Pushing forward Hall-Littlewood polynomials

Piotr Pragacz
(IM PAN, Warszawa)

Introduction

(This is a version of July 2021)

Introduction

(This is a version of July 2021)

Let $f : X \to Y$ be a proper map of varieties.

Introduction

(This is a version of July 2021)

Let $f : X \to Y$ be a proper map of varieties.

f gives rise to a map $f_* : A(X) \to A(Y)$ of the Chow groups induced by push-forward of cycles. It is called push-forward or Gysin map.

Introduction

(This is a version of July 2021)

Let $f : X \to Y$ be a proper map of varieties.

f gives rise to a map $f_* : A(X) \to A(Y)$ of the Chow groups induced by push-forward of cycles. It is called push-forward or Gysin map.

There are many formulas for Gysin maps. Those for degeneracy loci often involve determinants and Pfaffians.

Introduction

(This is a version of July 2021)

Let $f : X \to Y$ be a proper map of varieties.

f gives rise to a map $f_* : A(X) \to A(Y)$ of the Chow groups induced by push-forward of cycles. It is called push-forward or Gysin map.

There are many formulas for Gysin maps. Those for degeneracy loci often involve determinants and Pfaffians.

We shall use them simultaneously by means of Hall-Littlewood polynomials associated with a vector bundle $E \to X$ of rank n with Chern roots $x_1, \ldots x_n$.

Let $s_i(E)$ denotes the ith complete symmetric function in $x_1, \ldots, x_n$, or $(-1)^i$ (i-th Segre class from Fulton's IT).

Let $s_i(E)$ denotes the ith complete symmetric function in $x_1, \ldots, x_n$, or $(-1)^i$ (i-th Segre class from Fulton's IT).

Given a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$, we set

$$s_\lambda(E) = \left| s_{\lambda_i - i + j}(E) \right|_{1 \leq i,j \leq n}.$$

Let $s_i(E)$ denotes the ith complete symmetric function in $x_1, \ldots, x_n$, or $(-1)^i$ (i-th Segre class from Fulton's IT).

Given a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$, we set

$$s_\lambda(E) = \left| s_{\lambda_i - i + j}(E) \right|_{1 \leq i,j \leq n}.$$

For $q \leq n$, let $\pi : G^q(E) \to X$ be the Grassmann bundle parametrizing rank q quotients of E. It is endowed with the universal exact sequence of vector bundles

$$0 \longrightarrow S \longrightarrow \pi^* E \longrightarrow Q \longrightarrow 0,$$

where $\mathrm{rank}(Q) = q$. Let $r = n - q$.

Let $s_i(E)$ denotes the ith complete symmetric function in $x_1, \ldots, x_n$, or $(-1)^i$ (i-th Segre class from Fulton's IT).

Given a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$, we set

$$s_\lambda(E) = \left| s_{\lambda_i - i + j}(E) \right|_{1 \leq i,j \leq n}.$$

For $q \leq n$, let $\pi : G^q(E) \to X$ be the Grassmann bundle parametrizing rank q quotients of E. It is endowed with the universal exact sequence of vector bundles

$$0 \longrightarrow S \longrightarrow \pi^* E \longrightarrow Q \longrightarrow 0,$$

where $\mathrm{rank}(Q) = q$. Let $r = n - q$.

Then for any partitions $\lambda = (\lambda_1, \ldots, \lambda_q)$, $\mu = (\mu_1, \ldots, \mu_r)$,

$$\pi_*\big(s_\lambda(Q) \cdot s_\mu(S)\big) = s_{\lambda_1 - r, \ldots, \lambda_q - r, \mu_1, \ldots, \mu_r}(E).$$

Want: degeneracy loci formulas for maps with symmetries.
We shall use Pfaffians.

Want: degeneracy loci formulas for maps with symmetries.
We shall use Pfaffians.

Consider Schur P-functions $P_\lambda(E) = P_\lambda$ defined as follows.
For a strict partition $\lambda = (\lambda_1 > \ldots > \lambda_k > 0)$ with odd k,

$$P_\lambda = P_{\lambda_1} P_{\lambda_2,\ldots,\lambda_k} - P_{\lambda_2} P_{\lambda_1,\lambda_3,\ldots,\lambda_k} + \cdots + P_{\lambda_k} P_{\lambda_1,\ldots,\lambda_{k-1}} ,$$

Want: degeneracy loci formulas for maps with symmetries.
We shall use Pfaffians.

Consider Schur P-functions $P_\lambda(E) = P_\lambda$ defined as follows.
For a strict partition $\lambda = (\lambda_1 > \ldots > \lambda_k > 0)$ with odd k,

$$P_\lambda = P_{\lambda_1} P_{\lambda_2,\ldots,\lambda_k} - P_{\lambda_2} P_{\lambda_1,\lambda_3,\ldots,\lambda_k} + \cdots + P_{\lambda_k} P_{\lambda_1,\ldots,\lambda_{k-1}}\,,$$

and with even k,

$$P_\lambda = P_{\lambda_1,\lambda_2} P_{\lambda_3,\ldots,\lambda_k} - P_{\lambda_1,\lambda_3} P_{\lambda_2,\lambda_4,\ldots,\lambda_k} + \cdots + P_{\lambda_1,\lambda_k} P_{\lambda_2,\ldots,\lambda_{k-1}}\,.$$

Want: degeneracy loci formulas for maps with symmetries.
We shall use Pfaffians.

Consider Schur P-functions $P_\lambda(E) = P_\lambda$ defined as follows.
For a strict partition $\lambda = (\lambda_1 > \ldots > \lambda_k > 0)$ with odd k,

$$P_\lambda = P_{\lambda_1} P_{\lambda_2,\ldots,\lambda_k} - P_{\lambda_2} P_{\lambda_1,\lambda_3,\ldots,\lambda_k} + \cdots + P_{\lambda_k} P_{\lambda_1,\ldots,\lambda_{k-1}} ,$$

and with even k,

$$P_\lambda = P_{\lambda_1,\lambda_2} P_{\lambda_3,\ldots,\lambda_k} - P_{\lambda_1,\lambda_3} P_{\lambda_2,\lambda_4,\ldots,\lambda_k} + \cdots + P_{\lambda_1,\lambda_k} P_{\lambda_2,\ldots,\lambda_{k-1}} .$$

Here, $P_i = \sum s_\mu$, the sum over all hook partitions μ of i,

Want: degeneracy loci formulas for maps with symmetries.
We shall use Pfaffians.

