Rationality of cubic fourfolds via Trisecant Flops

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Section 1

Motivations

Rationality of smooth degree-d hypersurfaces in $\mathbb{P}^{n+1}_{\mathbb{C}}$

d	curves	surfaces	threefolds	fourfolds	5-folds	6-folds	7-folds
2	all are rational	all are rational	all are rational	all are rational	all are rational	all are rational	all are rational
3	all are irrational	all are rational	all are irrational	open problem (we have many rational examples and a precise conjecture)	open problem	open problem (we know just some very special rational examples)	open problem
4	all are irrational	all are irrational	all are irrational	the very general is irrational but rational ones are unknown	the very general is irrational but rational ones are unknown	open problem	open problem
5	all are irrational	all are irrational	all are irrational	all are irrational	the very general is irrational but rational ones are unknown	the very general is irrational but rational ones are unknown	the very general is irrational but rational ones are unknown
6	all are irrational	all are irrational	all are irrational	all are irrational	all are irrational	the very general is irrational but rational ones are unknown	the very general is irrational but rational ones are unknown
7	all are irrational	all are irrational	all are irrational	all are irrational	all are irrational	all are irrational	the very general is irrational but rational ones are unknown
8	all are irrational	all are irrational	all are irrational	all are irrational	all are irrational	all are irrational	all are irrational

Classical and easy.

Easy cases: $K_X = -i\, H_X$ with $i = n + 2 - \deg(X) \le 0$, hence $p_g(X) = h^0(X, K_X) \neq 0$.

Famous result of Clemens and Griffiths.

Hard result (contrib. by Noether, Fano, B. Segre, Iskovskikh, Manin, Corti, Pukhlikov, Cheltsov, de Fernex, Ein, Mustaţa, Zhuan).

Result of Totaro obtained by combining methods and results of Kollár, Voisin, Colliot–Thélène, Pirutka.

Recent result of Schreieder.

Very recent result of Nicaise and Ottem.

Some classical open problems, also listed by J. Kollár in 2001 and 2019

7. Open problems

In this section I list the main open problems in this area. The formulations are intentionally general. It is more important to understand "nice" examples of rationally connected varieties, but I want to emphasize the rather complete lack of good examples of the theory.

For me the most vexing open problem of the theory over $\mathbb{C}$ is the following:

Problem 55. Find examples of rationally connected varieties which are not unirational.

The classical candidates are general quartic 3–folds in $\mathbb{P}^4$. It may be, however, easier to deal with hypersurfaces of degree n in $\mathbb{P}^n$ for large n. These may have an even stronger property:

Problem 56. Find examples of rationally connected varieties which do not contain rational surfaces through every point. There may even be examples which do not contain any rational surface.

Our knowledge about rationality of hypersurfaces is also very limited. I formulate two of the strongest questions, though there is little evidence for them.

Problem 57. Prove that the general cubic 4-fold is not rational.

The rationality of many special cubic 4-folds is known; see [Hassett00].

Problem 58. Prove that a smooth hypersurface of degree at least 4 is never rational.

For every $n \geq 1$ there exist rational smooth cubic hypersurfaces $X^{2n} \subset \mathbb{P}^{2n+1}$

Take $L_1 = \mathbb{P}^n, L_2 = \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ such that $\langle L_1, L_2 \rangle = \mathbb{P}^{2n+1}$. Then:

- there exists a smooth cubic hypersurface

$$X \supset L_1 \cup L_2$$

- We get a birational map $L_1 \times L_2 \dashrightarrow X$:

$$(p_1, p_2) \to p_3 = (\langle p_1, p_2 \rangle \cap X) \setminus \{p_1, p_2\}.$$

The codimension of (the closure of) such X's inside $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{2n+1}}(3)))$ is $\frac{n(n^2-1)}{3}$.

For $n = 1$, a smooth cubic surface is rational.

For $n \geq 2$, the codimension is bigger and bigger. So **FAR FROM BEING GENERAL**.

Section 2

Generalities on cubic fourfolds

Parameter spaces of cubic fourfolds

A *cubic fourfold* is a smooth cubic hypersurface $X = V(f) \subset \mathbb{P}^5$.

