New directions in enumerative geometry

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IMPANGA
422

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- joint work with Yiyan Shou
- learned about branes from Lev Rozansky
- related works with

  Andrey Smirnov
  Alexander Varchenko
  Zijun Zhou
  Andrzej Weber

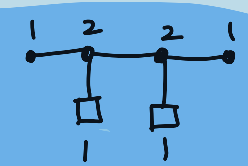
TOPIC-1

- characteristic classes of singularities
- coincidence

e.g.

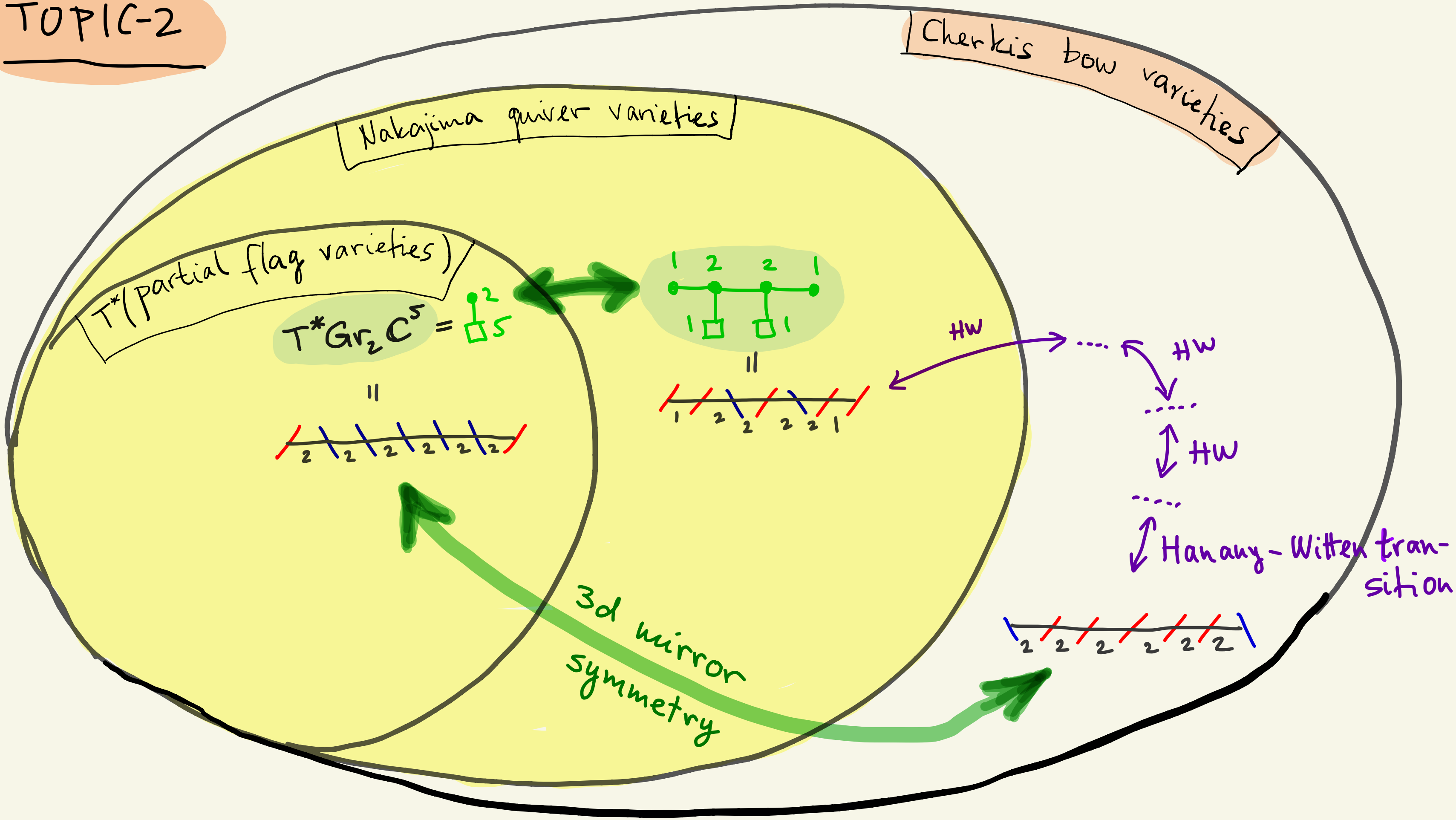
elliptic char. classes of singularities on $T^*Gr_2\mathbb{C}^5$

"coincide"

elliptic char. classes of singularities on $N\big($  $\big)$

"3D mirror symmetry for characteristic classes"

TOPIC-2



Cherkis bow varieties
Nakajima quiver varieties
T*(partial flag varieties)
T*Gr₂C⁵ =
HW
HW
HW
Hanany-Witten transition
3d mirror symmetry

Characteristic classes of singularities

$$T \circlearrowright \left(\begin{array}{ll} X & \text{smooth} \\ \cup_{i} & \\ \Sigma & \text{subvariety} \end{array} \right) \rightsquigarrow H_T^*(X) \ni \text{class of } \Sigma$$

a few different versions

today: "stable envelope class"

a deformation of

$[\Sigma]$ fundamental class

$[\Sigma] = i_* 1$

Examples

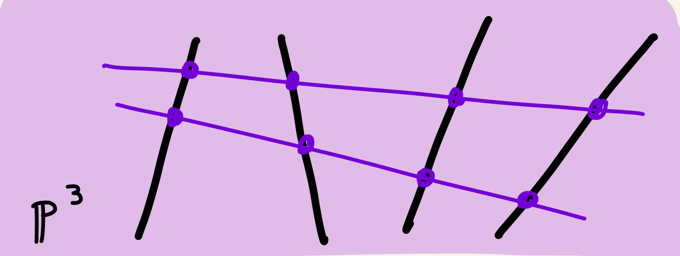
(1) $X := \mathbb{C}^{4\times 4} \circlearrowleft GL_4 \times T^4$

$\cup$

$\Sigma := \{\text{rank} \leq 2\}$

$$\underbrace{[\Sigma]}_{\cdots + 2t_1t_2t_3t_4 + \cdots} \in \underbrace{H^*_{GL_4 \times T^4}\left(\mathbb{C}^{4\times 4}\right)}_{\mathbb{Z}[\underbrace{c_1\ c_2\ c_3\ c_4}_{GL_4}\ \underbrace{t_1\ t_2\ t_3\ t_4}_{T^4}]}$$

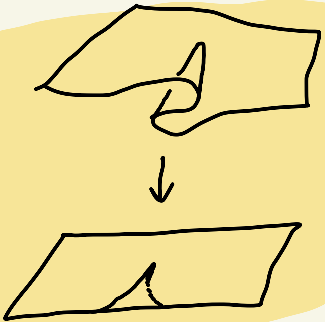
solution to Schubert problem:



(2)

$$X = \{ (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \text{ holo germs} \}$$

$$\cup$$

$$\Sigma = \{ \text{those} \sim (x,y) \mapsto (x^3 + xy, y)$$  $$\}$$

$$[\overline{\Sigma}] \in H^*_{GL_2 \times GL_2}(X)$$

$$\|$$

$$c_1^2 + c_2$$

<u>Cor</u> a generic map $\mathbb{RP}^2 \to \mathbb{R}^2$ has an odd number of cusps.

