Enriques surface fibrations with even index

Joint work with F. Suzuki.

John Christian Ottem

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B = a smooth curve
$f : X \to B$ a morphism

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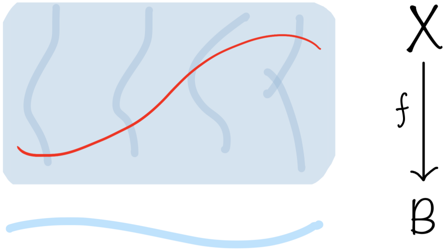


Graber–Harris–Starr theorem: If the general fiber of f is rationally connected, then f has a section.

$\therefore$ Any rationally connected variety X/K, over $K = k(B)$, has a K-point.

Serre (1958) (in a letter to Grothendieck):
Is the same conclusion true for varieties X/K with $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$?

Serre adds that it is "sans doute trop optimiste".

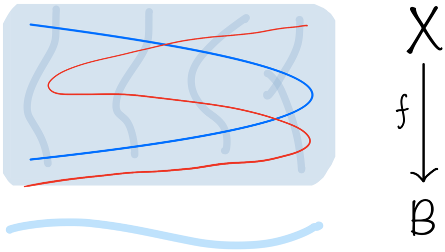
Graber–Harris–Mazur–Starr, Lafon, Starr ($\sim$ 2002)
No: There exist Enriques surface fibrations over curves with no section.

A question of Esnault:

For $f : X \to B$ with $\mathcal{O}$-acyclic fibers: Is the *index* of f equal to 1?

$$\text{index}(f) = \gcd\{\deg(C/B) \mid C \subset X \text{ a curve}\}$$

In other words, does X/K admit a 0-cycle of degree 1?



Main result of this talk:

Theorem (O.-Suzuki)

There exists an Enriques surface fibration

$$X \to \mathbb{P}^1$$

such that the index is even.
In other words, every curve $C \subset X$ has even degree over $\mathbb{P}^1$.

Thus, Serre's question has a negative answer even with 'rational point' replaced by '0-cycle of degree 1'.

Other consequences

The 3-fold X gives counterexamples to other questions:

1. The Integral Hodge conjecture
2. The Hasse principle for the reciprocity obstruction for varieties over function fields of curves
3. Murre's conjecture on the universality of Abel-Jacobi maps

The Integral Hodge Conjecture

Colliot-Thélène–Voisin: For $f : X \to B$ with $\mathcal{O}$-acyclic fibers:

$$f_* : H_2(X, \mathbb{Z}) \to H_2(B, \mathbb{Z})$$

is surjective.

Thus there is a homology class $\sigma \in H_2(X, \mathbb{Z})$ which has degree 1 on a fiber.
$\therefore$ "there is no topological obstruction to the existence of sections"

This class is automatically Hodge, so we obtain a counterexample to

The integral Hodge conjecture (IHC):

$$H^{k,k}(X, \mathbb{C}) \cap H^{2k}(X, \mathbb{Z})$$

is generated by classes of algebraic subvarieties.

In our example, 4σ is algebraic, but σ is not.

Enriques surfaces

Surfaces S with

- $\pi_1(S) = \mathbb{Z}/2$
- $2K_S = 0$

There is a universal cover $\pi : Z \to S$ where Z is a K3 surface

Example

Let $S \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the surface defined by the 2×2 minors of a generic matrix

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix} \qquad \begin{array}{rcl} p_i & = & p_i(x_0, x_1, x_2) \\ q_i & = & q_i(y_0, y_1, y_2) \end{array}$$

where $\deg p_i = (2, 0)$ and $\deg q_i = (0, 2)$. Then S is an Enriques surface.

Here is the K3 cover:
On $\mathbb{P}^5 = \operatorname{Proj} k[x_0, x_1, x_2, y_0, y_1, y_2]$, there is an involution

$$\iota : \mathbb{P}^5 \to \mathbb{P}^5$$

defined by $\iota^*(x_i) = x_i$, $\iota(y_i) = -y_i$.
Consider the quadrics

$$F_i = p_i + q_i \qquad \begin{array}{rcl} p_i &=& p_i(x_0, x_1, x_2) \\ q_i &=& q_i(y_0, y_1, y_2) \end{array}$$

These define a K3 surface

$$Z = \{F_0 = F_1 = F_2 = 0\} \subset \mathbb{P}^5$$

ι acts freely on Z, as Z is disjoint from

$$\operatorname{Fix}(\iota) = P_1 \cup P_2$$

where $P_1 = V(x_0, x_1, x_2)$ and $P_2 = V(y_0, y_1, y_2)$.

