Automorphism groups of affine spherical varieties

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Group of symmetries of a finite set S determines the set S (in the category of finite sets).

Erlangen Program

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In 1872 Felix Klein in his Erlangen Program proposed that group theory, a branch of mathematics that uses algebraic methods to abstract the idea of symmetry, was the most useful way of organizing geometrical knowledge.

Study geometrical objects via their transformation (diffeomorphisms, isometries, automorphism, etc.) groups.

Erlangen Program

This approach was very fruitful in many areas of mathematics, for example, to study manifolds via their diffeomorphism groups

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Theorem(R. P. Filipkewicz, 1982)

Let M and N be smooth (i.e. C^∞) manifolds without boundary and let $\mathrm{Diff}(M)$ and $\mathrm{Diff}(N)$ denote the groups of C^∞ diffeomorphisms of M and N respectively. If $\phi : \mathrm{Diff}(M) \xrightarrow{\sim} \mathrm{Diff}(N)$ is a group isomorphism then there is a C^∞ diffeomorphism $w : M \xrightarrow{\sim} N$.

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In algebraic geometry there are at least two natural possibilities for the group of symmetries:

Regular automorphism group $\mathrm{Aut}(X)$

Birational automorphism group $\mathrm{Bir}(X)$

$\mathrm{Aut}(X) \subset \mathrm{Bir}(X)$

Theorem(Cantat, Xie)

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$\mathbb{P}^n$ is uniquely determined (up to birational equivalence) among n-dimensional varieties by its group of birational transformations.

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Most toric projective varieties have automorphism group isomorphic to algebraic torus. Hence, projective toric variety is not determined by its automorphism group.

Let X be an affine irreducible variety with a "rich" automorphism group and Y be any irreducible variety such that there is an isomorphism $\varphi : \mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(Y)$. How similar X and Y are?

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Definition

Let G be a reductive, $B \subset G$ a Borel subgroup. An affine normal G-variety X is called spherical if B acts on X with an open orbit.

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There exists an affine surface and an isomorphism $\psi : \mathrm{Aut}(S) \xrightarrow{\sim} \mathrm{Aut}(S)$ such that for any algebraic subgroup $H \subset \mathrm{Aut}(S)$, $\psi(H)$ is not an algebraic subgroup of $\mathrm{Aut}(S)$.

Isomorphism that preserves algebraic subgroups

An isomorphism $\varphi : \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ that sends an algebraic subgroup $G \subset \mathrm{Aut}(X)$ to an algebraic subgroup $\varphi(G) \subset \mathrm{Aut}(Y)$ and restriction of φ to G is an isomorphism of algebraic groups

Characterization of spherical varieties

Weight Monoid

$\mathcal{O}(X)$ is a multiplicity free G-module, that is, the multiplicity of every irreducible module in $\mathcal{O}(X)$ is at most 1. By the weight monoid $\Lambda^+(X)$ of X we mean the set of all highest weights of the G-module $\mathcal{O}(X)$.

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Theorem(van Santen, R.)

Let X be a spherical affine variety different from algebraic torus and Y be an affine irreducible normal variety. If there is an isomorphism $\varphi\colon \mathrm{Aut}(Y) \simeq \mathrm{Aut}(X)$ that preserves algebraic subgroups, then Y is also spherical and $\Lambda^+(X) = \Lambda^+(Y)$.

Corollary

- if X is toric, then $Y \simeq X$.
- if X and Y are smooth, then $Y \simeq X$.
- in general, for a given X there are finitely many spherical varieties $Y_1, \ldots, Y_l$ such that $\mathrm{Aut}(Y_j) \simeq \mathrm{Aut}(X)$.

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Example (the case of algebraic torus)

Let T be an algebraic torus and let C be a smooth affine curve. If C has trivial automorphism group and no invertible global functions, then there is an isomorphism $\mathrm{Aut}(T) \to \mathrm{Aut}(C \times T)$ that preserves algebraic subgroups.