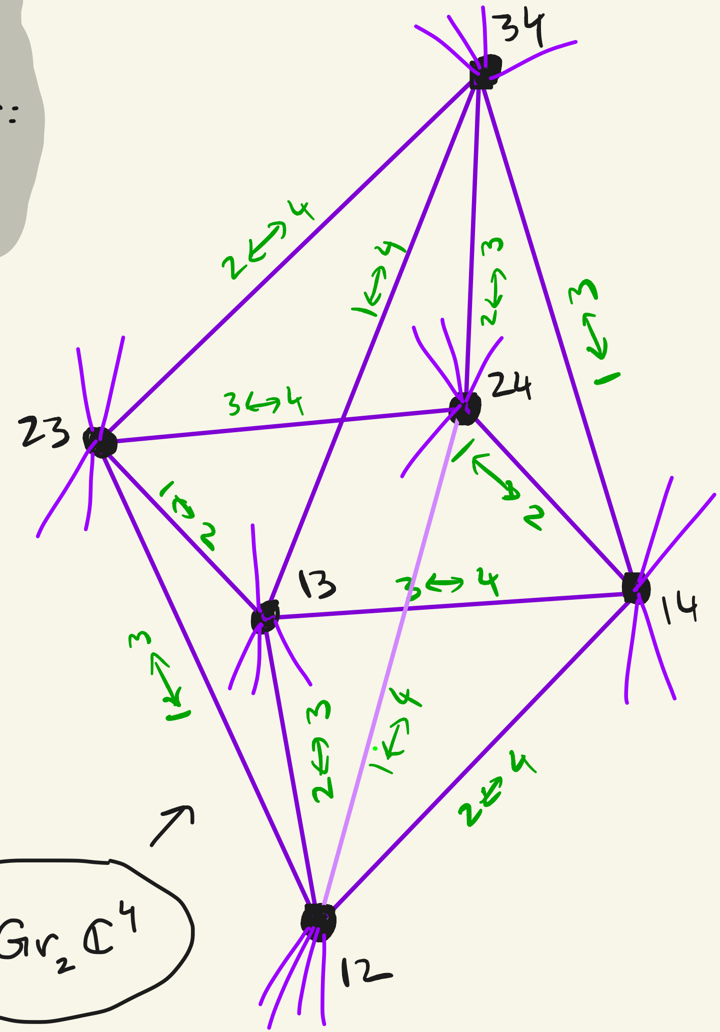
One key method to work with $H_T^*(X)$ and characteristic classes:

equivariant localization:

$$H^*_{T^n}(X) \xrightarrow{\text{Loc}} \bigoplus_{p \in X^T} \underbrace{H^*_T(p)}_{\mathbb{C}[u_1, u_2, \dots, u_n]}$$

$\mathrm{im}(\mathrm{Loc}) = ?$

"consistency" among components



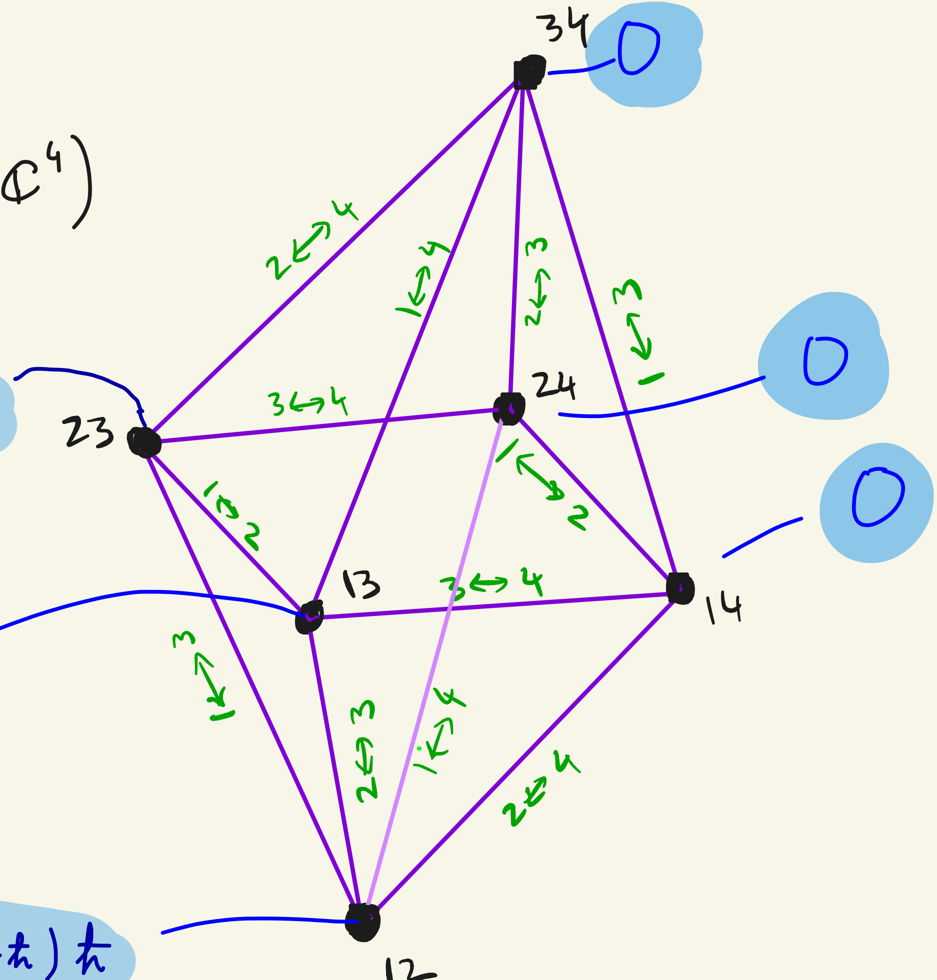
$T^*Gr_2\mathbb{C}^4$

For example, this 6-tuple is an element of $H_T^*(T^*Gr_2\mathbb{C}^4)$

$(u_4-u_3)(u_4-u_2)(u_2-u_1+\hbar)(u_3-u_1+\hbar)$

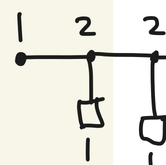
$(u_4-u_1)(u_4-u_3)(u_3-u_2+\hbar)\hbar$