Consider Schur P-functions $P_\lambda(E) = P_\lambda$ defined as follows.
For a strict partition $\lambda = (\lambda_1 > \ldots > \lambda_k > 0)$ with odd k,

$$P_\lambda = P_{\lambda_1} P_{\lambda_2,\ldots,\lambda_k} - P_{\lambda_2} P_{\lambda_1,\lambda_3,\ldots,\lambda_k} + \cdots + P_{\lambda_k} P_{\lambda_1,\ldots,\lambda_{k-1}}\,,$$

and with even k,

$$P_\lambda = P_{\lambda_1,\lambda_2} P_{\lambda_3,\ldots,\lambda_k} - P_{\lambda_1,\lambda_3} P_{\lambda_2,\lambda_4,\ldots,\lambda_k} + \cdots + P_{\lambda_1,\lambda_k} P_{\lambda_2,\ldots,\lambda_{k-1}}\,.$$

Here, $P_i = \sum s_\mu$, the sum over all hook partitions μ of i,

and for positive $i > j$ we set

$$P_{i,j} = P_i P_j + 2 \sum_{d=1}^{j-1} (-1)^d P_{i+d} P_{j-d} + (-1)^j P_{i+j}\,.$$

Want:

$$\pi_*\Big(c_{qr}(Q \otimes S)P_\lambda(Q)P_\mu(S)\Big) =?$$

Want:

$$\pi_*\big(c_{qr}(Q \otimes S)P_\lambda(Q)P_\mu(S)\big) = ?$$

If $l(\lambda) = q$, $\mu = 0$, we get $P_\lambda(E)$.

Want:

$$\pi_*\Big(c_{qr}(Q \otimes S)P_\lambda(Q)P_\mu(S)\Big) =?$$

If $l(\lambda) = q$, $\mu = 0$, we get $P_\lambda(E)$.

If $l(\lambda) = q - 1$, $\mu = 0$, we get $P_\lambda(E)$ for $n - q$ even.

Want:

$$\pi_*\Big(c_{qr}(Q \otimes S)P_\lambda(Q)P_\mu(S)\Big) = ?$$

If $l(\lambda) = q$, $\mu = 0$, we get $P_\lambda(E)$.

If $l(\lambda) = q - 1$, $\mu = 0$, we get $P_\lambda(E)$ for $n - q$ even.

If $l(\lambda) = q - 1$, $\mu = 0$, we get 0 for $n - q$ odd.

Want:

$$\pi_*\big(c_{qr}(Q \otimes S)P_\lambda(Q)P_\mu(S)\big) = ?$$

If $l(\lambda) = q$, $\mu = 0$, we get $P_\lambda(E)$.

If $l(\lambda) = q - 1$, $\mu = 0$, we get $P_\lambda(E)$ for $n - q$ even.

If $l(\lambda) = q - 1$, $\mu = 0$, we get 0 for $n - q$ odd.

If $n = 15$, $q = 7$, $l(\lambda) = 3$, $l(\mu) = 4$, then

$$\pi_*(c_{56}(Q \otimes S) \cdot P_{931}(Q) \cdot P_{7542}(S)) = (-6)P_{9754321}(E)\,.$$

Want:

$$\pi_*\big(c_{qr}(Q \otimes S)P_\lambda(Q)P_\mu(S)\big) = ?$$

If $l(\lambda) = q$, $\mu = 0$, we get $P_\lambda(E)$.

If $l(\lambda) = q - 1$, $\mu = 0$, we get $P_\lambda(E)$ for $n - q$ even.

If $l(\lambda) = q - 1$, $\mu = 0$, we get 0 for $n - q$ odd.

If $n = 15$, $q = 7$, $l(\lambda) = 3$, $l(\mu) = 4$, then

$$\pi_*(c_{56}(Q \otimes S) \cdot P_{931}(Q) \cdot P_{7542}(S)) = (-6)P_{9754321}(E)\,.$$

Like Schur S-polynomials govern Schubert calculus on classical Grassmannians,

Want:

$$\pi_*\big(c_{qr}(Q \otimes S)P_\lambda(Q)P_\mu(S)\big) = ?$$

If $l(\lambda) = q$, $\mu = 0$, we get $P_\lambda(E)$.

If $l(\lambda) = q - 1$, $\mu = 0$, we get $P_\lambda(E)$ for $n - q$ even.

If $l(\lambda) = q - 1$, $\mu = 0$, we get 0 for $n - q$ odd.

If $n = 15$, $q = 7$, $l(\lambda) = 3$, $l(\mu) = 4$, then

$$\pi_*(c_{56}(Q \otimes S) \cdot P_{931}(Q) \cdot P_{7542}(S)) = (-6)P_{9754321}(E)\,.$$

Like Schur S-polynomials govern Schubert calculus on classical Grassmannians,

Schur P-polynomials govern Schubert calculus on Lagrangian Grassmannians.

Let $\tau_E : Fl(E) \to X$ be the complete flag bundle parametrizing flags of quotients of E of ranks $n-1, n-2, \ldots, 1$.

Let $\tau_E : Fl(E) \to X$ be the complete flag bundle parametrizing flags of quotients of E of ranks $n-1, n-2, \ldots, 1$.

Let t be a variable. The main formula will be located in $A(X)[t]$ or $H(X)[t]$.

Let $\tau_E : Fl(E) \to X$ be the complete flag bundle parametrizing flags of quotients of E of ranks $n-1, n-2, \ldots, 1$.

Let t be a variable. The main formula will be located in $A(X)[t]$ or $H(X)[t]$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ be sequence of nonnegative integers. Set

$$R_\lambda(E;t) = (\tau_E)_* \Big(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j} (x_i - t x_j) \Big) \,,$$

where $(\tau_E)_*$ acts on each coefficient of the polynomial in t separately.

Proposition

If $\lambda \in \mathbb{Z}^q_{\geq 0}$ *and* $\mu \in \mathbb{Z}^{n-q}_{\geq 0}$ *then*

$$\pi_*\Big(R_\lambda(Q;t)R_\mu(S;t)\prod_{i\leq q<j}(x_i - tx_j)\Big) = R_{\lambda\mu}(E;t),$$

where $\lambda\mu = (\lambda_1, \ldots, \lambda_q, \mu_1, \ldots, \mu_r)$ *is the juxtaposition of* λ *and* μ*.*

Proposition

If $\lambda \in \mathbb{Z}^{q}_{\geq 0}$ *and* $\mu \in \mathbb{Z}^{n-q}_{\geq 0}$ *then*

$$\pi_*\Big(R_\lambda(Q;t)R_\mu(S;t)\prod_{i\leq q<j}(x_i - tx_j)\Big) = R_{\lambda\mu}(E;t),$$

where $\lambda\mu = (\lambda_1, \dots, \lambda_q, \mu_1, \dots, \mu_r)$ *is the juxtaposition of* λ *and* μ.

This is seen from a commutative diagram

$$\begin{array}{ccc} Fl(Q) \times_{G^q(E)} Fl(S) & \xrightarrow{\cong} & Fl(E) \\ \Big\downarrow{\scriptstyle \tau_Q \times \tau_S} & & \Big\downarrow{\scriptstyle \tau_E} \\ G^q(E) & \xrightarrow[\pi]{} & X \end{array}$$

which gives

$$\pi_*(\tau_Q \times \tau_S)_* = (\tau_E)_* \,.$$

Suppose that $x_1 \ldots, x_q$ are the Chern roots of Q and $x_{q+1}, \ldots, x_n$ are the ones of S.

