A tropical and logarithmic study of Milnor fibers

Patrick Popescu-Pampu

University of Lille, France

IMPANGA Seminar
of the Polish Academy of Sciences
Online talk

Friday, 22 May 2020

Our team

I will present results obtained in collaboration with :

- **Maria Angelica CUETO** (Ohio State University) ;
- **Dmitry STEPANOV** (Bauman Moscow State Technical University)

$(\mathbf{X}, \mathbf{o})$ denotes a **complex analytic normal surface singularity**, that is, a germ of normal complex analytic surface.

Definition

The **link** $\partial(\mathbf{X}, \mathbf{o})$ of (X, o) is the intersection of a representative $X \hookrightarrow \mathbb{C}^n$ of (X, o) with a sufficiently small euclidean sphere centered at the origin.

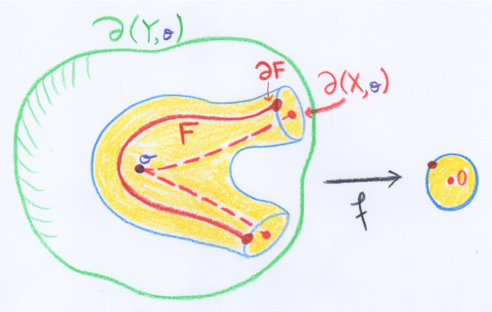
Using previous work of Waldhausen done in the 1960s, Neumann proved in 1981 that :

Theorem

The dual graph of any good resolution of (X, o) encodes the oriented topological type of the link of (X, o).

Definition

A **smoothing** of (X, o) is a germ of morphism $f : (Y, X) \to (\mathbb{D}, 0)$ with smooth generic fibers. Its **Milnor fiber** is a generic fiber of a convenient representative.



$$\partial \mathbf{F} \simeq_+ \partial(\mathbf{X}, \mathbf{o})$$

ICIS = isolated complete intersection singularity

Theorem

An ICIS has a well-defined Milnor fiber (independent of the smoothing).

Therefore, one may use **any** smoothing in order to study the structure of the Milnor fiber of an ICIS.

Neumann and Wahl formulated in 1990 :

Conjecture

Assume that (X, o) is an ICIS **with $\mathbb{Z}HS$ link**. *Then :*

$$\sigma(\mathbf{F}) = \mathbf{8} \cdot \lambda(\partial \mathbf{F}).$$

- $\sigma(\mathbf{F})$ is the **signature** of the intersection form on $H_2(F, \mathbb{R})$.
- $\lambda(\partial \mathbf{F})$ is the **Casson invariant** of ∂F.

This conjecture is still **open**. But it was proved for a large class of singularities, as we will see below.

$$\mathbf{E_8} := (Z(x^2 + y^3 + z^5), 0).$$

Theorem

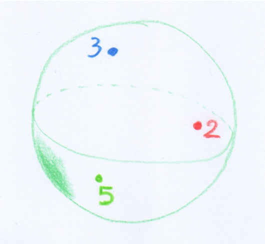
The link of the singularity E_8 is the Poincaré homology sphere.

Indeed, both are **Seifert fibered** oriented manifolds with the same Seifert invariants.

$$E_8 = (Z(\mathbf{x^2} + \mathbf{y^3} + \mathbf{z^5}), 0).$$

$$\left[\begin{array}{ccc} \mathbb{C}^*_t \times \mathbb{C}^3_{x,y,z} & \to & \mathbb{C}^3_{x,y,z} \\ (t, x, y, z) & \to & (t^{3\cdot 5}x, t^{2\cdot 5}y, t^{2\cdot 3}z) \end{array} \right.$$

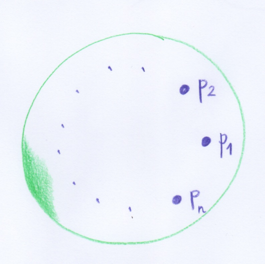
induces an $\mathbb{S}^1$-action on ∂E_8, with quotient space :