$(u_4-u_1)(u_4-u_2)(u_2-u_3+\hbar)\hbar$

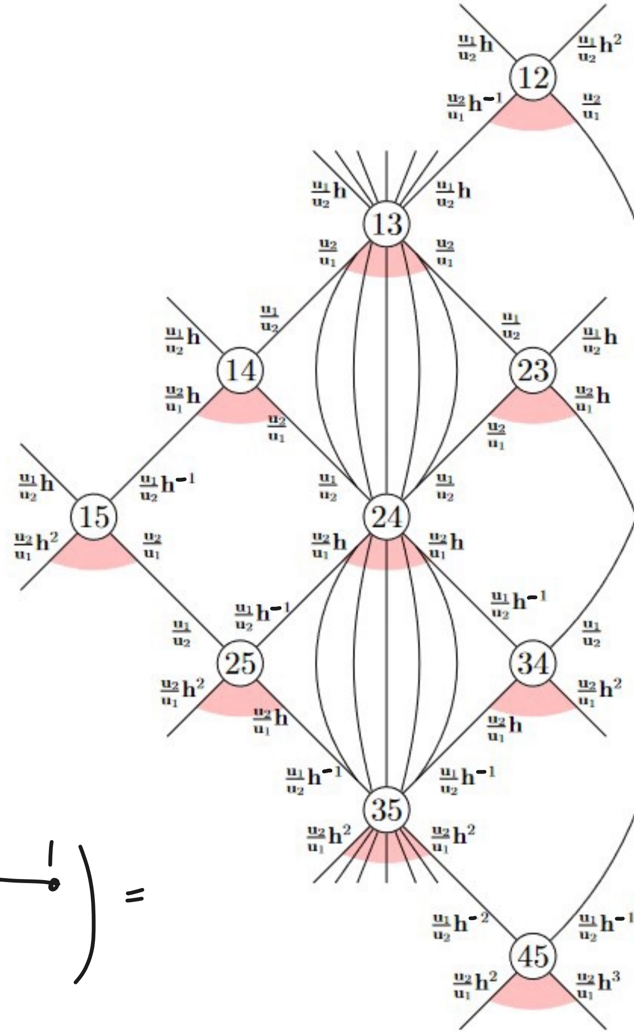


Warning

- $T^*\mathrm{Gr}_2\mathbb{C}^4$ was special ("GKM")
- In general the constraints among components are more restrictive

$\mathcal{N}\Big($ 

$\Big) =$



Fact: moment graph of X determines characteristic classes of $\Sigma \subset X$.

X $\longrightarrow$ moment graph of X $\longrightarrow$ characteristic classes of $\Sigma \subset X$

namely "stable envelope classes"

vertices : T fixed points

"edges" : invariant curves

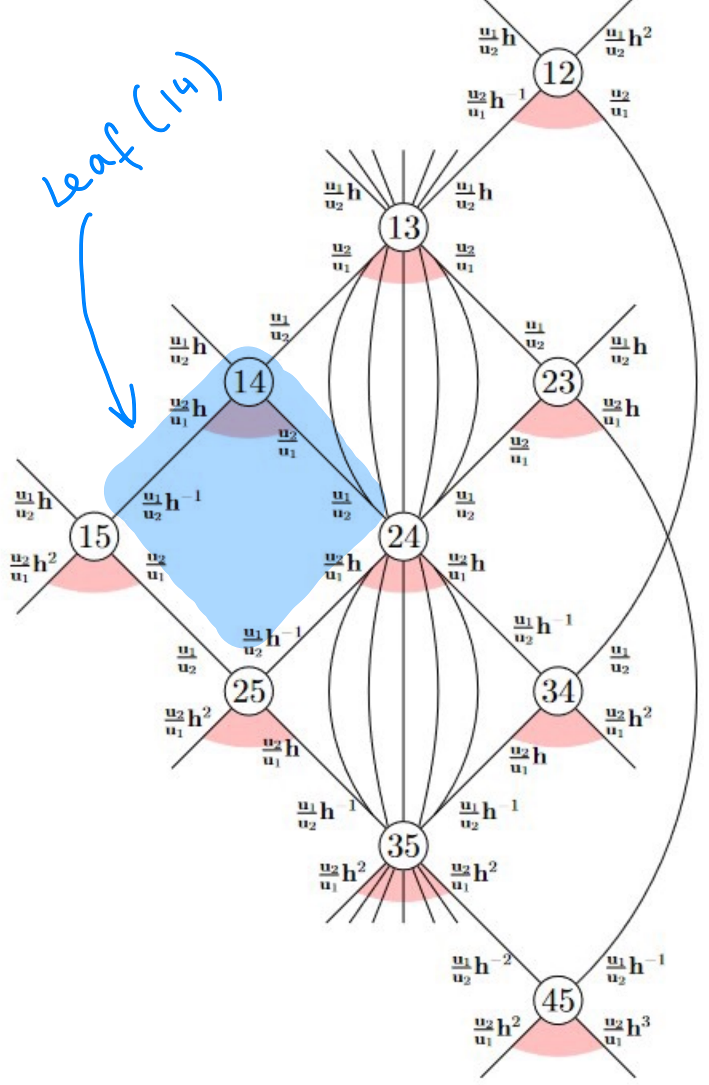
decorations on edges : T weights

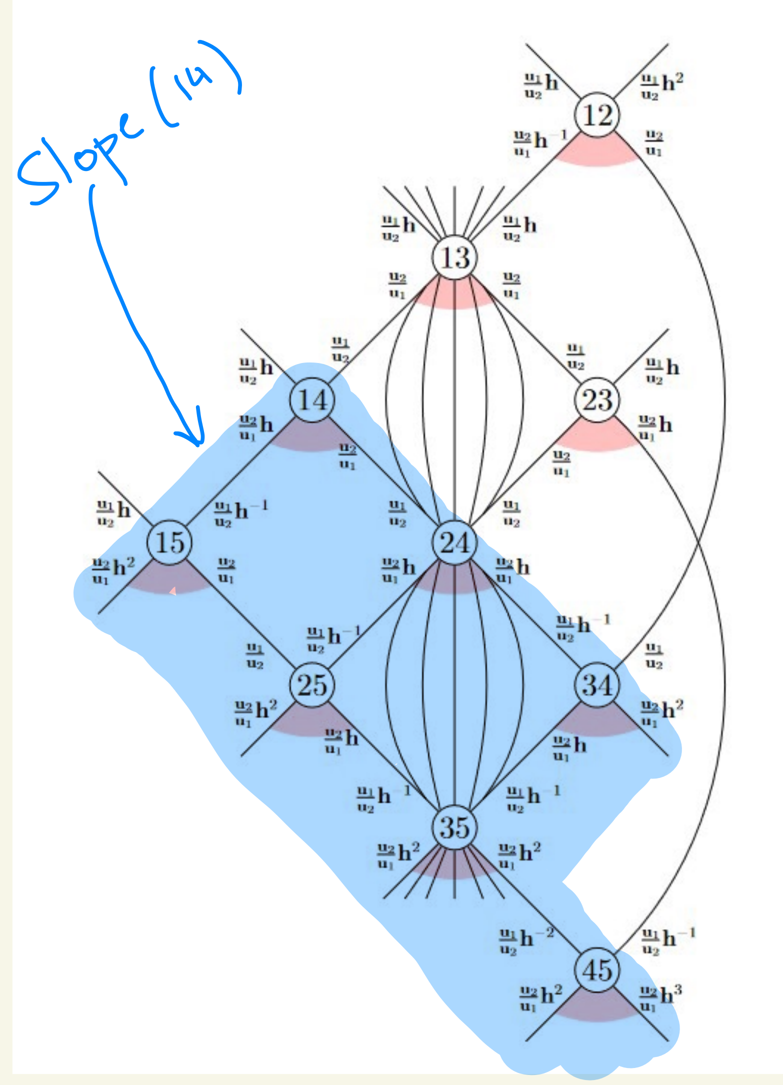
Stable Envelope classes:

