LECTURE NOTES ON $K3$ AND ENRIQUES SURFACES

SHIGERU MUKAI

Notes by Sławomir Rams

The main aim of these lectures is to study the connection between symplectic symmetries of $K3$ surfaces and the Mathieu group M_{24}, and its Enriques analogy, that is, a conjectural connection between semi-symplectic symmetries of Enriques surfaces and another Mathieu group M_{12}.

Algebraic varieties are the subject of study of algebraic geometers. The place of $K3$ and Enriques surfaces among them can be depicted in the following way:

A				
L				Kodaira dimension $\kappa = 0$
G				
E				
B				
R				
A	$\supset$	$\{$ algebraic varieties	$\ni$	Calabi-Yau varieties
I		of dimension ≥ 3 $\}$		holomorphic symplectic varieties
C				
V				
A	$\supset$	$\{$ algebraic surfaces $\}$	$\ni$	$K3$ surfaces: $K_X \sim 0, q = 0$
R				Enriques surfaces: $2K_S \sim 0, K_S \not\sim 0, q = 0$
I				
E				
T				
I				
E	$\supset$	$\{$ algebraic curves $\}$	$\ni$	elliptic curves (genus 1)
S				

Recall that both $K3$ and Enriques surfaces belong to the class of algebraic varieties with Kodaira dimension 0. The former satisfy $K_X = 0$, whereas the latter fulfill the conditions $2K_S = 0$ and $K_S \neq 0$, where K is the canonical class (and $q := h^1(\mathcal{O})$ is the irregularity). They can be seen as 2-dimensional analogues of elliptic curves. Moreover, these two kinds of surfaces are closely related to each other. For each Enriques surface S there exists a $K3$ surface X and a fixed-point-free involution $\varepsilon : X \to X$ such that

$$S = X/\varepsilon \, ,$$

namely every Enriques surface is a quotient of $K3$ surface by a fixed-point-free involution.

Notation: In what follows we will keep the notation X and S to denote a $K3$ and an Enriques surface respectively, which are connected by the relation described above.

Acknowledgement: In these lectures the author gave a short survey of [12] and [10, Appendix] on the automorphisms of $K3$ surfaces, and made an interim report of a work in progress toward a similar results to Enriques surfaces. This note, except for §9, is an output from the TEXnote taken by Prof. S. Rams, to whom he is very grateful. The author would like to thank Dr. H. Ohashi for his useful comments. Finally the author would like to thank the organizers for their hospitality.

1. Enriques surfaces and the Mathieu group M_{12}

The main subject of our considerations will be around the following

Conjecture 1.1. For a finite group G the following conditions are equivalent:[1]
[A] G has an M-semi-symplectic action on an Enriques surface, and
[B] G is a subgroup of one of G_i, where $i = 1, 2, 3, 4$ and the data concerning the groups G_i are collected in the following table:

	Group	Order	Decomposition type of $\Omega_+ \sqcup \Omega_-$	Root type
G_1	$\mathfrak{S}_5$	120	$(1+5+6) \quad + \quad (2+10)$	$A_1 + A_9$
G_2	$(\mathbb{Z}/3)^{\oplus 2} \rtimes D_8$	72	$(1+2+9) \quad + \quad (6+6)$	$A_5 + A_5$
G_3	$Q_8 \rtimes \mathfrak{S}_3$	48	$(1+3+8) \quad + \quad (4+8)$	$A_3 + A_7$
G_4	$\mathfrak{A}_6$	360	$(1+1+10) \quad + \quad (6+6)$	$A_5 + A_5$

In the table above $\mathfrak{S}_5$ (resp. $\mathfrak{A}_6$) stands for the symmetric (resp. the alternating) group, Q_8 is the quarternion group of order 8 and D_8 is the dihedral group of order 8. Other necessary definitions will be explained in further sections of the paper.

In particular we will sketch the proof of

Theorem 1.2. *The groups G_i, $i = 1, 2, 4$, have M-semi-symplectic actions on Enriques surfaces.*

2. Mathieu groups

Below we collect basic definitions and properties of Mathieu groups that we will use in the sequel. Recall that if we put $\Omega := \{1, \dots, 24\}$ then we have the inclusion

$$2^{\Omega} = \mathbb{F}_2^{24} \supset \mathcal{G}ol\,,$$

where $\mathcal{G}ol$ stands for the (extended binary) Golay code. For a vector/word $C := (c_1, \dots, c_{24}) \in \mathbb{F}_2^{24}$ one puts

$$|C| := \#\{j : c_j \neq 0\}$$

and calls $|C|$ the weight of the word/vector C. The Golay code is a 12-dimensional vector subspace of $\mathbb{F}_2^{24}$ with the weight enumerator

$$\sum_{C \in \mathcal{G}ol} t^{|C|} = 1 + 759 \cdot t^8 + 2576 \cdot t^{12} + 759 \cdot t^{16} + t^{24}.$$

[1] See the footnote in §7 and a new conjecture in §9.

Every vector/word C corresponds to a subset of Ω via $2^{\Omega} = \mathbb{F}_2^{24}$. Obviously the unique word C such that $|C| = 24$ corresponds to the set Ω, whereas 1 in the above equality stands for the empty set. The words of weight 8 (resp. 12) are called (special) octads (resp. (special) dodecads). The Golay code is characterized by the Steiner property $\mathrm{St}(5, 8; 24)$, i.e., for every subset $Q \subset \Omega$ that consists of five elements, there exists a unique octad that contains Q. In particular, the number of octads is exactly

$$\frac{\binom{24}{5}}{\binom{8}{5}} = 759$$

as above.