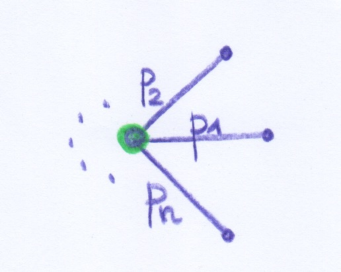
Seifert proved in 1933 :

Theorem

Given $p_1, \ldots, p_n \in \mathbb{N}^*$ which are pairwise coprime, there is a unique Seifert fibered $\mathbb{Z}$HS with base :



It turns out that there is a unique orientation turning this Seifert fibered $\mathbb{Z}$HS into a singularity link. Denote this oriented manifold by $\boldsymbol{\Sigma}(\mathbf{p_1}, \dots, \mathbf{p_n})$. In 1979, Siebenmann represented it by :



Neumann proved in 1977 that the previous oriented 3-dimensional manifold is an ICIS link :

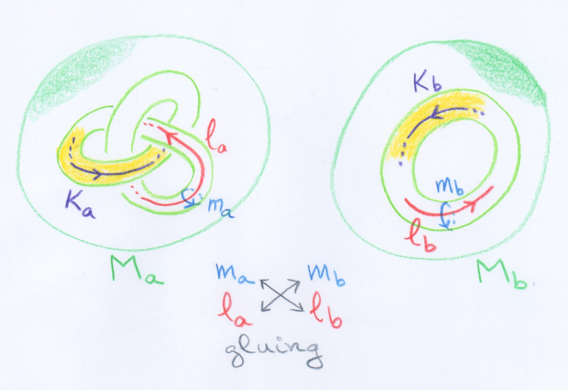
Theorem

$\Sigma(\mathbf{p_1}, \dots, \mathbf{p_n})$ *is the link of the ICIS defined by the following* **Pham-Brieskorn-Hamm system** *:*

$$\begin{cases} \mathbf{c_{1,1}}\, z_1^{\mathbf{p_1}} & + \cdots + & \mathbf{c_{1,n}}\, z_n^{\mathbf{p_n}} & = 0 \\ \vdots & \vdots & \vdots & \\ \mathbf{c_{n-2,1}}\, z_1^{\mathbf{p_1}} & + \cdots + & \mathbf{c_{n-2,n}}\, z_n^{\mathbf{p_n}} & = 0 \end{cases}$$

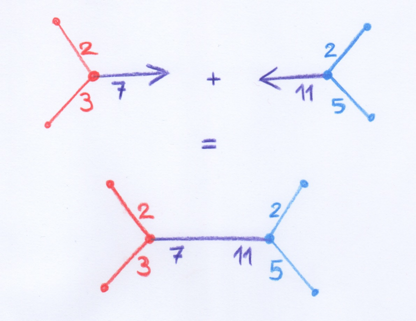
in which all maximal minors of the matrix $(\mathbf{c_{i,j}})_{\mathbf{i,j}}$ *are non-zero.*

Are there other $\mathbb{Z}$HS ? Many more, as shown by the following operation introduced by Dehn in 1907 :



Siebenmann called it **splicing** in 1979.

One may apply recursively such splicing operations to fibers of Seifert fibered $\mathbb{Z}$HS :

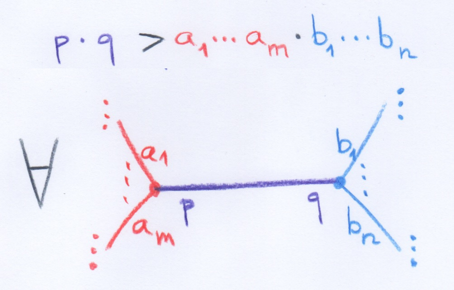


One gets a so-called **splice diagram**.

Eisenbud and Neumann proved in 1985 that :

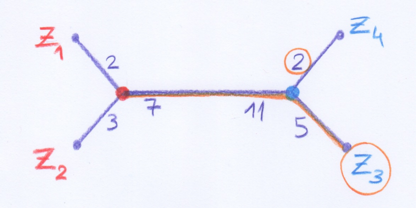
Theorem

The $\mathbb{Z}$HS which are singularity links are exactly those which may be described by splice diagrams satisfying :



For them, the splice diagram is **homeomorphic** to the dual graph of the minimal good resolution.