$\mathrm{Stab}_p \in H_T^*(X)$

- fix $\mathbb{C}^* \xrightarrow{\sigma} T^n$
  $z \longmapsto (z, z^2, z^3, \ldots, z^n)$
- $p \in X^T$ $\quad \mathrm{Leaf}(p) = \{x \in X : \lim_{z \to 0} \sigma(z)x = p\}$
- $p' \leq p$ if $\overline{\mathrm{Leaf}(p)} \ni p'$
- $\mathrm{Slope}(p) := \bigcup_{p' \leq p} \mathrm{Leaf}(p')$

Białynicki-Birula cell


Leaf(14)


Slope(14)

Maulik–Okounkov

def $\operatorname{Stab}_p \in H^*_T\big(\underset{X}{\cancel{\mathcal{C}(2)}}\big)$ is the unique class

$T = A \times \mathbb{C}^*_\hbar$

- **support axiom:**

  supported on $\operatorname{Slope}(p)$

- **normalization axiom:**

  $$\operatorname{Stab}_p\big|_p = e\big(\nu(\operatorname{Slope}_p)\big)$$

- **boundary axiom:**

  $\operatorname{Stab}_p\big|_q$ divisible by $\hbar$ for $p \neq q$

Stab_{14}

@ 12, 13, 23 $= 0$

@ 14 $= (u_1 - u_2)(u_1 - u_2 + \hbar)$

@ 15 divisible by $(u_1 - u_2 + \hbar)$
divisible by $\hbar$

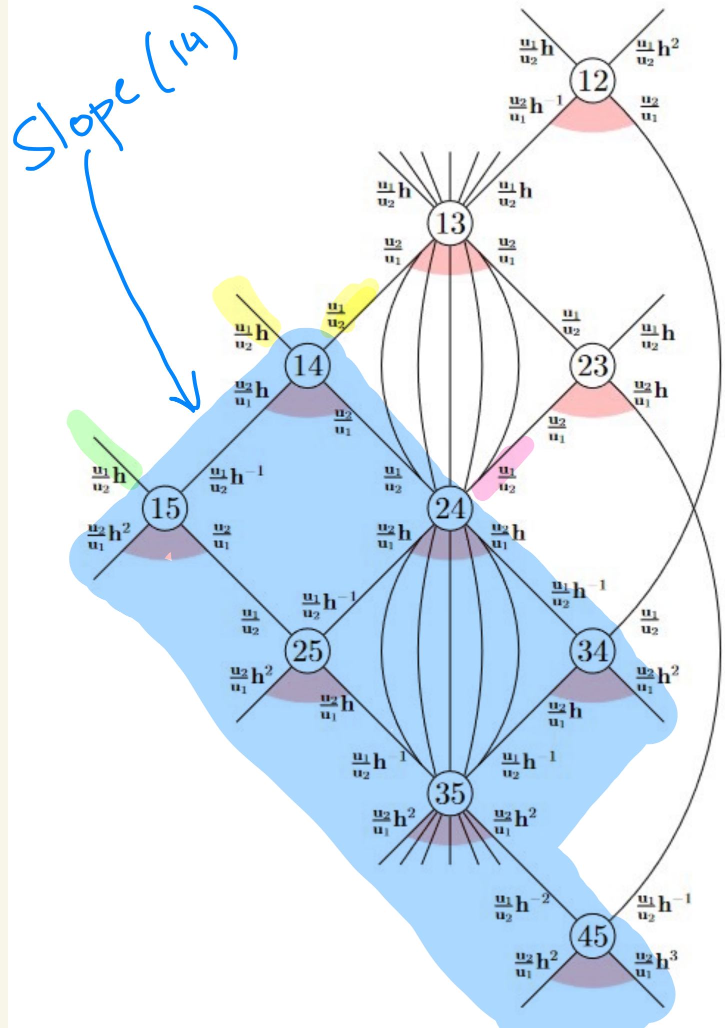
@ 24 divisible by $u_1 - u_2$
divisible by $\hbar$

@ 25 divisible by $\hbar$

@ 34 divisible by $(u_1 - u_2)$
divisible by $\hbar$

@ 35 divisible by $\hbar$

@ 45 divisible by $(u_1 - u_2 - \hbar)$
divisible by $\hbar$



So far:

$X \circlearrowleft T$ $\longrightarrow$ Moment Graph of X $\longrightarrow$ $H^*_T(X) \ni$ stable envelope classes

Fact

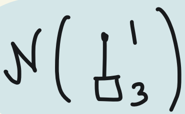
- $\exists$ elliptic version : elliptic stable envelopes

  polynomials in $u_1, \dots, u_n, \hbar$ $\longrightarrow$ ϑ- functions of $u_1 \dots u_n, \hbar$