Suppose that $x_1 \ldots, x_q$ are the Chern roots of Q and $x_{q+1}, \ldots, x_n$ are the ones of S. Let $x^\lambda = x_1^{\lambda_1} \cdots x_q^{\lambda_q}$ and $x^\mu = x_{q+1}^{\mu_1} \cdots x_n^{\mu_r}$.

Suppose that $x_1 \dots, x_q$ are the Chern roots of Q and $x_{q+1}, \dots, x_n$ are the ones of S. Let $x^\lambda = x_1^{\lambda_1} \cdots x_q^{\lambda_q}$ and $x^\mu = x_{q+1}^{\mu_1} \cdots x_n^{\mu_r}$.

It follows from the above equality:

$$\pi_*\big(R_\lambda(Q;t)R_\mu(S;t) \prod_{i \le q < j} (x_i - tx_j)\big)$$

Suppose that $x_1 \ldots, x_q$ are the Chern roots of Q and $x_{q+1}, \ldots, x_n$ are the ones of S. Let $x^\lambda = x_1^{\lambda_1} \cdots x_q^{\lambda_q}$ and $x^\mu = x_{q+1}^{\mu_1} \cdots x_n^{\mu_r}$.

It follows from the above equality:

$$\pi_*\Big(R_\lambda(Q;t)R_\mu(S;t) \prod_{i\le q<j}(x_i - tx_j)\Big)$$

$$= \pi_*\Big((\tau_Q)_*\big(x^\lambda \prod_{i<j\le q}(x_i - tx_j)\big) \cdot (\tau_S)_*\big(x^\mu \prod_{q<i<j}(x_i - tx_j)\big) \prod_{i\le q<j}(x_i - tx_j)$$

Suppose that $x_1 \dots, x_q$ are the Chern roots of Q and $x_{q+1}, \dots, x_n$ are the ones of S. Let $x^\lambda = x_1^{\lambda_1} \cdots x_q^{\lambda_q}$ and $x^\mu = x_{q+1}^{\mu_1} \cdots x_n^{\mu_r}$.

It follows from the above equality:

$$\pi_*\Big(R_\lambda(Q;t)R_\mu(S;t) \prod_{i\le q<j}(x_i - tx_j)\Big)$$

$$= \pi_*\Big((\tau_Q)_*\Big(x^\lambda \prod_{i<j\le q}(x_i - tx_j)\Big) \cdot (\tau_S)_*\Big(x^\mu \prod_{q<i<j}(x_i - tx_j)\Big) \prod_{i\le q<j}(x_i - tx_j)$$

$$= \pi_*(\tau_Q \times \tau_S)_*\Big(x^\lambda \prod_{i<j\le q}(x_i - tx_j)\; x^\mu \prod_{q<i<j}(x_i - tx_j) \prod_{i\le q<j}(x_i - tx_j)\Big)$$

Suppose that $x_1 \ldots, x_q$ are the Chern roots of Q and $x_{q+1}, \ldots, x_n$ are the ones of S. Let $x^\lambda = x_1^{\lambda_1} \cdots x_q^{\lambda_q}$ and $x^\mu = x_{q+1}^{\mu_1} \cdots x_n^{\mu_r}$.

It follows from the above equality:

$$\pi_*\Big(R_\lambda(Q;t)R_\mu(S;t) \prod_{i\le q<j}(x_i - tx_j)\Big)$$

$$= \pi_*\Big((\tau_Q)_*\Big(x^\lambda \prod_{i<j\le q}(x_i - tx_j)\Big) \cdot (\tau_S)_*\Big(x^\mu \prod_{q<i<j}(x_i - tx_j)\Big) \prod_{i\le q<j}(x_i - tx_j)$$

$$= \pi_*(\tau_Q \times \tau_S)_*\Big(x^\lambda \prod_{i<j\le q}(x_i - tx_j)\ x^\mu \prod_{q<i<j}(x_i - tx_j) \prod_{i\le q<j}(x_i - tx_j)\Big)$$

$$= (\tau_E)_*(x^\lambda\ x^\mu \prod_{i<j}(x_i - tx_j)) = R_{\lambda\mu}(E;t)\,.$$

Suppose that $x_1 \ldots, x_q$ are the Chern roots of Q and $x_{q+1}, \ldots, x_n$ are the ones of S. Let $x^\lambda = x_1^{\lambda_1} \cdots x_q^{\lambda_q}$ and $x^\mu = x_{q+1}^{\mu_1} \cdots x_n^{\mu_r}$.

It follows from the above equality:

$$\pi_*\Big(R_\lambda(Q;t)R_\mu(S;t) \prod_{i \le q < j} (x_i - tx_j)\Big)$$

$$= \pi_*\Big((\tau_Q)_*\big(x^\lambda \prod_{i<j\le q} (x_i - tx_j)\big) \cdot (\tau_S)_*\big(x^\mu \prod_{q<i<j} (x_i - tx_j)\big) \prod_{i\le q<j} (x_i - tx_j)$$

$$= \pi_*(\tau_Q \times \tau_S)_*\Big(x^\lambda \prod_{i<j\le q} (x_i - tx_j)\; x^\mu \prod_{q<i<j} (x_i - tx_j) \prod_{i\le q<j} (x_i - tx_j)\Big)$$

$$= (\tau_E)_*(x^\lambda\; x^\mu \prod_{i<j} (x_i - tx_j)) = R_{\lambda\mu}(E;t)\,.$$

$$\prod_{i<j\le q}(x_i - tx_j)\prod_{q<i<j}(x_i - tx_j)\prod_{i\le q<j}(x_i - tx_j) = \prod_{i<j}(x_i - tx_j)\,.$$

$$v_m(t) := \prod_{i=1}^{m} \frac{1-t^i}{1-t} = (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{m-1}).$$

$$v_m(t) := \prod_{i=1}^{m} \frac{1-t^i}{1-t} = (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{m-1}).$$

Let $\lambda \in \mathbb{Z}^n_{\geq 0}$. Consider the maximal subsets I_1,...,I_d in $\{1,\ldots,n\}$, where the sequence λ is constant.

$$v_m(t) := \prod_{i=1}^{m} \frac{1-t^i}{1-t} = (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{m-1}).$$

Let $\lambda \in \mathbb{Z}^n_{\geq 0}$. Consider the maximal subsets $I_1,\ldots,I_d$ in $\{1,\ldots,n\}$, where the sequence λ is constant.

Let $m_1,\ldots,m_d$ be the cardinalities of $I_1,\ldots,I_d$. So we have $m_1+\cdots+m_d = n$.

$$v_m(t) := \prod_{i=1}^{m} \frac{1-t^i}{1-t} = (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{m-1}).$$

Let $\lambda \in \mathbb{Z}^n_{\geq 0}$. Consider the maximal subsets $I_1,\dots,I_d$ in $\{1,\dots,n\}$, where the sequence λ is constant.