Cubic fourfolds are parametrized by an open set U of the projective space $\mathbb{P}(\mathbb{C}[x_0, \ldots, x_5]_3) \simeq \mathbb{P}^{55}$. Its complementary set $\Delta = \mathbb{P}^{55} \setminus U$ is the *discriminant hypersurface*, an irreducible hypersurface of degree 192.

The moduli space of cubic fourfolds is the quotient

$$\mathcal{C} = [U/\mathrm{PGL}_6],$$

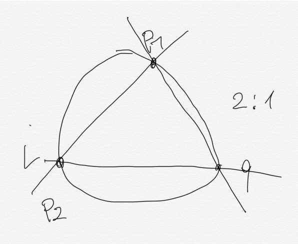
which is an irreducible quasi-projective variety of dimension

$$\dim(\mathcal{C}) = \dim(U) - \dim(\mathrm{PGL}_6) = 55 - 35 = 20.$$

Unirationality of cubic hypersurfaces of dimension at least three (M. Noether)

Let $X \subset \mathbb{P}^N$, $N \geq 4$, be a smooth cubic hypersurface, and let $L \subset X$ be a line passing through a general point of X.

- $$W = \{\text{tangent lines to } X \text{ along } L\} \dashleftarrow\dashrightarrow \mathbb{P}(T_{X_{|L}})$$
  is a rational irreducible variety of dimension $N-1$.
- We get a $2:1$ dominant rational map $W \dashrightarrow X$:
  $$[L'] \to q = (L' \cap X) \setminus L.$$



Section 3

Rational, special and admissible cubic fourfolds

General results for 2–cycles on cubic fourfolds by Voisin

For a cubic fourfold $X \subset \mathbb{P}^5$, we define

$$H^{2,2}(X,\mathbb{Z}) := H^4(X,\mathbb{Z}) \cap H^2(\Omega^2_X).$$

Theorem (Voisin, 1986)

If $[X] \in \mathcal{C}$ is very general, then $H^{2,2}(X,\mathbb{Z}) \simeq \mathbb{Z}\langle h^2\rangle$, where h denotes the class of a hyperplane section of X.

Theorem (Voisin, 2013 (Hodge Integral Conjecture for cubic fourfolds))

If $[X] \subset \mathbb{P}^5$ is a cubic fourfold, then every cycle in $H^{2,2}(X,\mathbb{Z})$ is algebraic.

If $H^{2,2}(X,\mathbb{Z}) \supsetneq \mathbb{Z}\langle h^2\rangle$, then there exists an "algebraic surface" (cycle of dimension 2 with integral coefficients) $T \subset X$ such that $H^{2,2}(X,\mathbb{Z}) \supseteq K$, with $K = \langle h^2, T\rangle$ and $\mathrm{rk}(K) = 2$. In this case, X is said to be a *special cubic fourfold* (more details below).

Rational cubic fourfolds

Let us define

$$\mathrm{Rat}(\mathcal{C}) = \{[X] \in \mathcal{C} : X \text{ is rational}\}.$$

As a consequence of a result of de Fernex and Fusi (2013) and of the specialization of rationality in smooth projective families proved by Kontsevich and Tschinkel in 2019, we have that

Proposition

$\mathrm{Rat}(\mathcal{C})$ *is a countable union of closed subsets.*

The general natural expectation is that

$$\mathrm{Rat}(\mathcal{C}) \subsetneq \mathcal{C},$$

that is, *a very general cubic fourfold $X \subset \mathbb{P}^5$ should be not rational.*

Below we shall specify better the conjectural description of $\mathrm{Rat}(\mathcal{C})$ by specifying a countable union of *admissible* divisors in $\mathcal{C}$.