- the elliptic version necessarily depends on new (Kähler / dynamical) variables $v_1 \dots v_m$

elliptic char. classes

	f_1	f_2	f_3
f_1	$\theta(\frac{\mathrm{u}_1}{\mathrm{u}_2})\theta(\frac{\mathrm{u}_1}{\mathrm{u}_3})\theta(\frac{\mathrm{v}_2}{\mathrm{v}_1}\mathrm{h}^4)$	0	0
f_2	$\theta(\mathrm{h})\theta(\frac{\mathrm{u}_1}{\mathrm{u}_3})\theta(\frac{\mathrm{u}_2\mathrm{v}_2}{\mathrm{u}_1\mathrm{v}_1}\mathrm{h}^3)$	$\theta(\frac{\mathrm{u}_1}{\mathrm{u}_2}\mathrm{h})\theta(\frac{\mathrm{u}_2}{\mathrm{u}_3})\theta(\frac{\mathrm{v}_2}{\mathrm{v}_1}\mathrm{h}^3)$	0
f_3	$\theta(\mathrm{h})\theta(\frac{\mathrm{u}_2}{\mathrm{u}_1}\mathrm{h})\theta(\frac{\mathrm{u}_3\mathrm{v}_2}{\mathrm{u}_1\mathrm{v}_1}\mathrm{h}^2)$	$\theta(\mathrm{h})\theta(\frac{\mathrm{u}_1}{\mathrm{u}_2}\mathrm{h})\theta(\frac{\mathrm{u}_3\mathrm{v}_2}{\mathrm{u}_2\mathrm{v}_1}\mathrm{h}^2)$	$\theta(\frac{\mathrm{u}_2}{\mathrm{u}_3}\mathrm{h})\theta(\frac{\mathrm{u}_1}{\mathrm{u}_3}\mathrm{h})\theta(\frac{\mathrm{v}_2}{\mathrm{v}_1}\mathrm{h}^2)$



$= T^*\mathbb{P}^2$

dim 4

coincidence

$(\,)^T$

$u_i \longleftrightarrow v_i'$

$v_i \longleftrightarrow u_i'$

$\hbar \longleftrightarrow \hbar^{-1}$

dim 2

$= \widetilde{\mathbb{C}^2/\mathbb{Z}_3}$

elliptic char. classes

	f_1'	f_2'	f_3'
f_1'	$\theta(\frac{\mathrm{u}_1'}{\mathrm{u}_2'}\mathrm{h}^4)\theta(\frac{\mathrm{v}_2'}{\mathrm{v}_1'})\theta(\frac{\mathrm{v}_3'}{\mathrm{v}_1'})$	$\theta(\mathrm{h})\theta(\frac{\mathrm{v}_3'}{\mathrm{v}_1'})\theta(\frac{\mathrm{v}_2'\mathrm{u}_2'}{\mathrm{v}_1'\mathrm{u}_1'}\mathrm{h}^{-3})$	$\theta(\mathrm{h})\theta(\frac{\mathrm{v}_2'}{\mathrm{v}_1'}\mathrm{h}^{-1})\theta(\frac{\mathrm{v}_3'\mathrm{u}_2'}{\mathrm{v}_1'\mathrm{u}_1'}\mathrm{h}^{-2})$
f_2'	0	$\theta(\frac{\mathrm{u}_1'}{\mathrm{u}_2'}\mathrm{h}^3)\theta(\frac{\mathrm{v}_2'}{\mathrm{v}_1'}\mathrm{h})\theta(\frac{\mathrm{v}_3'}{\mathrm{v}_2'})$	$\theta(\mathrm{h})\theta(\frac{\mathrm{v}_2'}{\mathrm{v}_1'}\mathrm{h})\theta(\frac{\mathrm{v}_3'\mathrm{u}_2'}{\mathrm{v}_2'\mathrm{u}_1'}\mathrm{h}^{-2})$
f_3'	0	0	$\theta(\frac{\mathrm{u}_1'}{\mathrm{u}_2'}\mathrm{h}^2)\theta(\frac{\mathrm{v}_3'}{\mathrm{v}_2'}\mathrm{h})\theta(\frac{\mathrm{v}_3'}{\mathrm{v}_1'}\mathrm{h})$

Jacobi theta function:

$$\vartheta(x) := \underbrace{\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)}_{\sim \sin(X)} \underbrace{\prod_{n \geq 1} (1 - q^n x)(1 - q^n x^{-1})}_{q\text{-decoration}}$$

$|q| < 1$ fixed

$\sim$ Euler class of line bundle in K-theory

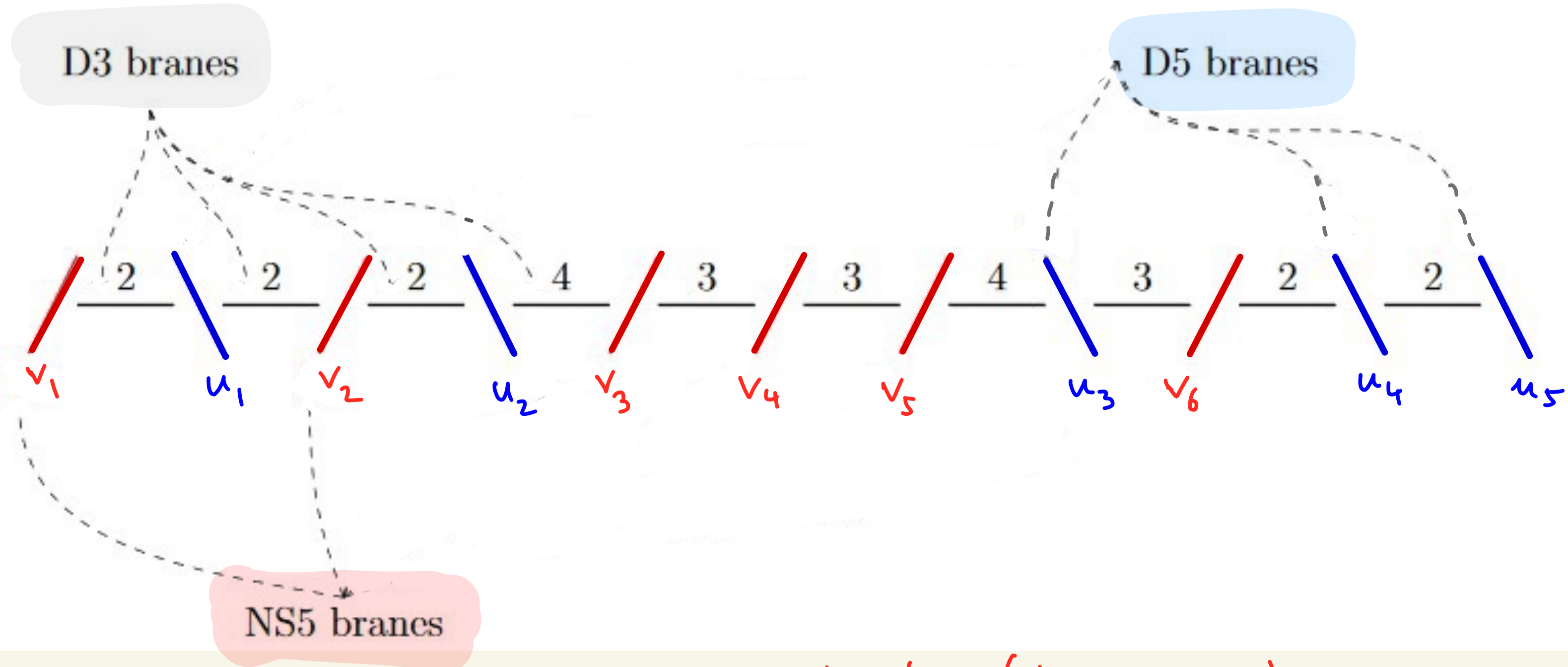
bow varieties

bow

arrows
(a.k.a. homomorphisms)

quiver

Brane diagrams



v_i : Kähler (dynamical) variables

u_i : equivariant variables

brane diagram $\mathcal{D}$ $\longrightarrow$ $\mathcal{C}(\mathcal{D})$

tautological bundles, one for each D3 brane

$\mathcal{C}(\mathcal{D})$ $\circlearrowleft$ $T^{\text{D5 branes}} \times \mathbb{C}^*_\hbar$