Let $m_1,\dots,m_d$ be the cardinalities of $I_1,\dots,I_d$. So we have $m_1+\cdots+m_d=n$.

Also we set $v_\lambda = v_\lambda(t) := \prod_{i=1}^{d} v_{m_i}(t)$.

Example. $\lambda = (9, 9, 9, 8, 0, 0, 8, 8, 8, 8, 9)$, $n = 11$

Example. $\lambda = (9,9,9,8,0,0,8,8,8,8,9)$, $n = 11$

$I_1 = \{1,2,3,11\}$, $I_2 = \{4,7,8,9,10\}$, $I_3 = \{5,6\}$

Example. $\lambda = (9, 9, 9, 8, 0, 0, 8, 8, 8, 8, 9)$, $n = 11$

$I_1 = \{1, 2, 3, 11\}$, $I_2 = \{4, 7, 8, 9, 10\}$, $I_3 = \{5, 6\}$

$m_1 = 4$, $m_2 = 5$, $m_3 = 2$

Example. $\lambda = (9, 9, 9, 8, 0, 0, 8, 8, 8, 8, 9)$, $n = 11$

$I_1 = \{1, 2, 3, 11\}$, $I_2 = \{4, 7, 8, 9, 10\}$, $I_3 = \{5, 6\}$

$m_1 = 4$, $m_2 = 5$, $m_3 = 2$

$v_\lambda = v_4 v_5 v_2$

Example. $\lambda = (9, 9, 9, 8, 0, 0, 8, 8, 8, 8, 9)$, $n = 11$

$I_1 = \{1, 2, 3, 11\}$, $I_2 = \{4, 7, 8, 9, 10\}$, $I_3 = \{5, 6\}$

$m_1 = 4$, $m_2 = 5$, $m_3 = 2$

$v_\lambda = v_4 v_5 v_2$

Example. Let $\nu = (\nu_1 > \ldots > \nu_k > 0)$ be a strict partition with $k \leq n$.

Example. $\lambda = (9, 9, 9, 8, 0, 0, 8, 8, 8, 8, 9)$, $n = 11$

$I_1 = \{1, 2, 3, 11\}$, $I_2 = \{4, 7, 8, 9, 10\}$, $I_3 = \{5, 6\}$

$m_1 = 4$, $m_2 = 5$, $m_3 = 2$

$v_\lambda = v_4 v_5 v_2$

Example. Let $\nu = (\nu_1 > \ldots > \nu_k > 0)$ be a strict partition with $k \leq n$.

Let $\lambda = \nu 0^{n-k}$ be the sequence ν with $n - k$ zeros added at the end.

Example. $\lambda = (9, 9, 9, 8, 0, 0, 8, 8, 8, 8, 9)$, $n = 11$

$I_1 = \{1, 2, 3, 11\}$, $I_2 = \{4, 7, 8, 9, 10\}$, $I_3 = \{5, 6\}$

$m_1 = 4$, $m_2 = 5$, $m_3 = 2$

$v_\lambda = v_4 v_5 v_2$

Example. Let $\nu = (\nu_1 > \ldots > \nu_k > 0)$ be a strict partition with $k \leq n$.

Let $\lambda = \nu 0^{n-k}$ be the sequence ν with $n-k$ zeros added at the end.

Then $d = k+1$, $(m_1, \ldots, m_d) = (1^k, n-k)$, $v_\lambda(t) = v_{n-k}(t)$.

Definition

Let λ be a sequence of nonnegative integers. Set

$$P_\lambda(E;t) = \frac{1}{v_\lambda(t)} R_\lambda(E;t)\,. \qquad (1)$$

Definition

Let λ be a sequence of nonnegative integers. Set

$$P_\lambda(E;t) = \frac{1}{v_\lambda(t)} R_\lambda(E;t)\,. \qquad (1)$$

If λ is a partition, then $P_\lambda(E;t)$ is a polynomial in t, called Hall-Littlewood polynomial (see Macdonald's book)

Definition

Let λ be a sequence of nonnegative integers. Set

$$P_\lambda(E;t) = \frac{1}{v_\lambda(t)} R_\lambda(E;t)\,. \qquad (1)$$

If λ is a partition, then $P_\lambda(E;t)$ is a polynomial in t, called Hall-Littlewood polynomial (see Macdonald's book)

Let $y_1, \ldots, y_n$ and t be independent indeterminates.

Definition

Let λ be a sequence of nonnegative integers. Set

$$P_\lambda(E;t) = \frac{1}{v_\lambda(t)} R_\lambda(E;t)\,. \qquad (1)$$

If λ is a partition, then $P_\lambda(E;t)$ is a polynomial in t, called Hall-Littlewood polynomial (see Macdonald's book)

Let $y_1, \ldots, y_n$ and t be independent indeterminates.

For a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of nonnegative integers, set

$$R_\lambda(y_1, \ldots, y_n; t) = \sum_{w \in S_n} w\left(y_1^{\lambda_1} \cdots y_n^{\lambda_n} \prod_{i<j} \frac{y_i - t y_j}{y_i - y_j} \right),$$

where S_n is the group of all bijections of $\{y_1, \ldots, y_n\}$.

Definition

Let λ be a sequence of nonnegative integers. Set

$$P_\lambda(E;t) = \frac{1}{v_\lambda(t)} R_\lambda(E;t)\,. \qquad (1)$$

If λ is a partition, then $P_\lambda(E;t)$ is a polynomial in t, called Hall-Littlewood polynomial (see Macdonald's book)

Let $y_1, \ldots, y_n$ and t be independent indeterminates.

For a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of nonnegative integers, set

$$R_\lambda(y_1, \ldots, y_n; t) = \sum_{w \in S_n} w\left(y_1^{\lambda_1} \cdots y_n^{\lambda_n} \prod_{i<j} \frac{y_i - ty_j}{y_i - y_j}\right),$$

where S_n is the group of all bijections of $\{y_1, \ldots, y_n\}$. (Specializing the y's to the Chern roots of E, $R_\lambda(y;t)$ becomes $R_\lambda(E;t)$.)

Computing with Maple, we get the following examples.

Example

For λ equal to $(0,2,0)$, $(0,2,2,0)$, $(0,2,3,0)$, $(0,2,2,3,3)$, $R_\lambda(y;t)$ is divisible by $v_\lambda(t)$.

Computing with Maple, we get the following examples.

Example

For λ equal to $(0, 2, 0)$, $(0, 2, 2, 0)$, $(0, 2, 3, 0)$, $(0, 2, 2, 3, 3)$, $R_\lambda(y; t)$ is divisible by $v_\lambda(t)$. For λ equal to $(0, 2, 0, 2)$, $(0, 2, 0, 0, 2)$, $(0, 2, 2, 0, 0, 0)$, $R_\lambda(y; t)$ is not divisible by $v_\lambda(t)$.

Computing with Maple, we get the following examples.