Let us fix a dodecad $\Omega_+ \subset \Omega$. We have the following definitions.

Definition 2.1.

$$\begin{aligned}
\mathrm{M}_{24} &:= \{g : g \text{ is an even permutation of } \Omega \text{ such that } g(\mathcal{G}ol) = \mathcal{G}ol\}\,, \\
\mathrm{M}_{23} &:= \text{stabilizer of the transitive action } \mathrm{M}_{24} \curvearrowright \Omega\,, \\
\mathrm{M}_{12} &:= \{g \in \mathrm{M}_{24} : g(\Omega_+) = \Omega_+\}\,, \\
\mathrm{M}_{11} &:= \text{stabilizer of the transitive action } \mathbf{M}_{12} \curvearrowright \Omega_+\,.
\end{aligned}$$

Observe that $\Omega_- := \Omega \setminus \Omega_+$ is also a dodecad. Thus by definition, M_{12} acts simultaneously on two dodecads Ω_+ and Ω_-. The action of M_{11} on Ω_- is transitive, while the action of M_{11} on Ω_+ acquires a fixed point. Thus the two actions are not equivalent.

Let G be a finite group. We consider the following condition:

[C]: The group G is embeddable into $M_{11} \subset M_{12}$ in such a way that G decomposes the dodecad Ω_+ into at least 3 orbits and decomposes Ω_- into at least 2 orbits.

In our proof of Thm. 1.2, we will use the following

Fact 2.2.

$$\text{Condition [B]} \Leftrightarrow \text{Condition [C]}\,.$$

3. $K3$ SURFACES AND M_{24}

We start this section with the following definition:

Definition 3.1. a) A surface X is called a $K3$ **surface** if it is a simply connected compact complex manifold of dimension 2 with a nowhere vanishing holomorphic 2-form ω.
b) We say that an automorphism $g \in \mathrm{Aut}(X)$ is **symplectic** (resp. **anti-symplectic**) if $g^*\omega = \omega$ (resp. $g^*\omega = -\omega$).

We have

Theorem 3.2. [12, Thm 0.3] *For a finite group G the following conditions are equivalent:*
[A]: *G has a symplectic action on some $K3$ surface,*
[B]: *G is embeddable into $M_{23} \subset M_{24}$ in such a way that G decomposes Ω into at least 5 orbits.*

As a result of the above theorem one obtains a complete classification of maximal finite groups acting symplectically on a $K3$ surface (see [12, Thm 0.6]). It should be pointed out that the groups G_1, G_2, G_3, G_4 appear in the list of 11 possible groups in [12]. Recall the following example from [12]:

Example 3.3. We consider the Fermat quartic X_4 in $\mathbb{P}^3$:

$$x^4 + y^4 + t^4 + z^4 = 0.$$

By adjunction formula and Lefschetz theorem it is a $K3$ surface. The nowhere vanishing holomorphic form on X_4 can be obtained as the residue of a rational 3-form on $\mathbb{P}^3$:

$$\omega = \mathrm{Res}_{X_4}^{\mathbb{P}^3}\left(\frac{d(x/t)\wedge d(y/t)\wedge d(z/t)}{(x/t)^4+(y/t)^4+(z/t)^4+1}\right).$$

One can see immediately that X_4 has many symmetries. The automorphism group of X_4 as a projective variety can be easily written down:

$$\mathrm{Aut}(X_4 \subset \mathbb{P}^3) = (\mathbb{Z}/4)^3 \rtimes \mathfrak{S}_4\,,$$

where the action of the group $(\mathbb{Z}/4)^3$ (resp. the symmetric group $\mathfrak{S}_4$) is induced by the multiplication of the coordinates by a primitive 4-th root of unity (resp. the permutation of coordinates). It can be checked that the action of $(\mathbb{Z}/4)^3$ is not symplectic. More precisely, the image of the natural homomorphism

$$\mathrm{Aut}(X_4 \subset \mathbb{P}^3) \longrightarrow \mathrm{Aut}(\mathbb{C}\omega) \simeq \mathbb{C}^*$$

is a cyclic group of order 4. We put F_{384} to denote its kernel. By definition $|\mathrm{F}_{384}| = (4^3{\cdot}4!)/4 = 384$ and this group acts symplectically on X_4. In this particular case one obtains the decomposition into orbits

$$\Omega = \amalg_{i=1}^5 \Omega_i \quad \text{of the type: } 24 = 1+1+2+4+16\,,$$

so that the first four orbits form a special octad. We have the isomorphism:

$$\mathrm{F}_{384} = \{g \in M_{24} : g(\Omega_i) = \Omega_i \quad \forall i = 1,\dots,5\}\,.$$

Basic Observation 3.4. Let $g \neq \mathrm{id}$ be a symplectic automorphism of a $K3$ surface of order $n < \infty$. Then we have $2 \le n \le 8$, the set $\mathrm{Fix}(g)$ is finite and the following equality holds

$$|\,\mathrm{Fix}(g)| = \frac{24}{n \cdot \Pi_{p|n}(1+\frac{1}{p})} =: \mu(n)\,.$$

Remark ([12, Obs. (0.2)]): Let $g' \in M_{23} \subset M_{24}$ be an element of order n. Then the following equality holds

$$|\,\mathrm{Fix}(g')| = \mu(n).$$

Namely, *the number of fixed points of* $g \curvearrowright X$ *and* $g' \curvearrowright \Omega$ *coincide.*

Let $S = X/\varepsilon$ be an Enriques surface. For each $g \in \mathrm{Aut}(S)$ there exists a lift $\tilde{g} \in \mathrm{Aut}(X)$. (There are two lifts of g. The other is $g\varepsilon$.)