Non-spherical case

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Example(Danielewski surfaces)

Let $D_p = \{xy = p(z)\} \subset \mathbb{A}^3$ be a Danielewski surface, where p is a polynomial without multible roots. Note that D_p is smooth. Then for two generic polynomials p and q, there is an isomorphism

$$\varphi\colon \mathrm{Aut}(D_p) \to \mathrm{Aut}(D_q)$$

that maps isomoprhically algebraic subgroups of $\mathrm{Aut}(D_p)$ to algebraic subgorups of $\mathrm{Aut}(D_q)$.

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Remark

$$\mathrm{Aut}(D_p) \simeq \mathrm{Aut}(D_q)$$

as a so-called ind-group if and only if $D_p \simeq D_q$.

Definition

Let X be an affine spherical G-variety. A unipotent subgroup $H \subset \mathrm{Aut}(X)$ is called a generalized root subgroup (with respect to B) if H is commutative and every one-dimensional subgroup of H is normalized by B.

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Proposition

Let Y be an irreducible normal affine G-variety. The following statements are equivalent:

- Y is G-spherical;
- there exists a constant C such that $\dim H \leq C$ for each generalized root subgroup $H \subset \mathrm{Aut}(Y)$.

Idea of the proof

Let $\varphi : \mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(Y)$ be an isomorphism that preserves algebraic groups.

For every algebraic subgroup $K \subset \mathrm{Aut}(X)$, the isomorphism φ restricts to an isomorphism of algebraic groups K and $\varphi(K)$. In particular, a generalized root subgroup of weight λ is maped to generalized root subgroup with the same weight.

The set of weights of generalized root subgroups determines the weight monoid of spherical variety.

What if $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$ are isomorphic only as abstract groups?

Characterization of affine spherical surfaces

Theorem(Liendo, Urech, R.)

Let X and Y be affine surfaces and there is an isomorphism $\varphi : \mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(Y)$ of groups. Then the following is satisfied.
(1) If H is a connected non-unipotent algebraic subgroup of $\mathrm{Aut}(X)$, then $\varphi(H)$ is an algebraic subgroup of $\mathrm{Aut}(Y)$ isomorphic to H.
(2) If X is spherical G-variety, then Y is also spherical G-variety that is isomorphic to X.

Topology on Bir(S)

Let A be a variety and $f : A \times S \to A \times S$ be an A-birational map, i.e.,

- f is the identity in the first factor
- f induces an isomorphism between open subsets U and V of $A \times S$ such that the projections from U and from V to A are both surjective.

Each $a \in A$ defines an element in Bir(S) and hence we obtain a map $A \to$ Bir(S) that we call a *morphism*.

The Zariski topology on Bir(S) is the finest topology making all such morphisms continuous.

Algebraic elements in $\mathrm{Bir}(S)$

Definition

An algebraic subgroup of $\mathrm{Bir}(S)$ is the image of an algebraic group G by a morphism $G \to \mathrm{Bir}(S)$ that is also an injective homomorphism of groups. An element $g \in \mathrm{Bir}(S)$ is called algebraic if it is contained in an algebraic subgroup.

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Lemma

Let S be a surface and $f \in \text{Bir}(S)$.
Then the following two conditions are equivalent:

- There exists a $k > 0$ such that f^k is divisible,
- f is algebraic.

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Lemma

Let S be an affine surface and let $g \in \text{Aut}(S)$ be an automorphism. Then g is an algebraic element in $\text{Bir}(S)$ if and only if g is an algebraic element in $\text{Aut}(S)$.

Algebraic elements are preserved

Proposition

Let X and Y be affine surfaces, $\varphi : \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ a group homomorphism, and $g \in \mathrm{Aut}(X)$ an algebraic element. Then $\varphi(g)$ is an algebraic element in $\mathrm{Aut}(Y)$.

Spherical surfaces

Any spherical surface different from toric surface is isomorphic either to $\mathrm{SL}(2, \mathbb{C})/D$ or to $\mathrm{SL}(2, \mathbb{C})/N$.

Therefore, we have to deal, mainly, with toric surfaces.

Torus goes to 2-dimensional torus

Lemma

Let X and Y be normal affine surfaces with X toric and $\varphi : Aut(X) \xrightarrow{\sim} Aut(Y)$ a group isomorphism. Then $\varphi(T)$ is a maximal subtorus in $Aut(Y)$.