Rational cubic fourfolds

Let $T_X \subset H^4(X, \mathbb{Z})$ denote the trascendental part of the cohomology of a cubic fourfold $X \subset \mathbb{P}^5$. If X is rational, we have a diagram

$$\begin{tikzcd} X' \arrow[d, "g"'] \arrow[dr, "f"] & \\ \mathbb{P}^4 \arrow[r, dashed, "\psi"] & X \end{tikzcd} \tag{1}$$

of birational maps with g the composition of blow-up's along smooth irreducible centers. Since the blow-up of points, of smooth curves and of smooth surfaces with $p_g = 0$ does not affect the trascendental cohomology, we get

$$T_{X'} \simeq \bigoplus_{j=1}^{m} T_{S_j}(-1)$$

where $p_g(S_j) > 0$ for every j and where $T_{S_j} \subset H^2(S_j, \mathbb{Z})$ is the trascendental part. So one *expects* to find a trace of some $T_{S_j}(-1)$ inside $f^*(T_X)$ and hence inside T_X as soon as X is rational.

Rationality *implies* special and admissible

The Hodge numbers $h^{p,q}$ of X with $p+q=4$ are 0,1, 21,1,0. After removing h^2 form $H^4(X,\mathbb{Z})$, we deduce that $H^4_{\mathrm{prim}}(X,\mathbb{Z})$ reminds the H^2 cohomology of a $K3$ surface S, modulo a Tate twist by -1.

The intersection forms have signatures: $(20,2)$ for $H^4_{\mathrm{prim}}(X,\mathbb{Z})$ and $(19,3)$ for $H^2(S,\mathbb{Z})(-1)$.

They become compatible as soon as we can find a common codimension one rank sublattice with signature $(19,2)$.

For a $K3$ surface S one can take $H^2_{\mathrm{prim}}(S,\mathbb{Z})(-1)$, the ortogonal complement to some primitive ample class l on S; for X the subspace of $H^4_{\mathrm{prim}}(X,\mathbb{Z})$ ortogonal to an algebraic cycle $T \in H^{2,2}(X,\mathbb{Z})$, yielding a rank two sublattice $\langle h^2, T\rangle = K \subseteq H^{2,2}(X,\mathbb{Z})$.

In conclusion: a rational cubic fourfold SHOULD be SPECIAL and SHOULD HAVE an extra compatibility condition with the cohomology of a $K3$ surface.

Hassett's Noether–Lefschetz loci

Inspired by the previous remarks, Hassett (1999) defined the Noether-Lefschetz loci $\mathcal{C}_d$ as

$$\mathcal{C}_d = \{[X] \in \mathcal{C} : \exists K \subseteq H^{2,2}(X, \mathbb{Z}),\ h^2 \in K,\ \mathrm{rk}(K) = 2, |K| = d\}.$$

The discriminant $|K|$ of the sublattice K is defined as the determinant of the intersection form on K. If $K = \langle h^2, T \rangle$, with T an algebraic surface, then

$$|K| = \det \begin{pmatrix} h^4 & h^2 \cdot T \\ T \cdot h^2 & (T)^2_X \end{pmatrix} = \det \begin{pmatrix} 3 & \deg(T) \\ \deg(T) & (T)^2_X \end{pmatrix} = 3(T)^2_X - (\deg(T))^2.$$

When T has smooth normalization $\tilde{T}$ and has only a finite number δ of nodes as singularities, the self-intersection $(T)^2_X$ can be explicitly calculated by the following *double point formula*:

$$(T)^2_X = 6h^2 + 3h \cdot K_{\tilde{T}} + K^2_{\tilde{T}} - \chi_{\tilde{T}} + 2\delta.$$

First properties of Hassett's Noether-Lefschetz loci

Theorem (Hassett, 1999, 2000)

1. *$\mathcal{C}_d$ is either empty or an irreducible divisor in $\mathcal{C}$.*
2. *$\mathcal{C}_d \neq \emptyset$ if and only if $d > 6$ and $d \equiv 0, 2 \mod 6$.*
3. *If $[X] \in \mathcal{C}_d$ is very general, then $H^{2,2}(X, \mathbb{Z}) = \langle h^2, T \rangle$, for some algebraic cycle T.*
4. *(Yang-Yu, 2019) If $\mathcal{C}_{d_1} \neq \emptyset$ and if $\mathcal{C}_{d_2} \neq \emptyset$, then $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \neq \emptyset$.*