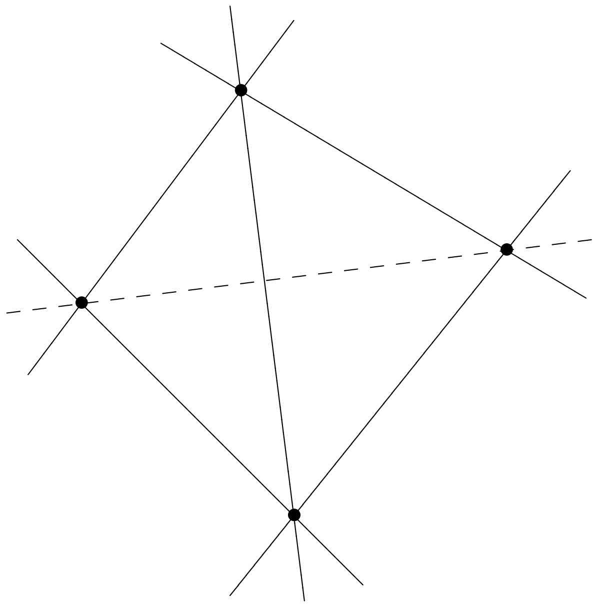
Definition 3.5. The automorphism $g \in \mathrm{Aut}(S)$ is **semi-symplectic** if the lift $\tilde{g}$ is either a symplectic or an anti-symplectic automorphism of the $K3$ surface X.

It should be pointed out that the natural analogue of Basic Observation 3.4 does not always hold for semi-symplectic automorphisms of Enriques surfaces. In particular, the set $\mathrm{Fix}(g)$ is not finite in general when the order of g is even.

4. $K3$ AND ENRIQUES SURFACES

There are many projective models of K3 surfaces - the quartics in $\mathbb{P}^3$ among them. Let us recall the Enriques' description of Enriques surfaces as polarized degree-6 surfaces:

Let $\overline{S}$ be a sextic surface mildly singular along 6 edges of the tetrahedron $x \cdot y \cdot z \cdot t = 0$:



Consider the normalization $\pi : S \longrightarrow \overline{S}$. Then $(S, \mathcal{O}_S(1))$ will be a (polarized) Enriques surface of degree 6 as we see.
Observe that $\overline{S}$ is given by the equation:

$$\overline{S} : \quad \underbrace{q(x,y,z,t)xyzt}_{\substack{q\,:\,\text{quadratic}\\ 10 \text{ monomials}}} + \underbrace{(ay^2z^2t^2 + bx^2z^2t^2 + cx^2y^2t^2 + dx^2y^2z^2)}_{4 \text{ monomials}} = 0.$$

In this way we have 14 monomials, and the 4-dimensional torus $(\mathbb{C}^*)^4$ of diagonal matrices acts on the space of equations. So the family of surfaces has dimension 10. We have

$$\mathrm{K}_S \sim \pi^*(\mathrm{K}_{\overline{S}}) - (\text{the edges of the tetrahedron})$$

and $K_{\overline{S}} \sim 2(\text{plane section})$, that is, the canonical divisors of the normalisation are cut out by quadrics passing through the edges of the tetrahedron. This implies $2K_S \sim 0$ and $K_S \not\sim 0$ (cf. [7, Chap. 4, §6]).

More explicitly and directly, we have

Proposition 4.1. *S is an Enriques and the covering $K3$ surface X is a divisor of tridegree $(2,2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ which is invariant under the involution*

$$\varepsilon : (u,v,w) \longrightarrow (-u,-v,-w),$$

where u,v,w are inhomogeneous coordinates of three $\mathbb{P}^1$-factors.

Proof. $\bar{S}$ is the image of

$$X : q(1, vw, uw, uv) + (au^2v^2w^2 + bu^2 + cv^2 + dw^2) = 0$$

of tridegree $(2,2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by the morphism associated with the linear system spanned by $\langle 1, vw, uw, uv\rangle \subset H^0(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(1,1,1))$. $\square$

Remark 4.2. By the standard Cremona transformation of $\mathbb{P}^3$, $\overline{S}$ is transformed to the sextic

$$\overline{S}' : q\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{t}\right) x^2y^2z^2t^2 + (ax^2 + by^2 + cz^2 + dt^2)xyzt = 0$$

of a similar kind. This is the image of the same $K3$ surface X by another morphism associated with $\langle u, v, w, uvw\rangle$. So this gives the same Enriques surface S after the normalization but the polarization (of degree 6) differs from that of $\overline{S}$ by the 2-torsion of $\operatorname{Pic} S$.

Remark 4.3. (Connection with Farkas' lecture) Under the Segre embedding X is a surface of degree 12:

$$X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7 .$$

Its hyperplane section is a degree-12 canonical curve $C_{12} \subset \mathbb{P}^6$ of genus 7. In this way by considering the hyperplane sections of the Enriques sextic we obtain genus-4 Prym canonical curves $D_6 \to \mathbb{P}^2$.

Going back to automorphisms, we have the following: Let $G \curvearrowright S = X/\varepsilon$ be an action of a finite group G on an Enriques surface and let ω be a nowhere vanishing holomorphic 2-form on the $K3$ cover X. Then the action of G is semi-symplectic if and only if $\tilde{G}$, the pull-back of G, preserves the set $\{\omega, -\omega = \varepsilon^*\omega\} \subset H^0(\Omega^2_S)$ (see Definition 3.5).