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Let X and Y be normal affine surfaces with X toric, $\varphi : Aut(X) \xrightarrow{\sim} Aut(Y)$ a group isomorphism, and $U \subset Aut(X)$ a root subgroup. Then $\varphi(U)$ is a root subgroup in $Aut(Y)$ with respect to $\varphi(T)$.

End of the proof

We know now that Y is a toric surface and we have a bijection on the root subgroups of $\text{Aut}(X)$ and $\text{Aut}(Y)$ with respect to T and $\varphi(T)$.

To finish the proof, it is enough to show that we can retrieve a toric surface X from the abstract group structure of its root subgroups and their relationship with the torus.

Recall that any affine toric surface X without torus factor is isomorphic to $X_{d,e}$, the quotient of $\mathbb{A}^2$ under the $\mu_d = \{\xi \in \mathbb{C}^* \mid \xi^d = 1\}$-action

$$g : (x, y) \mapsto (\xi^e x, \xi y)$$

where ξ is a d-th primitive root of unity $0 \leq e < d$, $(e, d) = 1$.

$X_{d,e}$ is isomorphic to $X_{d',e'}$ if and only if $d = d'$ and $e = e'$ or $d = d'$ and $e \cdot e' = 1$ mod d.

End of the proof

The center of $G_a \rtimes T$ is $\{0\} \times \ker \chi$, so we can recover $\ker \chi$.

There are two families $\mathcal{K}$ and $\mathcal{L}$ of commuting root subgroups in $\mathrm{Aut}(X)$. We define the following subsets of $\mathbb{Z}_{\geq 0}$:

$$K_U = \{| \ker \chi \cap \ker \chi' |, \forall U' \in \mathcal{L}\} \ \ \forall U \in \mathcal{K}$$

$$L_U = \{| \ker \chi \cap \ker \chi' |, \forall U' \in \mathcal{K}\} \ \ \forall U \in \mathcal{L}$$

After some finite part, they form arithmetic progressions.

The two shortest common differences in this arithmetic progressions are

$$d \text{ and } d + e \text{ or } d \text{ and } d + e' \text{ with } e \cdot e' = 1 \bmod d.$$

Hence, these sets uniquely determine X.

What about higher dimensional case?

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Theorem(Cantat, Xie, R.)

Let the base field be uncountable algberaically closed field (of any characteristic). Let X be a connected affine variety such that $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{A}^n)$, then $X \simeq \mathbb{A}^n$ as a variety.

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Remark

Note that if X is quasi-projective in the Theorem above, then the result does not hold: for example,

$$\mathrm{Aut}(\mathbb{A}^n) \simeq \mathrm{Aut}(\mathbb{A}^n \times Z),$$

where Z is projective with trivial automorphism group.
Moreover, the condition on X to be connected is crucial:

$$\mathrm{Aut}(\mathbb{A}^n) \simeq \mathrm{Aut}(\mathbb{A}^n \sqcup Z),$$

where Z is affine with trivial automorphism group.

We can weaken the requirement on uncountability of the base field (of characteristic zero) in the Theorem above, but in this case we need to require that $\dim X \leq n$.

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More precisely, we classify n-dimenisonal quasi-affine varieties X endowed with faithful action of a finite index subgroup $\Gamma < \mathrm{SL}_n(\mathbb{Z})$: X is either isomorphic to
(i) $\mathbb{A}^n/\mu_k$, where $\mu_k = \langle \xi \mid \xi^k = 1 \rangle$ acts on $\mathbb{A}^n$ by scalar multiplication:

$$\xi \cdot (x_1, \dots, x_n) = (\xi x_1, \dots, \xi x_n).$$

In this case Γ acts on $\mathbb{A}^n/\mu_k$ linearly.
(ii) G_m^n and the action of Γ on G_m^n is monomial.
(iii) $G_m^n/\langle \tau \rangle$, where $\tau : (x_1, \dots, x_n) \mapsto (x_1^{-1}, \dots, x_n^{-1})$ and Γ acts monomially.

As a consequence from this classification we prove the following result: If X is quasi-affine variety of dimension $\leq n$ and $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{A}^n)$ as an abstract group, then $X \simeq \mathbb{A}^n$ as a variety.