Geometric descriptions of some Hassett's divisors

$\mathcal{C}_8$: Cubic fourfolds containing a plane

$$\mathcal{C}_8 = \{[X] \in \mathcal{C} : X \supset P, P \text{ plane } \},$$

$$K_8 = \begin{array}{c|cc} & h^2 & P \\ \hline h^2 & 3 & 1 \\ P & 1 & 3 \end{array}$$

$\mathcal{C}_{12}$: Cubic fourfolds containing a cubic scroll

$$\mathcal{C}_{12} = \overline{\{[X] \in \mathcal{C} : X \supset S(1,2), S(1,2) \subset \mathbb{P}^5 \text{ smooth cubic rational normal scroll}\}},$$

$$K_{12} = \begin{array}{c|cc} & h^2 & S(1,2) \\ \hline h^2 & 3 & 3 \\ S(1,2) & 3 & 7 \end{array}$$

$\mathcal{C}_{14}$: Cubic fourfolds containing a smooth quartic rational normal scroll

$$\begin{aligned}\mathcal{C}_{14} &= \overline{\{[X] \in \mathcal{C} : X \supset S(2,2), S(2,2) \subset \mathbb{P}^5 \text{ smooth quartic rational scroll}\}}\\ &= \overline{\{[X] \in \mathcal{C} : X \supset D, D \subset \mathbb{P}^5 \text{ quintic del Pezzo surface}\}},\end{aligned}$$

$$K_{14} = \begin{array}{c|cc} & h^2 & S(2,2) \\ \hline h^2 & 3 & 4 \\ S(2,2) & 4 & 10 \end{array} \simeq \begin{array}{c|cc} & h^2 & D \\ \hline h^2 & 3 & 5 \\ D & 5 & 13 \end{array}, D + S(2,2) = 3h^2.$$

$\mathcal{C}_{20}$: Cubic fourfolds containing a Veronese surface

$$\mathcal{C}_{20} = \overline{\{[X] \in \mathcal{C} : X \supset V, V \text{ Veronese surface }\}},$$

$$K_{20} = \begin{array}{c|cc} & h^2 & V \\ \hline h^2 & 3 & 4 \\ V & 4 & 12 \end{array}$$

Associated K3 surfaces and admissible integers

Motivated by the previous analysis based on Clemens and Griffiths' proof of the irrationality of cubic 3-folds, Hassett considered an *associated K3 surface* to a cubic fourfold $[X] \in \mathcal{C}_d$ with rank two sublattice $K \subseteq H^{2,2}(X, \mathbb{Z})$ of discriminant d.

A polarized K3 surface (S, l) of degree $d = l^2$ is associated to X if there exists an isomorphism $H^2(S, \mathbb{Z})(-1) \supset l^{\perp} \xrightarrow{\simeq} K^{\perp}$ respecting Hodge structures.

Theorem (Hassett, 1999)

$[X] \in \mathcal{C}_d$ general admits an associated K3 surface if and only if d is admissible.

Definition

An even integer $d > 6$ is admissible if it is not divisible by 4, 9 or any odd prime congruent to 2 module 3.

Admissible values < 140

8 12 **14** 18 20 24 **26** 30 32 36 **38** **42** 44 48 50 54 56 60 **62** 66 68 72 **74** **78** 80 84 **86** 90 92 96 **98** 102 104 108 110 **114** 116 120 **122** 126 128 132 **134** 138

Kuznetsov's Derived Category associated K3 surface

Kuznetsov (2010) introduced another *associated K3 surface* to a cubic fourfold $[X] \in \mathcal{C}$ in terms of derived categories of coherent sheaves on X.

Kuznetsov conjectured that X is rational if and only if it admits a derived category associated K3 surface.

Theorem (Addington and Thomas, 2014)

If a cubic fourfold X has an associated K3 surface in the sense of Kuznetsov, then it has an associated K3 surface in the sense of Hassett, i.e. $[X] \in \mathcal{C}_d$ with d admissible.
Conversely, a generic $[X] \in \mathcal{C}_d$ with d admissible has an associated K3 surface in the sense of Kuznetsov.