Cherkis bow variety

<u>Cherkis:</u>
moduli space of unitary instantons on multi-Taub-NUT spaces
(key: Nahm's equation)

<u>Nakajima-Takayama</u>
Hamiltonian reduction of representations of certain quivers with relations

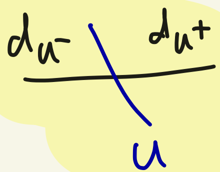
$\sim$

<u>Rozansky-R</u>
"symplectic intersection" of generalized Lagrange varieties

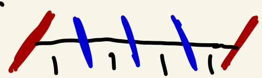
$\mathcal{C}(\mathcal{D})$

- smooth
- holomorphic symplectic
- "tautological" bundles — D3 branes
- torus action — D5 branes
- extra $\mathbb{C}^*_\hbar$-action

$$\dim\left(\mathcal{C}(D)\right) = \sum_{U\ \mathrm{D5}} \left[(d_{u_-}+1)\,d_{u_-} + (d_{u_+}+1)\,d_{u^+} \right] + \sum_{V\ \mathrm{NS5}} 2\, d_{v^+} d_{v^-} - 2 \sum_{X\ \mathrm{D3}} d_X^{2}$$



example

$\dim\Big(\mathcal{C}\big($  $\big)\Big)$

$T^*\mathbb{P}^2$

$$= 2\cdot 1 + 2\cdot 1 + 2\cdot 1 + 2\cdot 1 + 2\cdot 1 + 2\cdot 1 + 2\cdot 0\cdot 1 + 2\cdot 1\cdot 0 - 2\left(1^2+1^2+1^2+1^2\right)$$

$$= 4$$

How are $\mathcal{N}$(quiver) special cases?

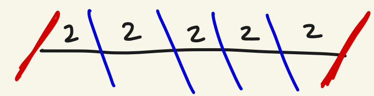
Examples

$T^*\mathbb{P}^1 = \mathcal{N}\Big($ 

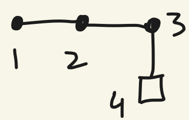
$\Big) = \mathcal{C}\Big($ 

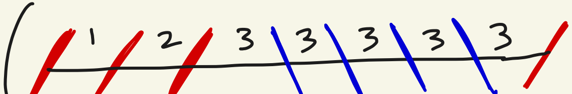
$\Big)$

$T^*Gr_2\mathbb{C}^4 = \mathcal{N}\Big($ 

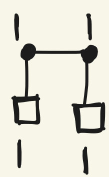
$\Big) = \mathcal{C}\Big($ 

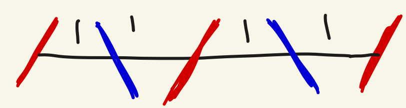
$\Big)$

$T^*\mathcal{F}_{1,2,3,4} = \mathcal{N}\Big($ 

$\Big) = \mathcal{C}\Big($ 

$\Big)$

$\mathcal{N}\Big($ 

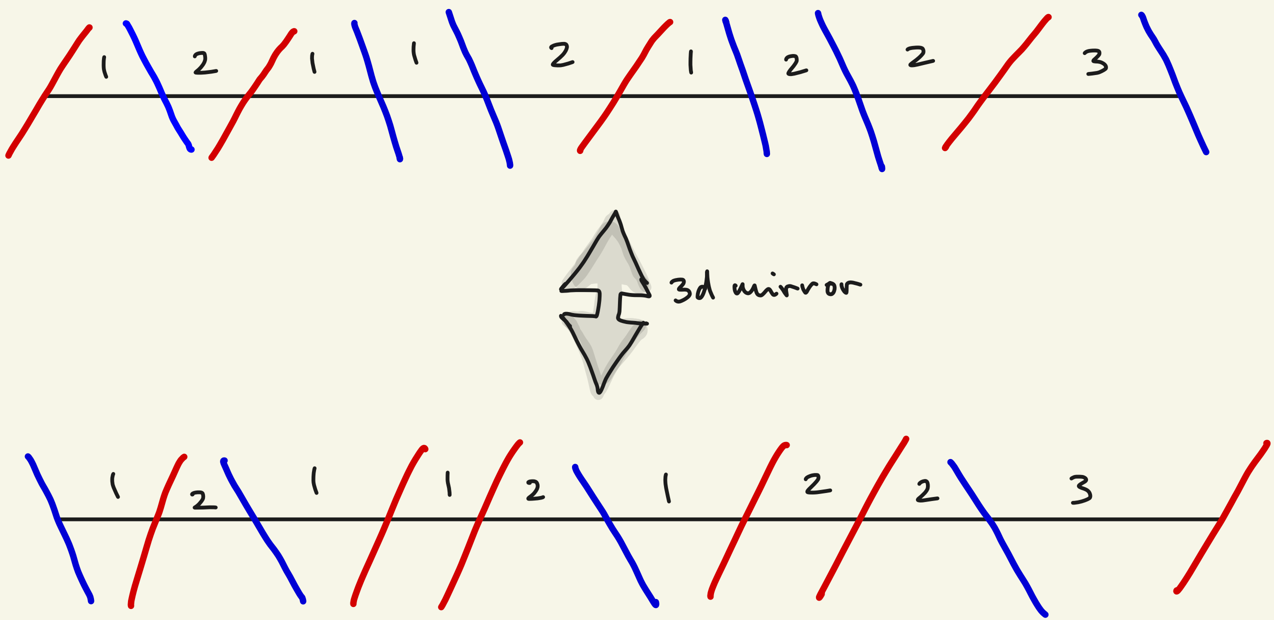
$\Big) = \mathcal{C}\Big($ 

$\Big)$

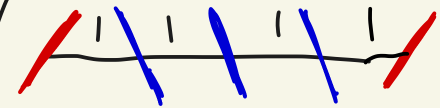
Observe 

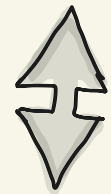
"cobalanced brane diagram"

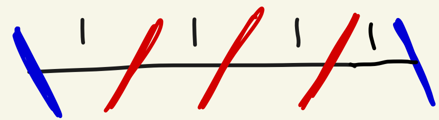
3D mirror symmetry for bow varieties:



Ex

$T^*\mathbb{P}^2 = \mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$ dim 4