Definition 4.4. Let $G \curvearrowright S = X/\varepsilon$ be a semi-symplectic action of a finite group G on an Enriques surface. We say that the action is **M-semi-symplectic** if

$$\chi_{\mathrm{top}}(\mathrm{Fix}(g)) = \begin{cases} 4 \text{ for every } g \in G \text{ of order } 2, 4. \\ 2 \text{ for every } g \in G \text{ of order } 8. \end{cases}$$

In this definition, "M" alludes the Mathieu group, in this case M_{12}, as in the following considerations.

Remark 4.5. Some data concerning $g' \in M_{11}$ are collected in the following table:

Order of g'	Permutation type	# of fixed points on Ω_+
1	1	12
2	$(2)^4$	4
3	$(3)^3$	3
4	$(4)^2$	4
5	$(5)^2$	2
6	$(6)(3)(2)$	1
8	$(8)(2)$	2

From this table and the Lefschetz formula, we can see that a semi-symplectic action on an Enriques surface is M-semi-symplectic if and only if

$$\chi_{\rm top}(\mathrm{Fix}(g)) = \mathrm{Trace}(g^* \curvearrowright \mathrm{H}^*(S,\mathbb{Q})) = \#(\mathrm{Fix}(g'))\,,$$

where $g' \in M_{11}$ has the same order as g.

Example 4.6. Let X be the quartic $\subset \mathbb{P}^3$ with 15 nodes given in $\mathbb{P}^4$ by the following equations

$$\begin{array}{rcl} x+y+z+u+v & = & 0 \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{u}+\frac{1}{v} & = & 0\,. \end{array}$$

Since each *biregular* involution of $\mathbb{P}^3$ has two lines of fixed points, its restriction to a quartic has eight fixed points. Therefore, in order to obtain fixed point free involutions one has to study rational maps.

Here we consider the *birational* involution:

$$\varepsilon\,:\,(x,y,z,u,v) \to \left(\frac{1}{x},\frac{1}{y},\frac{1}{z},\frac{1}{u},\frac{1}{v}\right)\,.$$

One can check that it defines a fixed point free involution on the quartic in question. This is a special case of the involution τ in the table in §8. As the quotient one obtains an Enriques surface with root type $E_6 + A_4$.

The action of the symmetric group $\mathfrak{S}_5$ on X defines a semi-symplectic $\mathfrak{S}_5$-action on S. We claim that the action of $\mathfrak{S}_5$ is not M-semi-symplectic. Indeed, consider the automorphism $g_{(12)}$ induced by the transposition $(12) \in \mathfrak{S}_5$. Then one can see that

$$\mathrm{Fix}(g_{(12)}) = (\text{plane quartic } \cup 8 \text{ points})/_{\varepsilon}\,,$$

which implies that

$$\chi_{\rm top}(\mathrm{Fix}(g_{(12)})) = 2 \neq 4\,.$$

On the other hand, the action of the alternative group $\mathfrak{A}_5$ is M-semi-symplectic since, for the automorphism $g_{(12)(34)}$ corresponding to the permutation $(12)(34) \in \mathfrak{A}_5$, one obtains

$$\mathrm{Fix}(g_{(12)(34)}) = (\text{elliptic curve } \cup 8 \text{ points})/_{\varepsilon}\,,$$

so $\chi_{\rm top}(\mathrm{Fix}(g_{(12)(34)})) = 4$.

Example 4.7. Let X be the complete intersection of three diagonal quadrics

$$\begin{array}{c} x_1^2+x_3^2+x_5^2 = x_2^2+x_4^2+x_6^2 \\ x_1^2+x_4^2 = x_2^2+x_5^2 = x_3^2+x_6^2 \end{array}$$

in $\mathbb{P}^5$. As is shown in [12, §2], this $K3$ surface has a symplectic action of one of the eleven maximal groups $H_{192} = 2^4 D_{12}$. The involution

$$\varepsilon : (x_1,x_2,x_3,x_4,x_5,x_6) \mapsto (-x_1,x_2,-x_3,x_4,-x_5,x_6)$$

defines a fixed point free involution and H_{192} acts semi-symplectically on the Enriques surface $S = X/\varepsilon$. But this action is not M-semi-symplectic.

Recall that a symplectic automorphism of a $K3$ surface of order two is called a Nikulin involution. Such an involution has exactly 8 fixed points. The Example 4.6 shows that an analogue of the above result (i.e. $\chi_{\rm top}(\mathrm{Fix}(g)) = 4$) does not always hold for semi-symplectic automorphisms of Enriques surfaces. This is the reason of Definition 4.4.

5. ROOT SYSTEMS OF ENRIQUES SURFACES

In Conjecture 1.1 we stated the following correspondence between finite groups G_i, $i = 1, 2, 3, 4$ which may act semi-symplectically on an Enriques surface S and the root system of S:

$$\begin{array}{ccc} G_1 & \longleftrightarrow & A_1 + A_9, \\ G_2 & \longleftrightarrow & A_5 + A_5, \\ G_3 & \longleftrightarrow & A_3 + A_7, \\ G_4 & \longleftrightarrow & A_5 + A_5. \end{array}$$

In this section we explain what a root system of an Enriques surface is. [2]

The universal K3-cover of an Enriques surface $X \xrightarrow{\pi} S$ induces the homomorphism of homology groups:

$$\begin{array}{ccc} \mathrm{H}_2(X, \mathbb{Z}) & \xrightarrow{\pi_*} & \mathrm{H}_2(S, \mathbb{Z}) \\ \| & & \| \\ (\mathbb{Z}^{22}, \text{intersect. prod.}) & & \mathbb{Z}^{10} \oplus \mathbb{Z}/2\mathbb{Z} \end{array}$$