Example

For λ equal to $(0,2,0)$, $(0,2,2,0)$, $(0,2,3,0)$, $(0,2,2,3,3)$, $R_\lambda(y;t)$ is divisible by $v_\lambda(t)$. For λ equal to $(0,2,0,2)$, $(0,2,0,0,2)$, $(0,2,2,0,0,0)$, $R_\lambda(y;t)$ is not divisible by $v_\lambda(t)$.

As a consequence of the Proposition we obtain the following result.

Computing with Maple, we get the following examples.

Example

For λ equal to $(0,2,0)$, $(0,2,2,0)$, $(0,2,3,0)$, $(0,2,2,3,3)$, $R_\lambda(y;t)$ is divisible by $v_\lambda(t)$. For λ equal to $(0,2,0,2)$, $(0,2,0,0,2)$, $(0,2,2,0,0,0)$, $R_\lambda(y;t)$ is not divisible by $v_\lambda(t)$.

As a consequence of the Proposition we obtain the following result.

Theorem

Suppose that $\lambda = (\lambda_1, \ldots, \lambda_q)$ *and* $\mu = (\mu_1, \ldots, \mu_r)$ *are sequences of nonnegative integers such that* $R_\lambda(Q;t)$ *is divisible by* $v_\lambda(t)$ *and* $R_\mu(S;t)$ *is divisible by* $v_\mu(t)$.

Computing with Maple, we get the following examples.

Example

For λ equal to $(0,2,0)$, $(0,2,2,0)$, $(0,2,3,0)$, $(0,2,2,3,3)$, $R_\lambda(y;t)$ is divisible by $v_\lambda(t)$. For λ equal to $(0,2,0,2)$, $(0,2,0,0,2)$, $(0,2,2,0,0,0)$, $R_\lambda(y;t)$ is not divisible by $v_\lambda(t)$.

As a consequence of the Proposition we obtain the following result.

Theorem

Suppose that $\lambda=(\lambda_1,\dots,\lambda_q)$ and $\mu=(\mu_1,\dots,\mu_r)$ are sequences of nonnegative integers such that $R_\lambda(Q;t)$ is divisible by $v_\lambda(t)$ and $R_\mu(S;t)$ is divisible by $v_\mu(t)$. Then for the polynomials $P_\lambda(Q;t)$ and $P_\mu(S;t)$ we have

$$\pi_*\Big(\prod_{i\le q<j}(x_i-tx_j)P_\lambda(Q;t)P_\mu(S;t)\Big)=\frac{v_{\lambda\mu}(t)}{v_\lambda(t)v_\mu(t)}P_{\lambda\mu}(E;t)\,.$$

In the sequel, the sequences λ and μ will be partitions.

In the sequel, the sequences λ and μ will be partitions.

We look at the specialization $t = 0$.

In the sequel, the sequences λ and μ will be partitions.

We look at the specialization $t = 0$.

We invoke the Jacobi-Trudi formula for $s_\lambda(E)$ with the help of the Gysin map associated to $\tau_E : Fl(E) \to X$:

$$s_\lambda(E) = (\tau_E)_*(x_1^{\lambda_1+n-1} \cdots x_n^{\lambda_n}).$$

In the sequel, the sequences λ and μ will be partitions.

We look at the specialization $t = 0$.

We invoke the Jacobi-Trudi formula for $s_\lambda(E)$ with the help of the Gysin map associated to $\tau_E : Fl(E) \to X$:

$$s_\lambda(E) = (\tau_E)_*(x_1^{\lambda_1+n-1} \cdots x_n^{\lambda_n}).$$

We see that $P_\lambda(E;t) = s_\lambda(E)$ for $t = 0$. Under this specialization, the Theorem becomes

$$\pi_*\big((x_1 \cdots x_q)^r s_\lambda(Q) s_\mu(S)\big) = \pi_*\big(s_{\lambda_1+r,\ldots,\lambda_q+r}(Q) s_\mu(S)\big) = s_{\lambda\mu}(E)\,.$$

(Józefiak-Lascoux-P)

If a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ is not a partition, then $s_\lambda(E)$ is either 0 or $\pm s_\mu(E)$ for some partition μ.

If a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ is not a partition, then $s_\lambda(E)$ is either 0 or $\pm s_\mu(E)$ for some partition μ.

One can rearrange λ by a sequence of operations $(\ldots, i, j, \ldots) \mapsto (\ldots, j-1, i+1, \ldots)$ applied to pairs of successive integers.

If a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ is not a partition, then $s_\lambda(E)$ is either 0 or $\pm s_\mu(E)$ for some partition μ.

One can rearrange λ by a sequence of operations $(\ldots, i, j, \ldots) \mapsto (\ldots, j-1, i+1, \ldots)$ applied to pairs of successive integers.

Either one arrives at a sequence of the form $(\ldots, i, i+1, \ldots)$, in which case $s_\lambda(E) = 0$, or one arrives in d steps at a partition μ, and then $s_\lambda(E) = (-1)^d s_\mu(E)$.

Pair of strict partitions ν, σ:

Pair of strict partitions ν, σ:

$$\nu = (\nu_1 > \ldots > \nu_k > 0), \quad k \leq q, \quad \lambda := \nu 0^{q-k}$$

Pair of strict partitions ν, σ:

$$\nu = (\nu_1 > \ldots > \nu_k > 0), \quad k \le q, \quad \lambda := \nu 0^{q-k}$$

$$\sigma = (\sigma_1 > \ldots > \sigma_h > 0), \quad h \le n-q, \quad \mu := \sigma 0^{n-q-h}$$

Pair of strict partitions ν, σ:

$$\nu = (\nu_1 > \ldots > \nu_k > 0), \quad k \le q, \quad \lambda := \nu 0^{q-k}$$

$$\sigma = (\sigma_1 > \ldots > \sigma_h > 0), \quad h \le n - q, \quad \mu := \sigma 0^{n-q-h}$$

$$v_\lambda = \prod_{i=1}^{q-k} \frac{1-t^i}{1-t}$$

Pair of strict partitions ν, σ:

$$\nu = (\nu_1 > \ldots > \nu_k > 0), \quad k \leq q, \quad \lambda := \nu 0^{q-k}$$

$$\sigma = (\sigma_1 > \ldots > \sigma_h > 0), \quad h \leq n-q, \quad \mu := \sigma 0^{n-q-h}$$

$$v_\lambda = \prod_{i=1}^{q-k} \frac{1-t^i}{1-t}$$

$$v_\mu = \prod_{i=1}^{n-q-h} \frac{1-t^i}{1-t}$$

Pair of strict partitions ν, σ:

$$\nu = (\nu_1 > \ldots > \nu_k > 0), \quad k \leq q, \quad \lambda := \nu 0^{q-k}$$

$$\sigma = (\sigma_1 > \ldots > \sigma_h > 0), \quad h \leq n-q, \quad \mu := \sigma 0^{n-q-h}$$

$$v_\lambda = \prod_{i=1}^{q-k} \frac{1-t^i}{1-t}$$

$$v_\mu = \prod_{i=1}^{n-q-h} \frac{1-t^i}{1-t}$$

$$v_{\lambda\mu} = \prod_{i=1}^{n-k-h} \frac{1-t^i}{1-t} \left(\prod_{i=1}^{2} \frac{1-t^i}{1-t}\right)^e$$

Pair of strict partitions ν, σ:

$$\nu = (\nu_1 > \ldots > \nu_k > 0), \quad k \le q, \quad \lambda := \nu 0^{q-k}$$

$$\sigma = (\sigma_1 > \ldots > \sigma_h > 0), \quad h \le n-q, \quad \mu := \sigma 0^{n-q-h}$$

$$v_\lambda = \prod_{i=1}^{q-k} \frac{1-t^i}{1-t}$$

$$v_\mu = \prod_{i=1}^{n-q-h} \frac{1-t^i}{1-t}$$

$$v_{\lambda\mu} = \prod_{i=1}^{n-k-h} \frac{1-t^i}{1-t} \left(\prod_{i=1}^{2} \frac{1-t^i}{1-t} \right)^e$$

Here e is the number of common parts of ν and σ.