Around 2000, Neumann and Wahl discovered a supplementary **sufficient** condition, which they called the **semigroup condition**, allowing to realize a $\mathbb{Z}$HS singularity link **by an ICIS**.

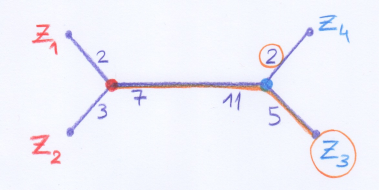


This splice diagram is a singularity link because :

$$7 \cdot 11 > 2 \cdot 3 \cdot 2 \cdot 5.$$

It satisfies also the semigroup condition :

$$\begin{cases} 7 & = \boxed{1} \cdot 2 + \boxed{1} \cdot 5 \\ 11 & = \boxed{1} \cdot 3 + \boxed{4} \cdot 2. \end{cases}$$



$$\begin{cases} 7 = \boxed{1} \cdot 2 + \boxed{1} \cdot 5 \\ 11 = \boxed{1} \cdot 3 + \boxed{4} \cdot 2. \end{cases}$$

These relations allow to write the following **splice type system** :

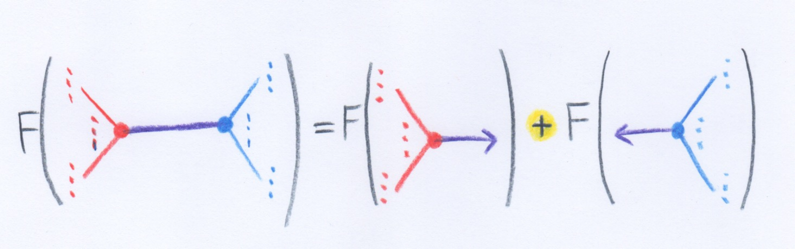
$$\begin{cases} \bullet z_1^2 + \bullet z_2^3 + \bullet\, z_3^{\boxed{1}} z_4^{\boxed{1}} = 0 \\ \bullet z_1^{\boxed{1}} z_2^{\boxed{4}} + \bullet\, z_3^5 + \bullet z_4^2 = 0. \end{cases}$$

In general, one has to impose also the Hamm non-vanishing condition to each subsystem associated to a fixed internal vertex.

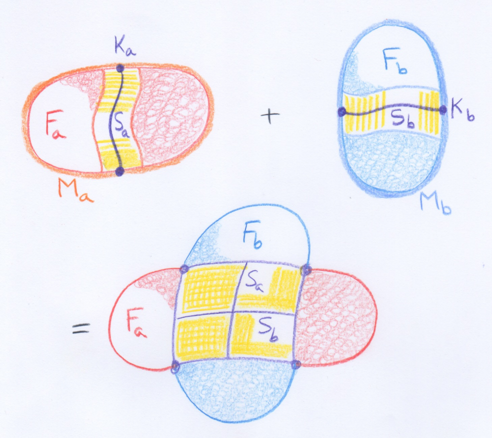
Are there other $\mathbb{Z}$HS singularity links ?

As a way to prove the Casson invariant conjecture for splice type singularities, Neumann and Wahl formulated around 2003 the following **"Milnor fiber conjecture"** :

Conjecture



The addition sign represents a **4-dimensional splicing operation**.



K_a
F_a
S_a
M_a
+
F_b
S_b
K_b
M_b
=
F_b
S_a
F_a
S_b

Némethi & Okuma proved in 2007 :

Theorem

The Casson invariant conjecture is true for splice type singularities.

Their proof translated the Casson invariant conjecture in terms of resolutions and avoided completely the Milnor fiber conjecture, which remained open.

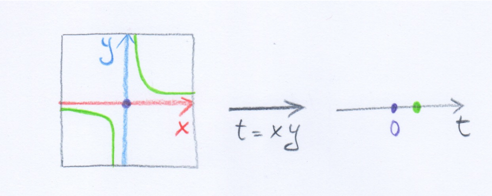
The Milnor fiber conjecture was proved only in special cases by Neumann–Wahl (2005) and by Lamberson (2009).