We define the rank-12 lattice

$$\mathrm{H}_2(S, \mathbb{Z}^\omega) := \ker\left[\mathrm{H}_2(X, \mathbb{Z}) \to \mathrm{H}_2(S, \mathbb{Z})\right]$$

with the bilinear form := $\frac{1}{2} \cdot$ (intersection product). This bilinear form in question is $\mathbb{Z}$-valued ([5]). Moreover, the resulting lattice is isomorphic to $\mathrm{I}_{2,10}$ - the unique odd unimodular lattice of signature $(2, 10)$, so in an appropriate basis $\alpha_1, \ldots, \alpha_{12}$ the intersection form on $\mathrm{H}_2(S, \mathbb{Z}^\omega)$ is given by the diagonal matrix:

$$\left(\begin{array}{cc|ccc} 1 & 0 & & \cdots & \\ 0 & 1 & & & \\ \hline & & -1 & & \\ \vdots & & & \ddots & \\ & & & & -1 \end{array}\right).$$

We define the **period** [3] of S as:

$$\left(\int_{\alpha_1} \omega, \ldots, \int_{\alpha_{12}} \omega\right) \in \mathbb{C}^{12}$$

This vector is uniquely determined up to constant multiplication and up to the action of the orthogonal group $\mathrm{O}(2, 10; \mathbb{Z})$ of the lattice $\mathrm{I}_{2,10}$.

The kernel of

$$\mathrm{H}_2(S, \mathbb{Z}^\omega) \longrightarrow \mathbb{C}, \quad \alpha \mapsto \int_\alpha \omega,$$

coincides with the kernel of the push-forward $\operatorname{Pic} X \to \operatorname{Pic} S$. This sublattice is denoted by $\operatorname{Pic}^\omega S$ and called the **twisted Picard lattice**. By the Riemann-Roch theorem, $\operatorname{Pic}^\omega S$ does not contain (-1)-elements. $\operatorname{Pic}^\omega S$ is an odd lattice if and only if the pull-back homomorphism $\operatorname{Br}(S) \to \operatorname{Br}(X)$ of the Brauer groups is zero by Beauville[4, Corollary 5.7, Lemma 5.9].

The **root system** R of S is the sublattice generated by (-2)-elements in the twisted Picard lattice $\operatorname{Pic}^\omega S$. If C is a smooth rational curve on S, then its pull-back to X is a disjoint union of

[2] Though defined differently, the root system of an Enriques surface was first introduced by Nikulin[16].

[3] In the literature this period of an Enriques surface was first considered by Allcock[1] for arithmetic reason.

two smooth rational curves C_+ and C_-. Hence the difference $[C_+] - [C_-]$ is a root of R. Roughly speaking, one has the following correspondences for a twisted 2-cycle $\alpha \in \mathrm{H}_2(S, \mathbb{Z}^\omega)$:

$$\begin{array}{rcl} \alpha \in \ker[\mathrm{H}_2(S,\mathbb{Z}^\omega) \longrightarrow \mathbb{C}] & \longleftrightarrow & \alpha \text{ is algebraic,} \\ \alpha \in \ker \text{ and } (\alpha^2) = -2 & \longleftrightarrow & \alpha \text{ produces a curve isomorphic to } \mathbb{P}^1 \text{ on } S. \end{array}$$

The root system R of an Enriques surface (together with an overlattice $\tilde{R} \supset R$ of index 1 or 2) describes the configuration of $\mathbb{P}^1$'s on S.

6. Leech lattice and $K3$ surfaces

In order to sketch the proof of Thm 3.2 we recall the definition of Leech lattice below. We maintain the notation of §1, in particular $\Omega = \{1, \dots, 24\}$ and $\mathcal{G}ol$ stands for the Golay code. We consider the free $\mathbb{Z}$-module

$$\mathbb{Z}\,\Omega = \bigoplus_{i \in \Omega} \mathbb{Z} e_i \quad \text{with inner product } (\cdot,\cdot) \text{ such that } e_i^2 = 2 \text{ and } (e_i, e_j) = 0 \text{ for } i \neq j\,,$$

and define the Leech lattice $\Lambda \subset \mathbb{Q}\,\Omega$ to be a lattice that is commensurable with $\mathbb{Z}\,\Omega$: [4]

$$\begin{array}{rll} \Lambda := \{ \frac{1}{2} \sum a_i e_i & : & \text{i)} \quad \text{all coordinates } a_i \text{ are even or all are odd,} \\ & & \text{ii)} \quad \{i : a_i \equiv k (\text{mod } 4)\} \in \mathcal{G}ol \quad \forall\, k = 0,1,2,3, \\ & & \text{iii)} \quad \displaystyle\sum_i a_i \equiv 4a_1 (\text{mod } 8)\, \}. \end{array}$$

The restriction of the inner product on $\mathbb{Q}\,\Omega$ to Λ is even (i.e. for each $v \in \Lambda$ one has $v^2 \in 2\mathbb{Z}$), positive-definite and unimodular. Moreover, it has no roots:

$$(v^2) \geq 4 \text{ for every } v \in \Lambda, v \neq 0\,.$$

By definition, the Mathieu group M_{24} acts on Λ isometrically. For a subgroup $G \subset M_{24}$ we define the invariant sublattice and the anti-invariant one by

$$\Lambda^G := \{v \in \Lambda \,|\, g.v = v \,\forall g \in G\} \quad \text{and} \quad \Lambda_G := (\Lambda^G)^\perp \subset \Lambda,$$

respectively.