We have

$$\frac{v_{\lambda\mu}}{v_\lambda v_\mu} = \frac{(1-t)\cdots(1-t^{n-k-h})}{(1-t)\cdots(1-t^{q-k})(1-t)\cdots(1-t^{n-q-h})}(1+t)^e$$

We have

$$\frac{v_{\lambda\mu}}{v_\lambda v_\mu} = \frac{(1-t)\cdots(1-t^{n-k-h})}{(1-t)\cdots(1-t^{q-k})(1-t)\cdots(1-t^{n-q-h})}(1+t)^e$$

This is the Gaussian polynomial $\begin{bmatrix} n-k-h \\ q-k \end{bmatrix}(t)$ times $(1+t)^e$.

We have

$$\frac{v_{\lambda\mu}}{v_\lambda v_\mu} = \frac{(1-t)\cdots(1-t^{n-k-h})}{(1-t)\cdots(1-t^{q-k})(1-t)\cdots(1-t^{n-q-h})}(1+t)^e$$

This is the Gaussian polynomial $\begin{bmatrix} n-k-h \\ q-k \end{bmatrix}(t)$ times $(1+t)^e$.

So by Theorem we have

$$\pi_*\Big(\prod_{i\le q<j}(x_i - tx_j)P_\lambda(Q;t)P_\mu(S;t)\Big) = \begin{bmatrix} n-k-h \\ q-k \end{bmatrix}(t)(1+t)^e P_{\lambda\mu}(E;t).$$

We have

$$\frac{v_{\lambda\mu}}{v_\lambda v_\mu} = \frac{(1-t)\cdots(1-t^{n-k-h})}{(1-t)\cdots(1-t^{q-k})(1-t)\cdots(1-t^{n-q-h})}(1+t)^e$$

This is the Gaussian polynomial $\left[{n-k-h \atop q-k}\right](t)$ times $(1+t)^e$.

So by Theorem we have

$$\pi_*\Big(\prod_{i\le q<j}(x_i - t x_j) P_\lambda(Q;t) P_\mu(S;t)\Big) = \begin{bmatrix} n-k-h \\ q-k \end{bmatrix}(t)(1+t)^e P_{\lambda\mu}(E;t).$$

- some zeros at the end of λ possible

We have

$$\frac{v_{\lambda\mu}}{v_\lambda v_\mu} = \frac{(1-t)\cdots(1-t^{n-k-h})}{(1-t)\cdots(1-t^{q-k})(1-t)\cdots(1-t^{n-q-h})}(1+t)^e$$

This is the Gaussian polynomial $\begin{bmatrix} n-k-h \\ q-k \end{bmatrix}(t)$ times $(1+t)^e$.

So by Theorem we have

$$\pi_*\Big(\prod_{i\leq q<j}(x_i-tx_j)P_\lambda(Q;t)P_\mu(S;t)\Big) = \begin{bmatrix} n-k-h \\ q-k \end{bmatrix}(t)(1+t)^e P_{\lambda\mu}(E;t).$$

- some zeros at the end of λ possible

We look at the specialization $t=-1$. Most interesting is the specialization of Gaussian polynomials.

Lemma

At $t = -1$, the Gaussian polynomial

$$\begin{bmatrix} a+b \\ a \end{bmatrix}(t)$$

specializes to zero if ab is odd and to the binomial coefficient

$$\binom{\lfloor (a+b)/2 \rfloor}{\lfloor a/2 \rfloor}$$

otherwise.

Lemma

At $t = -1$, the Gaussian polynomial

$$\begin{bmatrix} a+b \\ a \end{bmatrix}(t)$$

specializes to zero if ab is odd and to the binomial coefficient

$$\binom{\lfloor (a+b)/2 \rfloor}{\lfloor a/2 \rfloor}$$

otherwise.

(with Witold Kraśkiewicz)

Indeed, we have

$$\begin{bmatrix} a+b \\ a \end{bmatrix}(t) = \frac{(1-t)(1-t^2)\cdots(1-t^{a+b})}{(1-t)\cdots(1-t^a)(1-t)\cdots(1-t^b)}.$$

Indeed, we have

$$\begin{bmatrix} a+b \\ a \end{bmatrix}(t) = \frac{(1-t)(1-t^2)\cdots(1-t^{a+b})}{(1-t)\cdots(1-t^a)(1-t)\cdots(1-t^b)} .$$

Since $t = -1$ is a zero with multiplicity 1 of the factor $(1-t^d)$ for even d, and a zero with multiplicity 0 for odd d,

Indeed, we have

$$\begin{bmatrix} a+b \\ a \end{bmatrix}(t) = \frac{(1-t)(1-t^2)\cdots(1-t^{a+b})}{(1-t)\cdots(1-t^a)(1-t)\cdots(1-t^b)}\,.$$

Since $t=-1$ is a zero with multiplicity 1 of the factor $(1-t^d)$ for even d, and a zero with multiplicity 0 for odd d,

the order of the rational function $\left[\begin{smallmatrix} a+b \\ a \end{smallmatrix}\right](t)$ at $t=-1$ is equal to

$$\lfloor (a+b)/2 \rfloor - \lfloor a/2 \rfloor - \lfloor b/2 \rfloor\,.$$

Indeed, we have

$$\begin{bmatrix} a+b \\ a \end{bmatrix}(t) = \frac{(1-t)(1-t^2)\cdots(1-t^{a+b})}{(1-t)\cdots(1-t^a)(1-t)\cdots(1-t^b)}\,.$$

Since $t=-1$ is a zero with multiplicity 1 of the factor $(1-t^d)$ for even d, and a zero with multiplicity 0 for odd d,

the order of the rational function $\left[\begin{smallmatrix} a+b \\ a \end{smallmatrix}\right](t)$ at $t=-1$ is equal to

$$\lfloor (a+b)/2 \rfloor - \lfloor a/2 \rfloor - \lfloor b/2 \rfloor\,.$$

This order is equal to 1 when a and b are odd, and 0 otherwise.

