.3.3 is

$$I_{RV_2V_{23}V_{34}} = 1.2.3.3 \ldots$$