Recall that for a K3-surface X we have the lattice $\mathrm{H}_2(X, \mathbb{Z}) = \mathbb{Z}^{22}$ with the bilinear form given by intersection numbers. After those preparations we are in position to sketch the proof of the implication (Condition [B] $\Rightarrow$ Condition [A]) in Thm 3.2 (the proof appeared in [10, Appendix]):

Sketch of the proof: We assume that G is embeddable into $M_{23} \subset M_{24}$ in such a way that G decomposes Ω into at least 5 orbits. It implies that the group G acts on the Leech lattice, and we obtain the anti-invariant sublattice Λ_G.

The sublattice Λ_G has no roots and is definite. One can show that the assumption on the number of orbits yields that

$$\mathrm{rank}(\Lambda_G) \leq 19 = 24 - 5\,.$$

The latter combined with a computation of the discriminant group and discriminant form yields a primitive embedding (recall that for a $K3$ surface, the signature of the intersection product on $\mathrm{H}_2(X, \mathbb{Z})$ is $(3, 19)$):

$$\Lambda_G(-1) \hookrightarrow \mathrm{H}_2(\mathrm{K3}, \mathbb{Z})\,.$$

[4]For computational purpose the Niemeier lattice of type $(A_1)^{24}$ is more convenient as is used in [10, Appendix].

Using Torelli theorem for K3 surfaces ([2]) one shows that G has a symplectic action on a K3 surface. $\square$

7. Finite M-semi-symplectic actions on Enriques surfaces

The situation for Enriques surfaces is similar to the one for K3 surfaces except for the fact that we have to add the condition M-semi-symplectic. Recall that by Fact 2.2 the Conjecture 1.1 reads

Conjecture 1.1 *For a finite group G the following conditions are equivalent:*
[A]: *G has an M-semi-symplectic action on an Enriques surface,*
[C]: *G is embeddable into $M_{11} \subset M_{12}$ in such a way that G decomposes Ω_+ (resp. Ω_-) into at least* 3 *(resp.* 2*) orbits.*

Now we sketch the proof of Theorem 1.2. Consider a decomposition of Ω into a pair of complementary (special) dodecads:

$$\Omega = \Omega_+ \amalg \Omega_- .$$

By assumption, we have an action of the group $G = G_1, G_2$ or G_4:

$$G \subset M_{11} \curvearrowright \Omega_+, \quad \text{(resp. } \Omega_-) .$$

Now, we consider the sublattices of the Leech lattice Λ:

$$\Lambda^{\pm} := \Lambda \cap \mathbb{Q}\,\Omega_{\pm} \ \subset \ \mathbb{Q}\,\Omega .$$

Obviously their (orthogonal) sum is a sublattice of the Leech lattice

$$\Lambda \supset \Lambda^+ + \Lambda^- .$$

If we put $\Lambda_G^{\pm} := \Lambda_G \cap \Lambda^{\pm}$, then

$$\Lambda_G \supset \Lambda_G^+ + \Lambda_G^- ,$$

and the fact that Ω_+ (resp. Ω_-) consists of at least 3 (resp. 2) orbits implies that

$$\operatorname{rank}(\Lambda_G^+) \leq 9 = 12 - 3 \quad \text{and} \quad \operatorname{rank}(\Lambda_G^-) \leq 10 = 12 - 2 .$$

The above inequalities and a computation of discriminant forms enable us to obtain primitive embeddings[5]:

$$\begin{array}{ccc} \Lambda_G^+(-\frac{1}{2}) & \overset{\text{isometric}}{\hookrightarrow} & \mathrm{H}_2(\text{Enriques surf.}, \mathbb{Z}) \\ & & \text{signature } (1,9) \\ \Lambda_G^-(-\frac{1}{2}) & \hookrightarrow & \mathrm{H}_2(\text{Enriques surf.}, \mathbb{Z}^{\omega}) \\ & & \text{signature } (2,10) \end{array}$$

Recall that, by definition, $\mathrm{H}_2(\text{Enriques surf.}, \mathbb{Z}^{\omega})$ is the kernel of the map

$$\begin{array}{ccc} \mathrm{H}_2(K3 \text{ surf.}, \mathbb{Z}) & \xrightarrow{\pi_*} & \mathrm{H}_2(\text{Enriques surf.}, \mathbb{Z}) \\ \text{signature } (3,19) & & \text{signature } (1,9) \end{array}$$

induced by the universal K3 cover $\pi : X \to S$.

Then, using Torelli Theorem for Enriques surfaces ([3], [15], [2] and [1]), one shows that G has a semi-symplectic action on an Enriques surface. The fact that the action in question is M-semi-symplectic results almost immediately from the definition. $\square$

[5]For $G = G_3$, $\Lambda_G^-(-1/2)$ has no primitive embeddings into $\mathrm{H}_2(\text{Enriques surf.}, \mathbb{Z}^{\omega})$. The author made Conjecture 1.1 overlooking this fact. See §9.

Remark 7.1. The Enriques surface S with an M-semi-symplectic action of $G_1 = \mathfrak{S}_5$ constructed above is is an Enriques surface with finite automorphism group (type VII) studied by Kondo in [9]. In this case S contains exactly 20 smooth rational curves and has root type $A_1 + A_9$ (see [ibid., Main Thm.]). Furthermore $\mathfrak{S}_5$ is the full automorphism group of S.