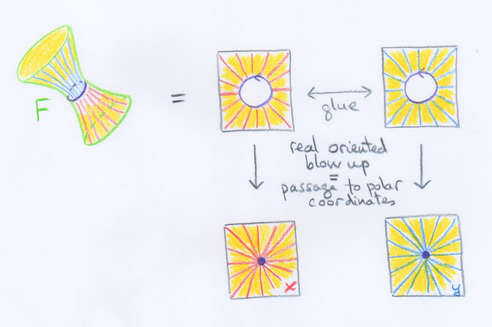
Maria Angelica Cueto, Dmitry Stepanov and myself proved (2020) :

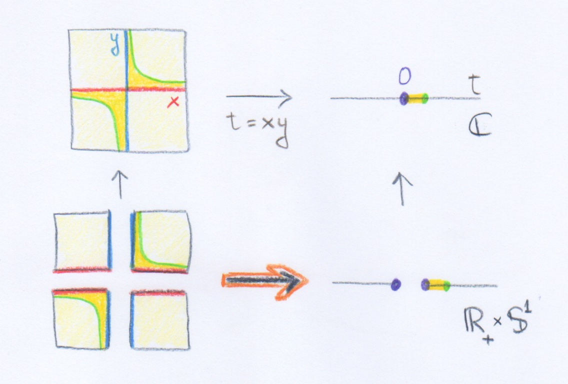
Theorem

The Milnor fiber conjecture is true.

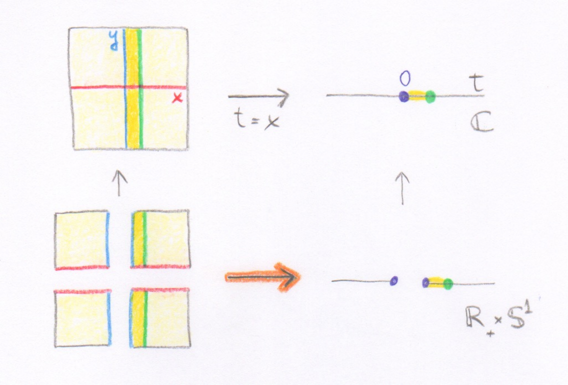
I will explain now the principle of the proof. Namely, I will describe how it combines constructions from **logarithmic geometry** and **tropical geometry**.


y
x
t = xy
0
t


F
=
glue
real oriented
blow up
=
passage to polar
coordinates
x
y



One gets a canonical representative of the Milnor fibration over the circle !



One needs to cut the source along a divisor which is **strictly larger than the special fiber** !

The standard **passage to polar coordinates** is :

$$\left[\begin{array}{ccc} \mathbb{R}_+ \times \mathbb{S}^1 & \to & \mathbb{C} \\ (r, u) & \to & r \cdot u \end{array}\right.$$

More generally, if $(P, +)$ is a **toric monoid** and $\mathcal{X}^{\mathbf{P}}$ is the corresponding **affine toric variety**, one has an associated **real oriented blow up** :

$$\begin{array}{ccc} \mathrm{Hom}(\mathbf{P}, \mathbb{R}_+) \times \mathrm{Hom}(\mathbf{P}, \mathbb{S}^1) & \longrightarrow & \mathcal{X}^{\mathbf{P}} \\ \| & & \| \\ \mathrm{Hom}(P, \mathbb{R}_+ \times \mathbb{S}^1) & \longrightarrow & \mathrm{Hom}(P, \mathbb{C}) \end{array}$$

One may glue such maps in order to get a real oriented blow up of any toric variety X along $\partial \mathbf{X} := X \setminus$ dense torus.

Real oriented blow ups can be performed still more generally, in the framework of Fontaine and Illusie's **logarithmic geometry**. Its foundational paper was published by Kato in 1988.