$\rightsquigarrow$ $S = Z/\iota$ is a smooth Enriques surface.

Enriques surface fibrations

Let $X \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ be the threefold defined by the 2×2 minors of a generic matrix

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix}$$

where $\deg p_i = (\mathbf{2}, \mathbf{2}, \mathbf{0})$ and $\deg q_i = (\mathbf{2}, \mathbf{0}, \mathbf{2})$.

Then X is a smooth threefold, and the first projection defines an Enriques surface fibration

$$p : X \to \mathbb{P}^1.$$

Also, let $Y \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ be defined by

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix}$$

where $\deg p_i = (\mathbf{1}, \mathbf{2}, \mathbf{0})$ and $\deg q_i = (\mathbf{1}, \mathbf{0}, \mathbf{2})$.

Properties of X

- X has Kodaira dimension 1
- X is simply connected and $H^*(X,\mathbb{Z})$ has no torsion.
- Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 50 & & 0 & \\ 0 & & 99 & & 99 & & 0 \end{array}$$

- $CH_0(X) = \mathbb{Z}$ (as expected by the Bloch conjecture)

Properties of Y

- Y has Kodaira dimension 1
- Y is simply connected and $H^*(X, \mathbb{Z})$ has no torsion.
- Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 26 & & 0 & \\ 0 & & 45 & & 45 & & 0 \end{array}$$

- $CH_0(Y) = \mathbb{Z}$ (as expected by the Bloch conjecture)

We first study the geometry of Y.

Thus $Y \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ is the codimension 2 subvariety defined by the minors of

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix}$$

where $\deg p_i = (\mathbf{1}, \mathbf{2}, \mathbf{0})$ and $\deg q_i = (\mathbf{1}, \mathbf{0}, \mathbf{2})$.

We then use this to study X; X will give the main counterexample to Esnault's question.

The geometry of Y

Let $F_i = p_i + q_i$, considered as a $(1,2)$ form on $\mathbb{P}^1 \times \mathbb{P}^5$.

$$Z = Bl_{\mathbb{P}^1 \times (P_1 \cup P_2)} Z_0$$

$$\swarrow \pi \qquad\qquad p \searrow$$

$$Z_0 = \{F_0 = F_1 = F_2 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^5 \qquad\qquad Y \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$$

- π is the blow-up of $\mathbb{P}^1 \times (P_1 \cup P_2) \cap Y_0$ $(= 12 + 12$ fixed points of $\iota)$.

  $\rightsquigarrow$ $12 + 12 = 24$ exceptional divisors

$$E_{i,j} \quad i = 1, 2, j = 1, \ldots, 12.$$

- p is a double cover, ramified along the $E_{i,j}$.

Out of the 24 $E_{i,j}$'s, we single out $E_{1,1}, \dots, E_{1,12}$.

When Y is defined by $\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix}$, the $E_{1,i}$ are the components of

$$E_1 = \{p_0 = p_1 = p_2 = 0\} \subset Y$$

Claim

For a curve $C \subset Y$ we have

$$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{j=1}^{12} E_{1,j} \right) \mod 2.$$

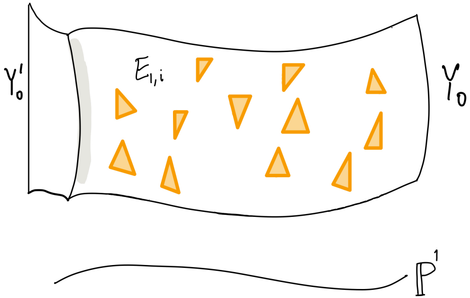
$\therefore$ If $C \subset Y$ is a section of $X \to \mathbb{P}^1$, then C has to intersect at least one of the $E_{1,j}$'s (!).

Y is very general $\rightsquigarrow$ we may prove this using a degeneration argument.

The degeneration: $\mathcal{Y} \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \operatorname{Spec} k[\epsilon]$ defined by the minors of

$$M_\epsilon = \begin{pmatrix} p_0 & p_1 & p_2 \\ sy_0^2 + \epsilon r_0 & sy_1^2 + \epsilon r_1 & sy_2^2 + \epsilon r_2 \end{pmatrix}$$

Special fiber (over $\epsilon = 0$):



- $Y_0 \cap Y_0' = \{s = 0\}$= an Enriques surface
- All the $E_{i,j}$ lie on Y_0 and do not intersect Y_0'.
- $V(p_0, p_1, p_2) = E_{1,1} \cup \cdots \cup E_{1,12}$.

$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{j=1}^{12} E_{1,j}\right) \mod 2$

Y_0 is defined by the matrix

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ y_0^2 & y_1^2 & y_2^2 \end{pmatrix}$$

Let $D_1 = \{p_0 = 0\}$; this is a divisor of type $(1, 2, 0)$.