Indeed, we have

$$\begin{bmatrix} a+b \\ a \end{bmatrix}(t) = \frac{(1-t)(1-t^2)\cdots(1-t^{a+b})}{(1-t)\cdots(1-t^a)(1-t)\cdots(1-t^b)}\,.$$

Since $t=-1$ is a zero with multiplicity 1 of the factor $(1-t^d)$ for even d, and a zero with multiplicity 0 for odd d,

the order of the rational function $\left[\begin{smallmatrix} a+b \\ a \end{smallmatrix}\right](t)$ at $t=-1$ is equal to

$$\lfloor (a+b)/2 \rfloor - \lfloor a/2 \rfloor - \lfloor b/2 \rfloor\,.$$

This order is equal to 1 when a and b are odd, and 0 otherwise.

In the former case, we get the claimed vanishing, and

in the latter one, the product of the factors with even exponents is equal to

$$\begin{bmatrix} \lfloor a + b/2 \rfloor \\ \lfloor a/2 \rfloor \end{bmatrix} (t^2) \, .$$

The value of this function at $t = -1$ is equal to $\begin{bmatrix} \lfloor a+b/2 \rfloor \\ \lfloor a/2 \rfloor \end{bmatrix} (1)$

in the latter one, the product of the factors with even exponents is equal to

$$\begin{bmatrix} \lfloor a + b/2 \rfloor \\ \lfloor a/2 \rfloor \end{bmatrix} (t^2) \, .$$

The value of this function at $t = -1$ is equal to $\left[\begin{smallmatrix} \lfloor a+b/2 \rfloor \\ \lfloor a/2 \rfloor \end{smallmatrix}\right] (1)$

which is the binomial coefficient

$$\binom{\lfloor (a + b)/2 \rfloor}{\lfloor a/2 \rfloor} \, .$$

in the latter one, the product of the factors with even exponents is equal to

$$\begin{bmatrix} \lfloor a+b/2 \rfloor \\ \lfloor a/2 \rfloor \end{bmatrix}(t^2)\,.$$

The value of this function at $t=-1$ is equal to $\left[\begin{smallmatrix} \lfloor a+b/2 \rfloor \\ \lfloor a/2 \rfloor \end{smallmatrix}\right](1)$

which is the binomial coefficient

$$\binom{\lfloor (a+b)/2 \rfloor}{\lfloor a/2 \rfloor}\,.$$

This is the requested value since the remaining factors with odd exponents give 2 in the numerator and the same number in the denominator. QED

Let $x_1, \ldots, x_n$ and t be independent indeterminates over $\mathbb{Z}$.

Let $x_1, \dots, x_n$ and t be independent indeterminates over $\mathbb{Z}$.

Schur in his 1911 paper on projective representations of the symmetric group showed that for any strict partition λ of length k,

$$P_\lambda(x_1, \dots, x_n) = \sum_{w \in S_n/(S_1)^k \times S_{n-k}} w\left(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j, i \le k} \frac{x_i + x_j}{x_i - x_j} \right)$$

Let $x_1, \ldots, x_n$ and t be independent indeterminates over $\mathbb{Z}$.

Schur in his 1911 paper on projective representations of the symmetric group showed that for any strict partition λ of length k,

$$P_\lambda(x_1, \ldots, x_n) = \sum_{w \in S_n/(S_1)^k \times S_{n-k}} w\Big(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j, i \le k} \frac{x_i + x_j}{x_i - x_j}\Big)$$

We have also, for a similar λ, the following formula for a Hall-Littlewood polynomial (see Mcd p. 208):

$$P_\lambda(x_1, \ldots, x_n; t) = \sum_{w \in S_n/(S_1)^k \times S_{n-k}} w\Big(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j, i \le k} \frac{x_i - t x_j}{x_i - x_j}\Big)$$

We pass now to the notation associated to a pair of strict partitions ν and σ. It follows from comparison of the last two formulas that for $\lambda = \nu 0^{q-k}$, we have $P_\lambda(Q; t)_{t=-1} = P_\nu(Q)$;

We pass now to the notation associated to a pair of strict partitions ν and σ. It follows from comparison of the last two formulas that for $\lambda = \nu 0^{q-k}$, we have $P_\lambda(Q;t)_{t=-1} = P_\nu(Q)$; and for $\mu = \sigma 0^{r-h}$, we have $P_\mu(S;t)_{t=-1} = P_\sigma(S)$.

We pass now to the notation associated to a pair of strict partitions ν and σ. It follows from comparison of the last two formulas that for $\lambda = \nu 0^{q-k}$, we have $P_\lambda(Q;t)_{t=-1} = P_\nu(Q)$; and for $\mu = \sigma 0^{r-h}$, we have $P_\mu(S;t)_{t=-1} = P_\sigma(S)$.

For $\lambda \in \mathbb{Z}^n_{\geq 0}$ set $\mathcal{P} := P_\lambda(E;t)_{t=-1}$.

We pass now to the notation associated to a pair of strict partitions ν and σ. It follows from comparison of the last two formulas that for $\lambda = \nu 0^{q-k}$, we have $P_\lambda(Q;t)_{t=-1} = P_\nu(Q)$; and for $\mu = \sigma 0^{r-h}$, we have $P_\mu(S;t)_{t=-1} = P_\sigma(S)$.

For $\lambda \in \mathbb{Z}^n_{\geq 0}$ set $\mathcal{P} := P_\lambda(E;t)_{t=-1}$.

Note that for $i+j>0$, we have $\mathcal{P}_{\dots,i,j,\dots} = -\mathcal{P}_{\dots,j,i,\dots}$.

We pass now to the notation associated to a pair of strict partitions ν and σ. It follows from comparison of the last two formulas that for $\lambda = \nu 0^{q-k}$, we have $P_\lambda(Q;t)_{t=-1} = P_\nu(Q)$; and for $\mu = \sigma 0^{r-h}$, we have $P_\mu(S;t)_{t=-1} = P_\sigma(S)$.

For $\lambda \in \mathbb{Z}^n_{\geq 0}$ set $\mathcal{P} := P_\lambda(E;t)_{t=-1}$.

Note that for $i+j>0$, we have $\mathcal{P}_{\dots,i,j,\dots} = -\mathcal{P}_{\dots,j,i,\dots}$.

Thus $\mathcal{P}_{\nu 0^{q-k}\sigma 0^{r-h}} = (-1)^{(q-k)h}\mathcal{P}_{\nu\sigma 0^{n-k-h}} = (-1)^{(q-k)h}\mathcal{P}_{\nu\sigma}$.

We pass now to the notation associated to a pair of strict partitions ν and σ. It follows from comparison of the last two formulas that for $\lambda = \nu 0^{q-k}$, we have $P_\lambda(Q;t)_{t=-1} = P_\nu(Q)$; and for $\mu = \sigma 0^{r-h}$, we have $P_\mu(S;t)_{t=-1} = P_\sigma(S)$.

For $\lambda \in \mathbb{Z}^n_{\geq 0}$ set $\mathcal{P} := P_\lambda(E;t)_{t=-1}$.