Definition

Let $(X, \mathcal{O}_X)$ be a ringed space. A **log structure** on X is a sheaf of monoids $\mathcal{M}_X$ on X together with a morphism

$$\mathcal{M}_{\mathbf{X}} \xrightarrow{\alpha_{\mathbf{X}}} (\mathcal{O}_{\mathbf{X}}, \cdot)$$

of sheaves of monoids such that $\alpha_X^{-1}(\mathcal{O}_X^*) \xrightarrow[\alpha_X]{\sim} \mathcal{O}_X^*$.

A **log space** is a ringed space endowed with a log structure.

Let X be a complex variety and $D \hookrightarrow X$ be a hypersurface (a reduced Weil divisor). Then :

$$\mathcal{O}^*_{\mathbf{X}}(-\mathbf{D}) := \{f \in \mathcal{O}_X,\ f|_{X \setminus D} \in \mathcal{O}^*_X\}$$

is a log structure, the map $\mathcal{O}^*_X(-D) \xrightarrow{\alpha_X} (\mathcal{O}_X, \cdot)$ being the inclusion. It is called the **divisorial log structure** determined by D on X.

Let A, B be ringed spaces, enriched into log spaces $\mathbf{A}^\dagger, \mathbf{B}^\dagger$.

Definition

A **log morphism** $A^\dagger \xrightarrow{\phi^\dagger} B^\dagger$ is a morphism $A \xrightarrow{\phi} B$ of ringed spaces together with a morphism of sheaves of monoids $\phi^{-1}\mathcal{M}_B \to \mathcal{M}_A$.

For instance, assume that $D_1 \hookrightarrow X_1$, $D_2 \hookrightarrow X_2$ are hypersurfaces in complex varieties and that $X_1 \xrightarrow{\phi} X_2$ is a complex morphism. Then the usual pull back of holomorphic functions induces a log morphism

$$(X_1, \mathcal{O}^*_{X_1}(-D_1)) \xrightarrow{\phi^\dagger} (X_2, \mathcal{O}^*_{X_2}(-D_2))$$

if and only if :

$$\phi^{-\mathbf{1}}(\mathbf{D_2}) \subseteq \mathbf{D_1}.$$

It is the **log enhancement of ϕ relative to D_1 and D_2.**

We will be interested in the special case in which the target pair is $(\mathbb{D}, 0)$. Namely, we will work with analytic morphisms

$$\tilde{f} : \tilde{Y} \to \mathbb{D}$$

and with divisors $D \subset \tilde{Y}$ such that $\mathbf{\tilde{f}^{-1}(0) \subseteq D}$. The corresponding log enhancement of $\tilde{f}$ is :

$$(\tilde{Y}, \mathcal{O}^*_{\tilde{Y}}(-D)) \xrightarrow{\mathbf{\tilde{f}^{\dagger}}} (\mathbb{D}, \mathcal{O}^*_{\mathbb{D}}(-0)).$$

One may **pull back** log structures, getting special kinds of log morphisms :

Definition

A morphism $A \xrightarrow{\phi} B$ of log spaces is called **strict** if it induces an isomorphism $\underline{\phi}^* \mathcal{M}_B \xrightarrow{\sim} \mathcal{M}_A$.

We will be mainly interested in **restrictions of divisorial log structures to the inducing divisors** :

$$(D, \mathcal{O}^*_{\mathbf{X|D}}(-\mathbf{D})) \to (X, \mathcal{O}^*_X(-D)).$$

Note that the structure morphism $\mathcal{O}^*_{X|D}(-D) \xrightarrow{\alpha_{X|D}} (\mathcal{O}_D, \cdot)$ is not injective any more, in contrast to $\mathcal{O}^*_X(-D) \xrightarrow{\alpha_X} (\mathcal{O}_X, \cdot)$.