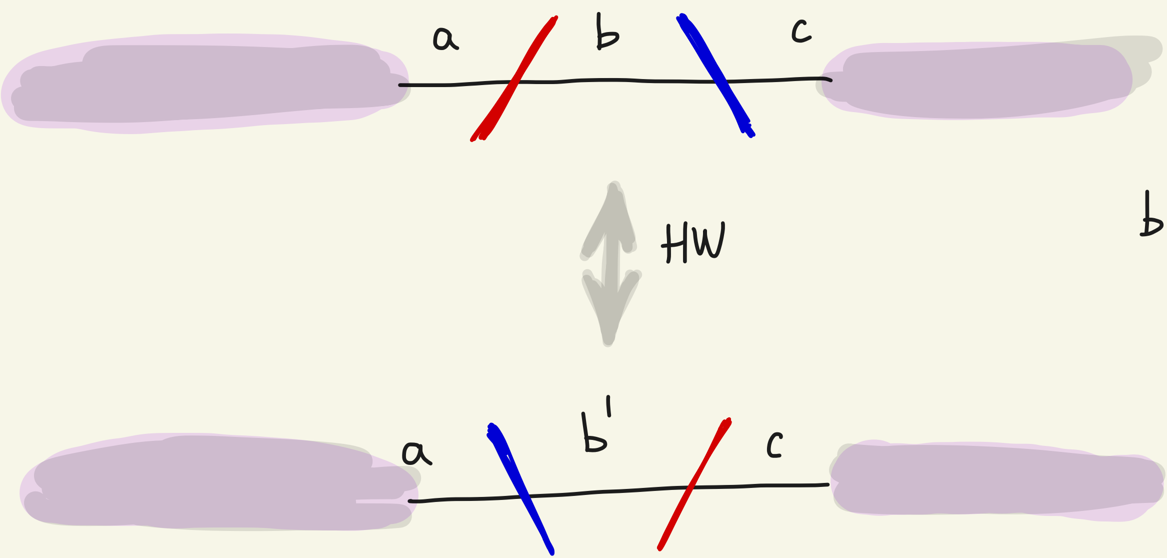
 3d mirror

$\mathcal{C}\Big($  $\Big)$ dim 2

not cobalanced, ie not $\mathcal{N}(\dots)$

... but ... <to be continued>

Hanany-Witten transition on brane diagrams.

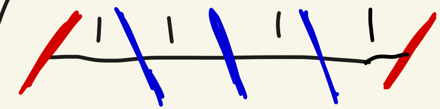


$b + b' = a + c + 1$

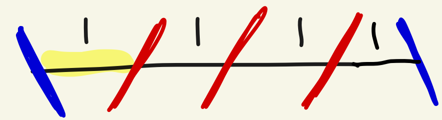
(why? later: "brane charge")

<u>Thm</u> $\mathcal{C}(\mathcal{D}) \cong \mathcal{C}(\,HW(\mathcal{D})\,)$

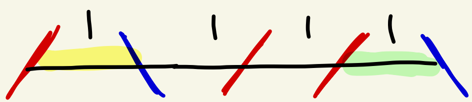
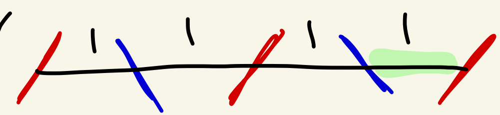
Ex

$T^*\mathbb{P}^2 = \mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

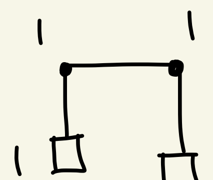
$\Big\Updownarrow$ 3d mirror

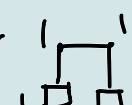
$\mathcal{C}\Big($  $\Big)$

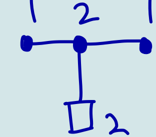
HW

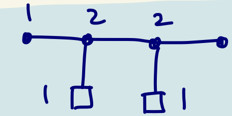
$\mathcal{C}\Big($  $\Big) \overset{\text{HW}}{=} \mathcal{C}\Big($  $\Big)$

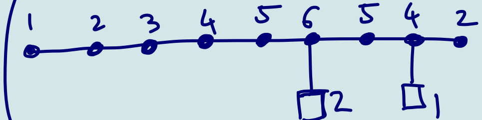
$\parallel$

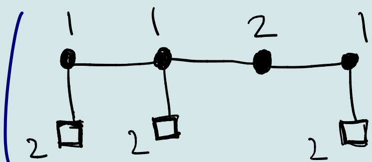
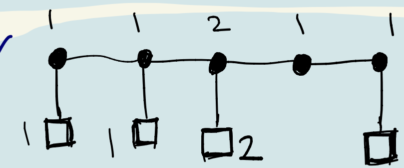
$\mathcal{N}\Big($  $\Big)$

$\Rightarrow$ $\boxed{T^*\mathbb{P}^2 \xleftrightarrow{\text{3d mirror}} \mathcal{N}\Big(\text{}} $  $\boxed{\Big)}$

8 $T^*\mathrm{Gr}_2\mathbb{C}^4 \longleftrightarrow \mathcal{N}\Big($  $\Big)$ 4 [RSVZ]

12 $T^*\mathrm{Gr}_2\mathbb{C}^5 \longleftrightarrow \mathcal{N}\Big($  $\Big)$ 4

64 $T^*\mathcal{F}_{2,6,10} \longleftrightarrow \mathcal{N}\Big($  $\Big)$ 16 [RS]

8 $\mathcal{N}\Big($  $\Big) \longleftrightarrow \mathcal{N}\Big($  $\Big)$ 10

$T^*G/B \longleftrightarrow T^*G^L/B^L$ [R-Weber 2020]

32 $T^*\mathcal{F}_{2,5,7} \longleftrightarrow \mathcal{N}\Big($  $\Big)$

$\boxed{\text{dim}}$

Conjecture (work in progress)

[known in special cases: R-Smirnov-Varchenko-Zhou (2x)
Smirnov-Zhou
R-Weber]

$$\mathcal{D} \xleftrightarrow[\text{3D mirror}]{} \mathcal{D}'$$

$$\Downarrow$$

elliptic characteristic classes on $\mathcal{C}(\mathcal{D})$ $\xleftrightarrow[\text{"same"}]{}$ elliptic characteristic classes on $\mathcal{C}(\mathcal{D}')$

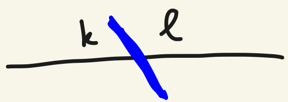
- transposition
- equivariant $\leftrightarrow$ Kähler
- $\hbar \leftrightarrow \hbar^{-1}$