Moreover, it is recently shown by Bayer-Lahoz-Macrì-Nuer-Perry-Stellari (2019) that in the above result the word "*generic*" can be replaced by "*every*".

Hassett–Kuznetsov Rationality Conjecture

General expectation:

$$\mathrm{Rat}(\mathcal{C}) \subseteq \bigcup_{d>6} \mathcal{C}_d.$$

Hassett and Kuznetsov analysis predicts:

Rationality Conjecture:

$$\mathrm{Rat}(\mathcal{C}) = \bigcup_{d \text{ admissible}} \mathcal{C}_d.$$

For example, a very general cubic fourfold containing either a plane or a Veronese surface is irrational.

Analysis of known rational cubic fourfolds

Theorem (Fano, 1943)

A generic $[X] \in \mathcal{C}_{14}$ is rational. (The word "generic" can be replaced by "every", by [Bolognesi, R., Staglianò, 2017] or [Kontsevich and Tschinkel, 2019].)

If $T = S(2,2) \subset \mathbb{P}^5$, then $|H^0(\mathcal{I}_T(2))| = \mathbb{P}^5$ and it defines a rational map $\phi : \mathbb{P}^5 \dashrightarrow Q \subset \mathbb{P}^5$, Q smooth quadric hypersurface. The fiber through a general $p \in \mathbb{P}^5$ is the unique secant line to T passing through p. HENCE the restriction of ϕ to a general $X \in |H^0(\mathcal{I}_T(3))|$ induces a diagram of birational maps:

$$\begin{tikzcd} & \mathrm{Bl}_T X = \mathrm{Bl}_S Q \arrow[dl, "g"'] \arrow[dr, "\tilde{\phi}"] & \\ X \arrow[rr, dashed, "\phi"] & & Q \end{tikzcd} \quad . \tag{2}$$

The morphism $\tilde{\phi}$ is the blow-up of a smooth surface $S \subset Q$ of degree 10 and sectional genus 8, which is isomorphic to the blow-up of a minimal K3 surface $\tilde{S} \subset \mathbb{P}^8$ of genus 8 and degree 14 at a point $q \in \tilde{S}$. The surface S is the projection of $\tilde{S} \subset \mathbb{P}^8$ from the tangent plane at q.

Analysis of known rational cubic fourfolds

Similarly, If $D \subset \mathbb{P}^5$ is a smooth quintic del Pezzo surface, then $|H^0(\mathcal{I}_D(2))| = \mathbb{P}^4$ and it defines a rational map $\phi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$. The closure of the fiber through a general $p \in \mathbb{P}^5$ is the unique secant line to D passing through p.

HENCE the restriction of ϕ to a general $X \in |H^0(\mathcal{I}_D(3))|$ induces a diagram of birational maps:

$$\begin{tikzcd} & \mathrm{Bl}_D X = \mathrm{Bl}_S \mathbb{P}^4 \arrow[dl, "g"'] \arrow[dr, "\tilde{\phi}"] & \\ X \arrow[rr, dashed, "\phi"] & & \mathbb{P}^4 \end{tikzcd} \quad . \tag{3}$$

The morphism $\tilde{\phi}$ is the blow-up of a smooth surface $S \subset \mathbb{P}^4$ of degree 9 and sectional genus 8, which is isomorphic to the blow-up of a minimal K3 surface $\tilde{S} \subset \mathbb{P}^8$ of genus 8 and degree 14 at five points, whose linear span is $L = \mathbb{P}^3$. The surface S is the projection of $\tilde{S} \subset \mathbb{P}^8$ from the linear space L.