8. Motivation and Background

The above study has been motivated by research on involutions of Enriques surfaces.

The so-called Horikawa model is an important tool to understand the behaviour of involutions on Enriques surfaces (below we consider one of two Horikawa expressions in [8]).

Consider a double quadric

$$X \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^1 (\subset \mathbb{P}^3)$$

with branch divisor B of bi-degree $(4, 4)$ with only ADE singularities. By ramification formula X is a K3 surface. Assume that B is invariant under the small involution

$$\mathbb{P}^1 \times \mathbb{P}^1 \ni (x, y) \xrightarrow{e} (-x, -y) \in \mathbb{P}^1 \times \mathbb{P}^1 .$$

The involution e has two lifts to X. Let ε be the anti-symplectic one. Then $\mathrm{Fix}(\varepsilon) = \emptyset$ (unless B passes through $\mathrm{Fix}(e)$) and results in an Enriques surface

$$S := X/\varepsilon .$$

S has an involution σ induced by the covering involution of $X \to \mathbb{P}^1 \times \mathbb{P}^1$. The fixed point locus of σ is given as

$$\mathrm{Fix}(\sigma) = (B \sqcup (8 \text{ points}))/_{\varepsilon} , \tag{1}$$

because the involution we started from has 4 fixed points on $\mathbb{P}^1 \times \mathbb{P}^1$, so we get 8 fixed points on X.

Now there are various cases to consider. The first extremal case is the generic one:

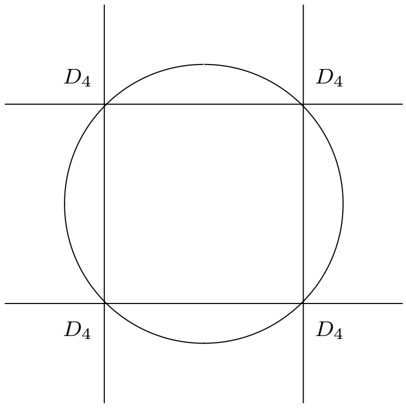
Example 8.1. The curve B is smooth. In this case $p_g(B) = 9$, so the Euler number is $\chi_{\mathrm{top}}(B) = -16$. Therefore, from (1), we have

$$\chi_{\mathrm{top}}(\mathrm{Fix}(\sigma)) = \frac{-16 + 8}{2} = -4 .$$

Observe that the above (generic) case is parametrized by 10 moduli.

The other extreme is

Example 8.2. (Barth-Peters Enriques surface) The curve B consists of a quadrangle and a smooth $(2, 2)$-curve running through its four vertices:



On the minimal resolution of the singularities of the double cover branched along B we obtain the branch locus $B' = \amalg_1^8 \mathbb{P}^1 +$ (an elliptic curve) so

$$\chi_{\mathrm{top}}(\mathrm{Fix}(\sigma)) = \frac{16+8}{2} = 12\,.$$

This surface was first studied by Horikawa[8], later by Barth-Peters[3] and Mukai-Namikawa[11]. Barth-Peters[3] found out that the involution σ acts trivially on $\mathrm{H}_2(S,\mathbb{Z})$. Observe that the number of moduli is 2 in this case.

The above examples are two extremal cases. In general, if g is an involution on an Enriques surface then

$$-4 \leq \chi_{\mathrm{top}}(\mathrm{Fix}(g)) \leq 12$$

and the condition "M-semi-symplectic" is exactly central in all possible involutions:

$$\text{the action of } g \text{ is M-semi-symplectic } \Leftrightarrow \chi_{\mathrm{top}}(\mathrm{Fix}(g)) = 4$$

with 6 moduli.

While studying the geometry of Enriques surfaces we found out that the Barth-Peters Enriques surface is characterized by the condition

$$\begin{array}{cccc} E_8 & \subset & \ker[\mathrm{H}_2(\text{Enriques surf.}, \mathbb{Z}^\omega) & \longrightarrow & \mathbb{C}\,] \\ & & \alpha & \mapsto & \int_\alpha \omega\,. \end{array}$$

Therefore, the interplay between the geometry of Enriques surfaces and the root systems should be interesting.

In the table below we collect the facts that (will) appear in various papers:

Enriques surface S	Root system of S
G_i ($i = 1, 2, 4$) has an M-semi-symplectic action on S	$A_1 + A_9, \quad A_5 + A_5$
S has a cohomologically trivial involution (Example 8.2)	E_8
S has an involution acting trivially on $H^2(S, \mathbb{Q})$ (see [13] and [9, Theorem 1.7])	$E_8, \quad E_7 + A_1, \quad D_8$
S = quotient of $\{\lambda_1(xt + yz) + \lambda_2(yt + xz) + \lambda_3(zt + xy)\}^2$ $+\lambda_4 xyzt = 0$ with four rational double points of type D_4 by $(x, y, z, t) \mapsto (1/x, 1/y, 1/z, 1/t)$	E_7
$S = H/\tau$, where H : = Hessian quartic of a cubic surface also given by the equations $\sum_{i=1}^5 x_i = \sum_{i=1}^5 \lambda_i/x_i = 0$ in $\mathbb{P}^4$ (see [6, § 1]) and $\tau(x_1, \ldots, x_5) = (1/x_1, \ldots, 1/x_5)$	E_6 ($E_6 + A_4$ if all λ_i's are equal)
Quotients of Jacobian Kummer surfaces by an involution ε, *i.e.* $S = \overline{\mathrm{Km}}(\mathrm{J}(C))/\varepsilon$ with $\varepsilon := \varepsilon_G$, where G := Göpel subgroup of $\mathrm{J}(C)_2$ $\varepsilon := \varepsilon_\eta$, where η is an even theta characteristic (see [14] for ε_G)	 $D_6 + A_1$ A_7