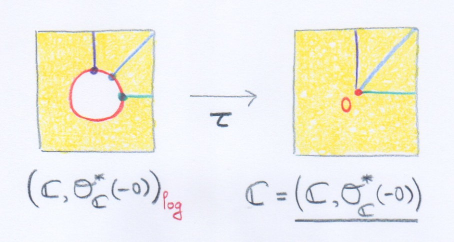
Kato and Nakayama (1999) generalized as follows the notion of **real oriented blow up** :

Definition

Let $X^\dagger$ be a log complex space. Its **Betti realization $\mathbf{X}^\dagger_{\log}$** is :

$$\{(x,u), x \in X, u \in \mathrm{Hom}((\mathcal{M}_{X,x},\cdot),\mathbb{S}^1), u(f) = f/|f|, \forall\, f \in \mathcal{O}^*_{X,x}\}.$$

The **Betti realization map $\mathbf{X}^\dagger_{\log} \xrightarrow{\tau_{X^\dagger}} \mathbf{X}$** is defined by $(x,u) \to x$.



Proposition

Betti realization is functorial. Moreover, for any log morphism $A \xrightarrow{\phi^\dagger} B$, the following diagram in the topological category is commutative :

$$\begin{array}{ccc} A^\dagger_{log} & \xrightarrow{\phi^\dagger_{log}} & B^\dagger_{log} \\ \downarrow \tau_{A^\dagger} & & \downarrow \tau_{B^\dagger} \\ A & \xrightarrow[\phi]{} & B \end{array} .$$

If $\phi^\dagger$ is **strict***, then this diagram is* **cartesian***.*

- Start from a **suitable smoothing** $Y \xrightarrow{f} \mathbb{D}$ of the singularity (X, o).
- Consider a **modification** $\tilde{Y} \xrightarrow{\pi} Y$ such that the composition $\tilde{\mathbf{f}} := f \circ \pi : \tilde{Y} \to \mathbb{D}$ is "nice" (semistable would be fine).
- Transform the commutative diagram of inclusion of the special fiber of $\tilde{f}$ in its total family into a diagram of **log morphisms** :

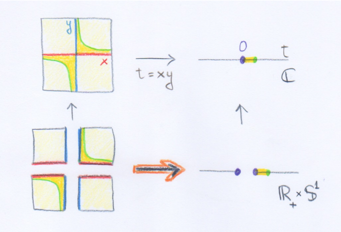
$$\begin{array}{ccc} \left(\mathbf{Z}(\tilde{\mathbf{f}}), \mathcal{O}^*_{\tilde{Y}|Z(\tilde{f})}(-Z(\tilde{f}))\right) & \longrightarrow & \left(\tilde{\mathbf{Y}}, \mathcal{O}^*_{\tilde{Y}}(-Z(\tilde{f}))\right) \\ \downarrow & & \downarrow \tilde{\mathbf{f}}^{\dagger} \\ \left(\mathbf{0}, \mathcal{O}^*_{\mathbb{D}|0}(-0)\right) & \longrightarrow & (\mathbb{D}, \mathcal{O}^*_{\mathbb{D}}(-0)) \end{array} .$$

- Consider the **Betti realization** of the previous diagram.

One gets a **cartesian** diagram :

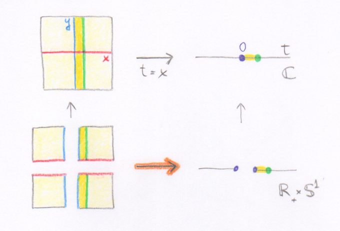
$$\begin{array}{ccc}
\left(\mathbf{Z}(\tilde{\mathbf{f}}), \mathcal{O}^*_{\tilde{Y}|Z(\tilde{f})}(-Z(\tilde{f}))\right)_{log} & \longrightarrow & \left(\tilde{\mathbf{Y}}, \mathcal{O}^*_{\tilde{Y}}(-Z(\tilde{f}))\right)_{log} \\
\downarrow & & \downarrow \tilde{\mathbf{f}}^{\dagger}_{\mathbf{log}} \\
\left(\mathbf{0}, \mathcal{O}^*_{\mathbb{D}|0}(-0)\right)_{log} & \longrightarrow & (\mathbb{D}, \mathcal{O}^*_{\mathbb{D}}(-0))_{log}
\end{array} .$$