Analysis of known rational cubic fourfolds

Let $T^{[2]}$, respectively $D^{[2]}$, be the blow-up of the symmetric product $T^{(2)}$, respectively $D^{(2)}$, along the diagonal. These fourfolds parametrize secant and tangent lines to the surfaces T, respectively D. Thus we have diagrams:

$$\begin{tikzcd} \mathcal{L} \arrow[d, "\pi"'] \arrow[drr, "\psi"] & & \\ T^{[2]} & & \mathbb{P}^5 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L} \arrow[d, "\pi"'] \arrow[drr, "\psi"] & & \\ D^{[2]} & & \mathbb{P}^5 \end{tikzcd} \tag{4}$$

where $\mathcal{L}$ is the pull-back of the universal family over $\mathbb{G}(1,5)$ and where $\psi : \mathcal{L} \to \mathbb{P}^5$ are the tautological morphisms. The maps ψ's are birational because through a general point of $\mathbb{P}^5$ there passes a unique secant line to T, respectively D. In general, one can compute the degree of ψ via Double Point Formula.

We can think to the maps ϕ's defined above as a concrete projective incarnation of the more abstract birational maps $\pi \circ \psi^{-1}$. The restriction of ϕ to a general X through the surface can be interpreted as the fact that $\psi^{-1}(X)$ becomes a section of π.

Hassett–Kuznetsov's conjecture for $d = 14, 26, 38, 42$

Until July 2017, the Rationality Conjecture was known to be true only for the first admissible value $d = 14$, proved by Fano in 1943.

Theorem (Russo and Staglianò, 2019a (preprint in 2017))

A generic cubic fourfold in $\mathcal{C}_{26}$ and $\mathcal{C}_{38}$ is rational. (As before, "generic" can be replaced by "every", by [Kontsevich and Tschinkel, 2019].)

This result follows from the discovery of so-called *congruences of 5-secant conics*, which will be illustrated in the following slides and whose definition has been inspired by the above diagrams associated to $T^{[2]}$ and to $D^{[2]}$.

Moreover, very recently we also showed the following:

Theorem (Russo and Staglianò, 2019b)

A generic (hence every) cubic fourfold in $\mathcal{C}_{42}$ is rational.

Thus, for the moment, we have:

$$\mathcal{C}_{14} \cup \mathcal{C}_{26} \cup \mathcal{C}_{38} \cup \mathcal{C}_{42} \subset \mathrm{Rat}(\mathcal{C}).$$

Section 4

Rationality via congruences of $(3e-1)$-secant curves of degree e

Congruences of $(3e-1)$-secant curves of degree e

Let $\mathcal{H}$ be an irreducible proper family of rational (or of a fixed arithmetic genus) curves of degree $e \geq 1$ in $\mathbb{P}^5$ whose general element $[C] \in \mathcal{H}$ is irreducible.

We have a universal family $\mathcal{D}$ and two natural projections:

$$\begin{tikzcd} \mathcal{D} \arrow[d, "\pi"'] \arrow[dr, "\psi"] & \\ \mathcal{H} & \mathbb{P}^5 \end{tikzcd} \tag{5}$$

such that $\psi(\pi^{-1}([C])) = C \subset \mathbb{P}^5$.

Definition

Let $S \subset \mathbb{P}^5$ be an irreducible surface. We say that (5) is a *congruence of* $(3e-1)$*-secant curves of degree* e to S if the following hold:

1. ψ is birational;
2. for $[C] \in \mathcal{H}$ general, the intersection $C \cap S$ consists of $3e-1$ points.

Let $S \subset \mathbb{P}^5$ be a surface admitting a congruence of $(3e-1)$-secant curves of degree e parametrized by $\mathcal{H}$, and let $X \in |H^0(\mathcal{I}_S(3))|$ be an irreducible cubic hypersurface containing S.

If $[C] \in \mathcal{H}$ is general, from Bézout's theorem there exists a unique point $p \in X$ such that $\{p\} = (C \cap X) \setminus (C \cap S)$. Thus we have a rational map:

$$\alpha : \mathcal{H} \dashrightarrow X, \quad \alpha([C]) = (C \cap X) \setminus (C \cap S).$$

If $p \in X$ is a general point, from Zariski's Main Theorem there exists a unique $[C_p] \in \mathcal{H}$ such that $p \in C_p$. Thus we have another rational map:

$$\beta : X \dashrightarrow \mathcal{H}, \quad \beta(p) = [C_p].$$

We say that X is *trasversal to the congruence* $\mathcal{H}$ if for the general point $p \in X$ the curve $\beta(p)$ is not contained in X, *i.e.* if the composition $\alpha \circ \beta$ is a well-defined rational map.