To give a more precise explanation of the last row in the above table let us recall the following theorem:

Theorem 8.3 (Ohashi[17])**.** *Let C be a smooth projective curve of genus 2 such that $X := Km(J(C))$ is Picard general, i.e. $\rho(\overline{Km}(\mathrm{J}(C))) = 17$, where $\overline{X}$ stands for the minimal desingularization of X. Then, the number of fixed point free involutions of X (up to conjugacy in $\mathrm{Aut}(X)$) equals*

$$31 = 2^5 - 1 = 15 + 10 + 6\,.$$

Moreover, the number of the involutions of the ε_G-type is 15, whereas the number of the involutions of the ε_η-type is 10.

Remark 8.4. The types of fixed point free involutions in the above theorem correspond to index-2 sublattices of E_7. By removing an appropriate vertex of the extended Dynkin diagram $\tilde{E}_7$ one obtains an index-2 root sublattice of the lattice in question. In this way one obtain two index-2 sublattices $D_6 + A_1$ and A_7. One can show that the following correspondence holds.

$$\begin{aligned} D_6 + A_1 \quad &\longleftrightarrow \quad \varepsilon_G\,, \\ A_7 \quad &\longleftrightarrow \quad \varepsilon_\eta\,. \end{aligned}$$

Finally, the lattice E_7 contains an index-2 sublattice L that is not a root lattice. The sublattice of L generated by roots is E_6. The involutions corresponding to the sublattice L are so-called Hutchinson-Weber involutions, which is a special case of the second last raw of the above table.

9. New conjecture (by S. Mukai and H. Ohashi)

Some progress is made on the Conjecture 1.1 after the lectures. On one hand, we could construct M-semi-symplectic actions of the two groups $(\mathbb{Z}/2)^3$ and $\mathbb{Z}/2 \times \mathbb{Z}/4$ on Enriques surfaces. On the other hand, we could exclude such actions of the groups of order 16 and two groups of order 8; the cyclic group $\mathbb{Z}/8$ and the quaternion group Q_8. Thus the Conjecture 1.1 does not hold true and the list of maximal groups G_i's in the conjecture should be modified. The following is our working hypothesis at present.

Conjecture 9.1. For a finite group G the following conditions are equivalent:
[A] G has an M-semi-symplectic action on an Enriques surface, and
[B] G is a subgroup of $G_1, G_2, G_4, \mathbb{Z}/2 \times \mathfrak{A}_4$ or $G \simeq \mathbb{Z}/2 \times \mathbb{Z}/4$. (In particular G is a proper subgroups of the symmetric group $\mathfrak{S}_6$.)

References

[1] Allcock, D. *The period lattice for Enriques surfaces.* Math. Ann. 317(2000), 483–488.
[2] Barth, W.: Peters, C.: and Van de Ven, A. *Compact Complex Surfaces.* Springer-Verlag, 1984.
[3] Barth, W.; Peters, C. *Automorphisms of Enriques surfaces.* Invent. Math. 73 (1983), 383–411.
[4] Beauville, A. *On the Brauer group of Enriques surfaces.* Math. Res. Lett. 16(2009), 1001–1008.
[5] Degtyarev, A.: Itenberg, I.: Kharlamov, V. *Real Enriques surfaces.* Lecture Notes in Math. 1746 (2000), Springer Verlag.
[6] Dolgachev, I.; Keum, J. *Birational automorphisms of quartic Hessian surfaces.* Trans. Amer. Math. Soc. 354 (2002), 3031–3057.
[7] Griffiths, P.: Harris, J. *Principles of Algebraic Geometry.* John Wiley & Sons, Inc., New York, 1978.
[8] Horikawa, E. *On the periods of Enriques surfaces. I.* Math. Ann. 234(1978), 73–88; *II* Math. Ann. 235(1978), 217–256.
[9] Kondo, S. *Enriques surfaces with finite automorphism groups.* Japan. J. Math. (N.S.) 12 (1986), 191–282.
[10] Kondo, S. *Niemeier lattices, Mathieu groups, and finite groups of symplectic automorphisms of $K3$ surfaces.* With an appendix by Shigeru Mukai. Duke Math. J. 92 (1998), 593–603.
[11] Mukai, S.; Namikawa, Y. *Automorphisms of Enriques surfaces which act trivially on the cohomology groups.* Invent. Math. 77 (1984), 383–397.
[12] Mukai, S. *Finite groups of automorphisms of $K3$ surfaces and the Mathieu group.* Invent. Math. 94 (1988), 183–221.
[13] Mukai, S. *Numerically trivial involutions of Kummer type of an Enriques surface.* Kyoto Math. Journal 50(2010), 889–902.
[14] Mukai, S. *Kummer's quartics and numerically reflective involutions of Enriques surfaces.* RIMS prepint # 1633, to appear in J. Math. Soc. Japan.
[15] Namikawa, Y. *Periods of Enriques surfaces.* Math. Ann. 270(1985), 201–222.
[16] Nikulin, V.V. *On the description of the groups of automorphisms of Enriques surfaces.* Dokl. Akad. Nauk SSSR 277(1984), 1324–1327, (in Russian).
[17] Ohashi, H. *Enriques surfaces covered by Jacobian Kummer surfaces.* Nagoya Math. J. 195 (2009), 165–186.

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502, Japan
e-mail address : mukai@kurims.kyoto-u.ac.jp