In 2010, Nakayama and Ogus proved that under "nice" hypotheses (**exactness, relative log smoothness**), the right vertical map $\tilde{\mathbf{f}}^{\dagger}_{\mathbf{log}}$ is a **topologically locally trivial fibration**, as in the example :



The theorem of Nakayama and Ogus applies also to situations where the source $\tilde{Y}$ is endowed with a divisorial log structure generated by a divisor which is **strictly larger** than the special fiber $Z(\tilde{f})$:

$$\begin{array}{ccc} \left(\mathbf{Z}(\tilde{\mathbf{f}}), \mathcal{O}^*_{\tilde{Y}|Z(\tilde{f})}(-Z(\tilde{f}\tilde{\mathbf{g}})\right)_{log} & \longrightarrow & \left(\tilde{\mathbf{Y}}, \mathcal{O}^*_{\tilde{Y}}(-Z(\tilde{f}\tilde{\mathbf{g}}))\right)_{log} . \\ \downarrow & & \downarrow \tilde{\mathbf{f}}^{\dagger}_{\mathbf{log}} \\ \left(\mathbf{0}, \mathcal{O}^*_{\mathbb{D}|0}(-0)\right)_{log} & \longrightarrow & (\mathbb{D}, \mathcal{O}^*_{\mathbb{D}}(-0))_{log} \end{array}$$



Q : When is the log enhancement

$$\tilde{f}^{\dagger} : \left(\tilde{Y}, \mathcal{O}^{*}_{\tilde{Y}}(-Z(\tilde{f}\tilde{g}))\right) \to (\mathbb{D}, \mathcal{O}^{*}_{\mathbb{D}}(-0))$$

"nice" (exact and relatively log smooth) in the sense of Kato and Nakayama ?

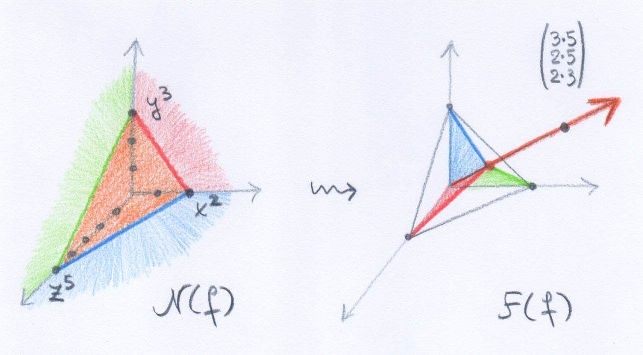
A : When one may find a divisor $\partial \tilde{Y}$ of $\tilde{Y}$ which contains $Z(\tilde{f}\tilde{g})$, such that $\tilde{f} : (\tilde{Y}, \partial \tilde{Y}) \to (\mathbb{D}, 0)$ is **toroidal** (locally monomial) and is locally "not too different from $Z(\tilde{f}\tilde{g})$".

Toroidal geometry is not too far from **tropical geometry** ...

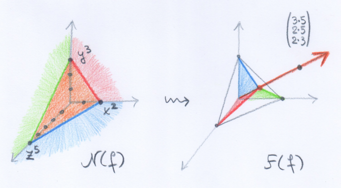
Let us pass to the **tropical aspects** of our proof.

Recall that :

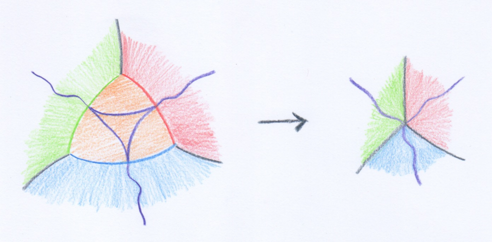
$$E_8 = (Z(x^2 + y^3 + z^5), 0).$$



The **Newton polyhedron** $\mathcal{N}(f)$ determines the **Newton fan** $\mathcal{F}(f)$.



Note that **the transversal section of the 2-dimensional skeleton of the Newton fan is isomorphic to the splice diagram of E_8**.



We proved that :

Theorem

The splice diagram of a splice type singularity is a transversal section of its local tropicalization.