If X is transversal to the congruence, then α and β are inverse to each other.

Genericity implies transversality

Proposition

Let $S \subset \mathbb{P}^5$ be a surface admitting a congruence of $(3e-1)$-secant curves of degree e parametrized by $\mathcal{H}$. Assume that the linear system $|H^0(\mathcal{I}_S(3))|$ of cubics through S defines a birational map $\Phi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^N$. Then a general $X \in |H^0(\mathcal{I}_S(3))|$ is transversal to $\mathcal{H}$, in particular X is birational to $\mathcal{H}$.

The proof follows easily from the fact that Φ induces a 1–1 correspondence:

$$\bigcup_{e \geq 1} \left\{ \begin{array}{c} (3e-1)\text{-secant curves of degree } e \text{ to } S \\ \text{passing through a general point } p \in \mathbb{P}^5 \end{array} \right\} \to \left\{ \begin{array}{c} \text{lines contained in } Z \\ \text{passing through } \Phi(p) \end{array} \right\}$$

Indeed, around a general point $p \in \mathbb{P}^5$ the map Φ is an embedding.

Therefore, an irreducible (rational) curve $C \subset \mathbb{P}^5$ through p maps birationally onto a line $L = \Phi(C) \subset Z$ passing through $q = \Phi(p)$ if and only if $\#(C \cap S) = 3e - 1$.

For $e = 1$ the curves C are the secant lines to S passing through p. For $e > 1$ one expects that, in general, there are no such curves.

How to apply congruences

Let $S \subset \mathbb{P}^5$ be a surface admitting a congruence of $(3e-1)$-secant curves of degree e parametrized by $\mathcal{H}$, as in the previous Proposition.

Suppose we are able to show at least one of the following:

1. $\mathcal{H}$ is rational (resp. irrational);
2. there exists a particular singular cubic hypersurface $X \in |H^0(\mathcal{I}_S(3))|$ trasversal to $\mathcal{H}$ which is rational (resp., irrational), for instance $\mathrm{Sing}(X)$ consists of a unique double point (resp., of a unique triple point).

Then we can conclude that a general cubic hypersurface $X \in |H^0(\mathcal{I}_S(3))|$ is rational (resp., irrational).

Fano's classical applications of congruences ($e = 1$)

The surfaces $S \subset \mathbb{P}^5$ admitting a congruence of 2-secant lines are usually called OADP (*one apparent double point*) surfaces because they acquire a *double point* by projection from a general point (the double point appears only after the projection and so is only *apparent*).

The OADP surfaces have been completely classified in the works by Severi (1901), Russo (2000), and Ciliberto and Russo (2011). Those contained in a cubic fourfold are only the following:

- quintic del Pezzo surfaces;
- smooth quartic rational normal scrolls;
- quintic rational scroll with a line of singular points.

Cubic fourfolds through these surfaces describe only the divisor $\mathcal{C}_{14}$, as it was firstly remarked by Fano (1943).

Until 2017 no congruence of $(3e-1)$-secant curves of degree $e \geq 2$ to an irreducible surface $S \subset \mathbb{P}^5$ had been discovered.

A congruence of 5-secant conics for the rationality of cubic fourfolds in $\mathcal{C}_{26}$

The first example of congruence of $(3e-1)$-secant curves of degree $e > 1$

Let $S_{26} \subset \mathbb{P}^5$ be a rational septimic scroll with three nodes, which is the projection of the rational normal scroll $S(3,4) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(4)) \subset \mathbb{P}^8$ from a plane intersecting the secant variety $\mathrm{Sec}(S(3,4))$ at three general points.

Farkas and Verra (2018) showed that the cubic fourfolds containing such a surface S_{26} describe the divisor $\mathcal{C}_{26}$.