The last result is mainly based on p-adic analisys and birational geometry.

Ind-groups

Definition

By an ind-variety we mean a set V together with an ascending filtration $V_0 \subset V_1 \subset V_2 \subset \dots$ such that the following is satised:
(1) $V = \cup_{k \in \mathbb{N}} V_k$;
(2) each V_k has the structure of an algebraic variety.
(3) for all $k \in \mathbb{N}$ the inclusion $V_k \subset V_{k+1}$ is a closed in Zariski topology.

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Topology

An ind-variety $V = \cup_k V_k$ has a natural Zariski topology: $S \subset V$ is closed (resp. open) if $S_k = S \cap V_k \subset V_k$ is closed (resp. open) for every k.

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Morphism

A map $\Phi\colon V \to W$ between ind-varieties $V = \cup_k V_k$ and $W = \cup_l W_l$ is a morphism if for each k there is $l \in \mathbb{N}$ such that $\Phi(V_k) \subset W_l$ and the induced map $\Phi_k\colon V_k \to W_l$ is a morphism of algebraic varieties.

Isomorphisms of ind-varieties are defined in the usual way.

Ind-groups

Definition

The product of two ind-varieties is defined in the obvious way. An ind-variety G is called an ind-group if the underlying set G is a group such that the map $G \times G \to G$, dened by $(g, h) \mapsto gh^{-1}$, is a morphism of ind-varieties.

Theorem

Let X be an affine variety. Then $\mathrm{Aut}(X)$ has the structure of an ind-group acting "morphically" on X; this means that the action $\mathrm{Aut}(X) \times X \to X$ of $\mathrm{Aut}(X)$ on X induces a morphism of algebraic varieties $\mathrm{Aut}(X)_i \times X \to X$ for every $i \in \mathbb{N}$, where

$$\mathrm{Aut}(X)_1 \subset \mathrm{Aut}(X)_2 \subset \dots$$

is a filtration of the ind-group $\mathrm{Aut}(X)$.

Example

Ind-structure on $\mathrm{Aut}(\mathbb{A}^n)$.

For example, if $X = \mathbb{A}^n$, the ind-group filtration $(\mathrm{Aut}(\mathbb{A}^n)_d)_{d\geq 1}$ of $\mathrm{Aut}(\mathbb{A}^n)$ is defined in the following way:

$$\mathrm{Aut}(\mathbb{A}^n)_d = \{f = (f_1, \ldots, f_n) \in \mathrm{Aut}(\mathbb{A}^n) \mid \deg f = \max_i \deg f_i, \deg f^{-1} \leq d\}$$

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Proposition

Let X be an affine variety and $V \subset \mathrm{Aut}(X)$ be an irreducible subvariety that consists of commuting elements and identity element belongs to V. Then the group $\langle V \rangle$ is an algebraic subgroup of $\mathrm{Aut}(X)$.

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Remark

It is crucial to assume that $\mathrm{id} \in V$. Indeed, one can pick a single automorphism f with $\{\deg f^n \mid n \in \mathbb{N}\}$ unbounded. Then $\{f^n \mid n \in \mathbb{Z}\}$ is not algebraic.
Note that such f exists, take for example $f\colon (x, y) \mapsto (y, x + y^2)$.
In this case $\deg f^n = 2^n$.

Proposition does not hold for the group of birational transformations

For $X = \mathbb{A}^2$ consider

$$V = \{(x, y) \mapsto (x, (1 + ax)y) \mid a \in \mathbb{C}\}.$$

Then $\mathrm{id} \in V$, but $\langle V \rangle$ is not an algebraic group since it is not of bounded degree.

Corollary

Connected commutative ind-subgroup of $\mathrm{Aut}(X)$ is the inductive limit of commutative linear algebraic groups.

Idea of the proof of Corollary

By definition, a connected commutative ind-subgroup $G \subset \mathrm{Aut}(X)$ is a union of closed irreducible algberaic subvarieties G_i. By Proposition above $\langle G_i \rangle = G_i$ is a connected algebraic group.

THANK YOU FOR YOUR ATTENTION