The notion of **local tropicalization**, adapted to the study of singularities, was introduced by Stepanov and myself in 2013 by analogy with the very actively studied **global tropicalization** of subvarieties of algebraic tori.

Here is a geometric interpretation of local tropicalization :

Proposition

Let $(X, o) \hookrightarrow (\mathbb{C}^n, 0)$ be a holomorphic germ and $\mathcal{F}$ a fan contained in $(\mathbb{R}_+)^n$. Consider the toric birational morphism $\pi_{\mathcal{F}} : \mathcal{X}_{\mathcal{F}} \to \mathbb{C}^n$ defined by $\mathcal{F}$. Let $\pi : \tilde{X} \to X$ be its restriction to the strict transform of X. Then the following conditions are equivalent :

- **π is a modification ;**
- **the support of $\mathcal{F}$ contains the local tropicalization $\mathbf{Trop}_{(\mathbb{C}^n, 0)}(\mathbf{X})$.**

Proposition

Assume that the previous conditions are satisfied. Then the following conditions are equivalent :

- $\mathcal{F}$ **subdivides** $Trop_{(\mathbb{C}^n,0)}(X)$ **;**
- **the strict transform** $\tilde{X}$ **of** X **by** $\pi_{\mathcal{F}}$ **intersects all the orbits of** $\mathcal{X}_{\mathcal{F}}$ **in subspaces of constant codimension.**

If moreover all these subspaces are **smooth**, one says that $(X, 0) \hookrightarrow (\mathbb{C}^n, 0)$ is **Newton non-degenerate**. In our case is satisfied the stronger condition of **Newton non-degenerate complete intersection**.

The principle of our proof of the Milnor fiber conjecture

- We construct **a special smoothing** $f : Y \to \mathbb{D}$ of the given splice type singularity $X \hookrightarrow \mathbb{C}^n$, by adding suitable powers of a new variable.
- We compute **a fan $\mathcal{F}$ subdividing the local tropicalization** of the total space $Y \hookrightarrow \mathbb{C}^{n+1}$ of the smoothing f.
- We prove that $Y \hookrightarrow \mathbb{C}^{n+1}$ is a **Newton non-degenerate complete intersection**.
- Consider the strict transform $\tilde{Y}$ of Y by the toric birational morphism $\mathcal{X}_{\mathcal{F}} \to \mathbb{C}^{n+1}$. Then $\pi : \tilde{Y} \to Y$ is a modification and $\tilde{f} := f \circ \pi : \tilde{Y} \to \mathbb{D}$ **satisfies the "nice" hypotheses of Nakayama and Ogus**.
- We perform **similar analyses for the a-side and b-side**.
- We relate the **log special fibers** of $\tilde{f}$, $\tilde{f}_a$, $\tilde{f}_b$.

$$\begin{array}{ccc} \left(Z(\tilde{f}), \mathcal{O}^*_{\tilde{Y}|Z(\tilde{f})}(-Z(\tilde{f}))\right)_{log} & \xrightarrow{\tau} & Z(\tilde{f}) \\ \downarrow & & \downarrow \\ \left(0, \mathcal{O}_{\mathbb{D}|0}(-0)\right)_{log} & \xrightarrow{\tau} & 0 \end{array}$$

Using our deformed splice type system, we see that :

$$Z(\tilde{f}) = \tilde{X} + \partial_{\mathbf{a}}\tilde{\mathbf{Y}} + \partial_{\mathbf{ab}}\tilde{\mathbf{Y}} + \partial_{\mathbf{b}}\tilde{\mathbf{Y}},$$

where **the dual complex of $\partial_{\mathbf{a}}\tilde{\mathbf{Y}} + \partial_{ab}\tilde{Y} + \partial_{\mathbf{b}}\tilde{\mathbf{Y}}$ is isomorphic to the given splice diagram, rooted at an interior point of $[a, b]$.**

We study analogous diagrams for ***a*-side** and ***b*-side** systems. We relate them to the previous diagram using adequate toric morphisms. **On the sources of the *a*-side and *b*-side we consider divisors which are strictly greater than the special fibers.**

Thank you very much for your attention !