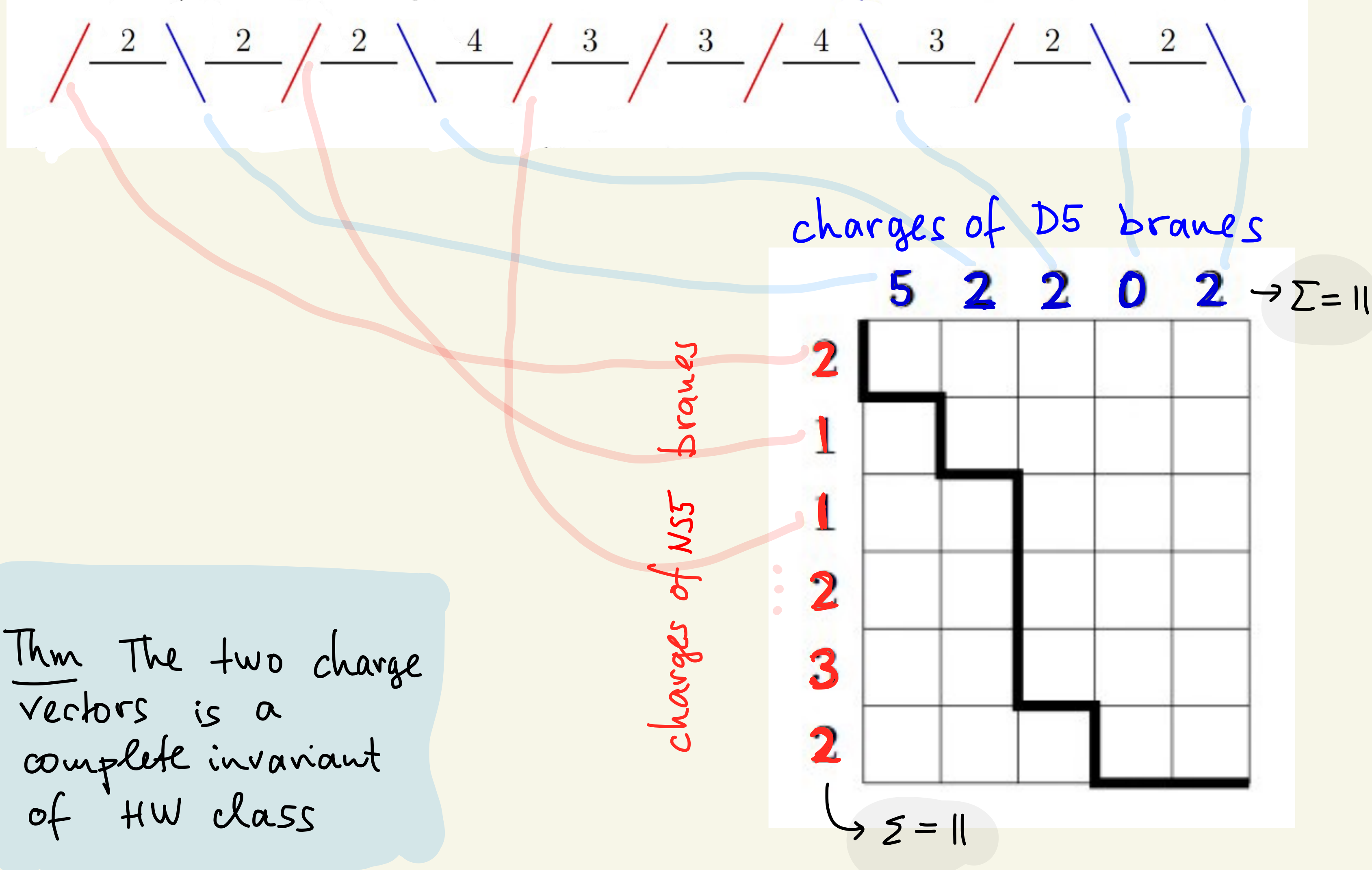
Relation to geometric representation theory

def brane charge

$$\mathrm{charge}\left(\text{NS5 brane}\right) := \ell - k + \#\{\text{D5-branes left of it}\}$$

$$\mathrm{charge}\left(\text{D5 brane}\right) := k - \ell + \#\{\text{NS5-branes right of it}\}$$

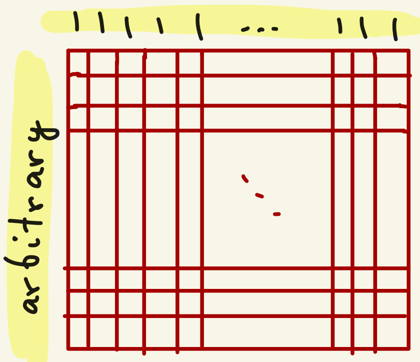


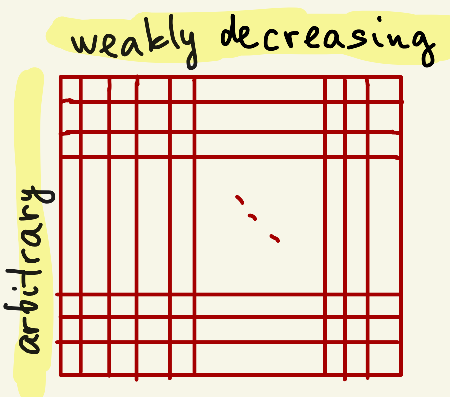


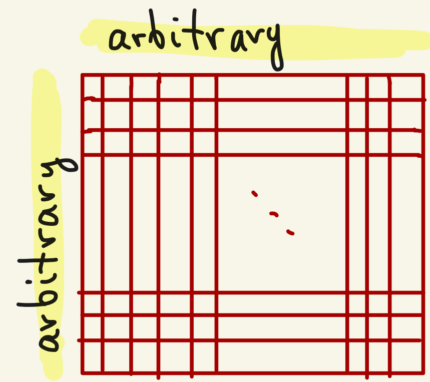
<u>Thm</u> The two charge vectors is a complete invariant of HW class

Thm (up to Hanany-Witten transition)

$$T^*G/P \subset \mathcal{N}(\text{quiver}) \subset \mathcal{C}(\text{brane diagram})$$







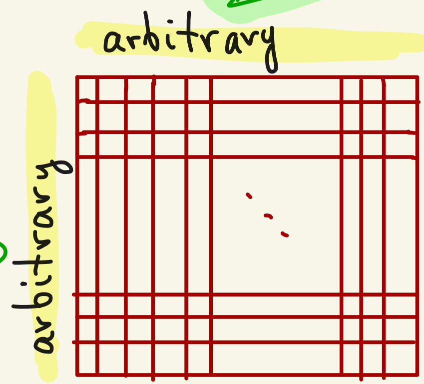
closed for transpose!

Expectation:

$H^*_T(\mathcal{C}(D)) \longleftrightarrow$ a weight space of a representation of a quantum group

(+ Lie superalgebra version R-Rozansky)

2 which representation



1 size : which quantum group

3 which weight space of the representation