For $C \subset Y_0$ a curve,

$$\deg(C/\mathbb{P}^1) \equiv D_1 \cdot C \mod 2$$

On the other hand,

$$D_1 = \{y_0^2 = 0\} + \sum_{j=1}^{12} E_{1,j}$$

This gives the desired congruence.

The threefold X and proof of the main theorem

Theorem

Let X be defined by a very general matrix in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix}$$

where $\deg p_i = (\mathbf{2}, \mathbf{2}, \mathbf{0})$ *and* $\deg q_i = (\mathbf{2}, \mathbf{0}, \mathbf{2})$.
Then any curve $C \subset X \to \mathbb{P}^1$ *has even degree over* $\mathbb{P}^1$.

On $X \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ there are now $24 + 24 = 48$ exceptional divisors

$$E_{i,j} \qquad i = 1, 2,\ j = 1, \ldots, 24$$

We focus on $E_{1,1}, \ldots, E_{1,24}$; the components of

$$E_1 = \{p_0 = p_1 = p_2 = 0\}.$$

Basic strategy: Prove the following *key congruence*:

$$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{k=1}^{12} E_{1,j_k} \right) \mod 2 \tag{1}$$

for any 12-tuple $1 \leq j_1 < \ldots < j_{12} \leq 24$.

This would imply the theorem: We would get that

$$C \cdot E_{1,1} \equiv \cdots \equiv C \cdot E_{1,24} \mod 2,$$

and hence that $\deg(C/\mathbb{P}^1)$ is even.

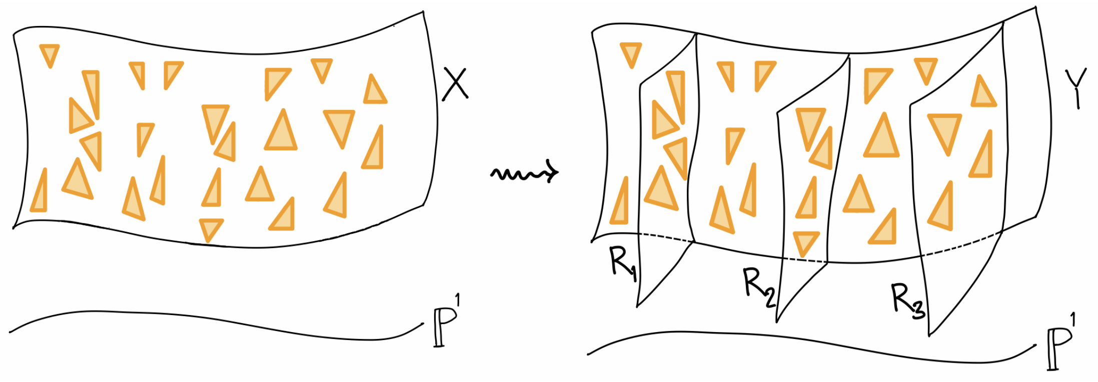
We want to prove that

$$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{j=1}^{12} E_{1,j_k} \right) \mod 2 \tag{2}$$

1. **Monodromy argument**: Reduce to proving (2) for *some* 12-tuple $j_1 < \ldots < j_{12}$.
2. **Specialization argument**: Prove (2) for some $(j_1, \ldots, j_{12})$ by analyzing a certain degeneration of X.

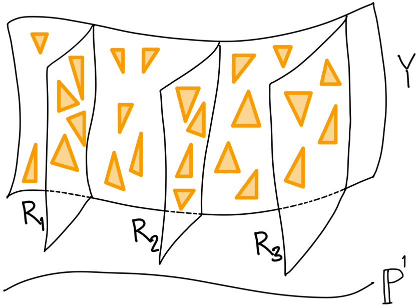
Here is the degeneration:

$$M = \begin{pmatrix} sp_0 + \epsilon r_0 & (s-t)p_1 + \epsilon r_1 & (s+t)p_2 + \epsilon r_2 \\ stq_0 + \epsilon s_0 & t(s-t)q_1 + \epsilon s_1 & t(s+t)q_2 + \epsilon s_2 \end{pmatrix}$$



The special fiber over $\epsilon = 0$ is a union

$$Y \cup R_1 \cup R_2 \cup R_3$$



- Y is the previous Enriques surface fibration with 12 planes $E_{1,j_1}, \ldots, E_{1,j_{12}}$
- On Y we know that

$$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{k=1}^{12} E_{1,j_k} \right) \mod 2 \tag{3}$$

(2) follows from this.

Thank you for the attention!