Note that for $i+j>0$, we have $\mathcal{P}_{\dots,i,j,\dots} = -\mathcal{P}_{\dots,j,i,\dots}$.

Thus $\mathcal{P}_{\nu 0^{q-k}\sigma 0^{r-h}} = (-1)^{(q-k)h}\mathcal{P}_{\nu\sigma 0^{n-k-h}} = (-1)^{(q-k)h}\mathcal{P}_{\nu\sigma}$.

If $e>0$, then $P_{\nu\sigma}(E) = 0$; so we can assume $e=0$ without loss of generality.

Specializing $t = -1$ we get from the Theorem by virtue of the Lemma the following result.

Specializing $t = -1$ we get from the Theorem by virtue of the Lemma the following result.

For strict partitions ν, σ with $l(\nu) = k \leq q$ and $l(\sigma) = h \leq n - q$, we have

Specializing $t = -1$ we get from the Theorem by virtue of the Lemma the following result.

For strict partitions ν, σ with $l(\nu) = k \leq q$ and $l(\sigma) = h \leq n - q$, we have

$$\pi_*\big(c_{qr}(Q \otimes S)P_\nu(Q)P_\sigma(S)\big) = d_{\nu,\sigma} \cdot P_{\nu\sigma}(E)\,,$$

Specializing $t = -1$ we get from the Theorem by virtue of the Lemma the following result.

For strict partitions ν, σ with $l(\nu) = k \leq q$ and $l(\sigma) = h \leq n - q$, we have

$$\pi_*\big(c_{qr}(Q \otimes S)P_\nu(Q)P_\sigma(S)\big) = d_{\nu,\sigma} \cdot P_{\nu\sigma}(E)\,,$$

where $d_{\nu,\sigma} = 0$ if $(q-k)(r-h)$ is odd, and

Specializing $t = -1$ we get from the Theorem by virtue of the Lemma the following result.

For strict partitions ν, σ with $l(\nu) = k \leq q$ and $l(\sigma) = h \leq n - q$, we have

$$\pi_*\big(c_{qr}(Q \otimes S)P_\nu(Q)P_\sigma(S)\big) = d_{\nu,\sigma} \cdot P_{\nu\sigma}(E)\,,$$

where $d_{\nu,\sigma} = 0$ if $(q-k)(r-h)$ is odd, and

$$d_{\nu,\sigma} = (-1)^{(q-k)h} \binom{\lfloor (n-k-h)/2 \rfloor}{\lfloor (q-k)/2 \rfloor}$$

otherwise.

It can happen that $\nu\sigma$ is not a strict partition.

It can happen that $\nu\sigma$ is not a strict partition.

If there are repetitions of indices in $\nu\sigma$ then the RHS is assumed to be zero.

It can happen that $\nu\sigma$ is not a strict partition.

If there are repetitions of indices in $\nu\sigma$ then the RHS is assumed to be zero.

If not, then $P_{\nu\sigma}(E) = (-I)^l P_{\kappa}(E)$, where l is the lenghth of the permutation which rearranges $\nu\sigma$ into the corresponding strict partition κ.

History of Hall-Littlewood polynomials (very brief):

Investigation of combinatorial structure of the lattice of finite p-groups (Philip Hall about 1950):

History of Hall-Littlewood polynomials (very brief):

Investigation of combinatorial structure of the lattice of finite p-groups (Philip Hall about 1950):

p - prime number, M finite Abelian p-group,
$M = \oplus_{i=1}^{r} \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$,

History of Hall-Littlewood polynomials (very brief):

Investigation of combinatorial structure of the lattice of finite p-groups (Philip Hall about 1950):

p - prime number, M finite Abelian p-group,
$M = \oplus_{i=1}^{r} \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$,

$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ - partition ("type of M").

History of Hall-Littlewood polynomials (very brief):

Investigation of combinatorial structure of the lattice of finite p-groups (Philip Hall about 1950):

p - prime number, M finite Abelian p-group,
$M = \oplus_{i=1}^{r} \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$,

$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ - partition ("type of M").

Hall algebra : λ, μ, ν three partitions. Let M be of type λ.

$$G^{\lambda}_{\mu\nu} := \mathrm{card}\{N \subset M : \mathrm{type}(N) = \nu,\ \mathrm{type}(M/N) = \mu\}$$

History of Hall-Littlewood polynomials (very brief):

Investigation of combinatorial structure of the lattice of finite p-groups (Philip Hall about 1950):

p - prime number, M finite Abelian p-group,
$M = \oplus_{i=1}^{r} \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$,

$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ - partition ("type of M").

Hall algebra : λ, μ, ν three partitions. Let M be of type λ.

$$G^{\lambda}_{\mu\nu} := \mathrm{card}\{N \subset M : \mathrm{type}(N) = \nu,\ \mathrm{type}(M/N) = \mu\}$$

Let H be a free $\mathbb{Z}$-module with basis $\{u_\lambda\}$,

History of Hall-Littlewood polynomials (very brief):

Investigation of combinatorial structure of the lattice of finite p-groups (Philip Hall about 1950):

p - prime number, M finite Abelian p-group,
$M = \oplus_{i=1}^{r} \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$,

$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ - partition ("type of M").

Hall algebra : λ, μ, ν three partitions. Let M be of type λ.

$$G^{\lambda}_{\mu\nu} := \mathrm{card}\{N \subset M : \mathrm{type}(N) = \nu,\ \mathrm{type}(M/N) = \mu\}$$

Let H be a free $\mathbb{Z}$-module with basis $\{u_\lambda\}$,

$$u_\mu \cdot u_\nu := \sum G^{\lambda}_{\mu\nu} u_\lambda \,.$$

Theorem

H is a commutative ring, and is generated as a $\mathbb{Z}$-algebra by $\{u_{(1^r)}\}$ (algebraically independent).

Theorem

H is a commutative ring, and is generated as a $\mathbb{Z}$-algebra by $\{u_{(1^r)}\}$ (algebraically independent).

Theorem

The $\mathbb{Q}$-linear map $\psi : H \otimes \mathbb{Q} \to \Lambda \otimes \mathbb{Q}$ (symmetric functions) such that

$$\psi(u_\lambda) = p^{-\sum(i-1)\lambda_i} P_\lambda(y_1, y_2, \ldots ; p^{-1})$$

is a ring isomorphism.

Theorem

H is a commutative ring, and is generated as a $\mathbb{Z}$-algebra by $\{u_{(1^r)}\}$ (algebraically independent).

Theorem

The $\mathbb{Q}$-linear map $\psi : H \otimes \mathbb{Q} \to \Lambda \otimes \mathbb{Q}$ (symmetric functions) such that

$$\psi(u_\lambda) = p^{-\sum(i-1)\lambda_i} P_\lambda(y_1, y_2, \ldots ; p^{-1})$$

is a ring isomorphism.

J.A. Green, D.E. Littlewood: Representation theory of GL_n over finite fields.

End