1. A surface $S_{26} \subset \mathbb{P}^5$ as above is cut out by 13 cubics which give a birational map $\Phi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^{12}$ onto a fivefold $Z \subset \mathbb{P}^{12}$ such that through a general point $\Phi(p)$, $p \in \mathbb{P}^5$ general, there pass 8 lines contained in Z.
2. Since $S_{26} \subset \mathbb{P}^5$ has only 7 secant lines through p (double point formula), there is an *exceeding line* $L \subset Z$ passing through $\Phi(p)$, which does not come from secant lines to S_{26}. This line is the image of a 5-secant conic to S_{26}.
3. The generality of $\Phi(p)$ assures the existence of a congruence $\mathcal{H}$ of 5–secant conics to S_{26}. We proved that the corresponding $\mathcal{H}$ is birational to $S_{26}^{(2)}$ (and hence rational).

Section 5

Explicit Rationality and Rationality via Trisecant Flops

The proof of rationality of cubic fourfolds in $\mathcal{C}_{26}$ and in $\mathcal{C}_{38}$ via congruences of $(3e-1)$-secant curves of degree e has two main disadvantages:

- it does not give information about the explicit birational maps to $\mathbb{P}^4$;
- it does not clarify the relation with the associated K3 surfaces.

To illustrate these fundamental relations, firstly in [Russo and Staglianò, 2018] we found explicit birational maps from a general cubic in $\mathcal{C}_{26}$ and in $\mathcal{C}_{38}$ to $\mathbb{P}^4$.

Then in [Russo and Staglianò, 2019b] we introduced the trisecant flop for the study of the rationality of cubic fourfolds.

We shall explain these methods in the following slides but before we motivate their appearance.

Explicit realizations of the congruences

Suppose we have a congruence of $(3e-1)$–secant curves of degree e to $S \subset \mathbb{P}^5$:

$$\begin{tikzcd} \mathcal{D} \arrow[d, "\pi"'] \arrow[dr, "\psi"] & \\ \mathcal{H} & \mathbb{P}^5, \end{tikzcd}$$

The fiber through a general point $p \in \mathbb{P}^5$ of the rational map

$$\mu = \pi \circ \psi^{-1} : \mathbb{P}^5 \dashrightarrow \mathcal{H},$$

$F = \overline{\mu^{-1}(\mu(p))}$, is the unique curve of the congruence passing through p.

If $D \in |H^0(\mathcal{I}_S^e(3e-1))|$, then

$$D \cdot F = (3e-1) \cdot e - e \cdot (3e-1) = 0.$$

Explicit realizations of the congruences

The first surprise came from the following computations for the surface $S_{26} \subset \mathbb{P}^5$:

- $|H^0(\mathcal{I}^2_{S_{26}}(5))| = \mathbb{P}^4$;
- the associated dominant rational map $\phi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ has the 5-secant conics to S_{26} as general fibers so that it coincides with μ;
- the restriction of $\mu = \phi$ to a general $X \in |H^0(\mathcal{I}_{S_{26}}(3))|$ is an explicit birational map onto $\mathbb{P}^4$.

Similarly, let $S_{38} \subset \mathbb{P}^5$ be the smooth surface of degree 10 and sectional genus 6, obtained as the image of $\mathbb{P}^2$ via the linear system of curves of degree 10 with 10 general triple points. By a parameter count, Nuer (2015) showed that

$$\mathcal{C}_{38} = \overline{\{[X] \in \mathcal{C} \ : \ X \subset S_{38}\}}.$$

Again, quite surprisingly, we get:

- $|H^0(\mathcal{I}^2_{S_{38}}(5))| = \mathbb{P}^4$;
- the associated dominant rational map $\phi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ has the 5-secant conics to S_{38} as general fibers so that it coincides with μ;
- the restriction of $\mu = \phi$ to a general $X \in |H^0(\mathcal{I}_{S_{38}}(3))|$ is an explicit birational map onto $\mathbb{P}^4$.

Explicit realizations of the congruences

In the sequel we shall develop a theoretical framework for these phenomena in order to be able to understand also the birational maps

$$\mu : X \dashrightarrow W \subseteq |H^0(\mathcal{I}_S^e(3e-1))|.$$

There are some important remarks:

1. the associated $K3$ surface should be RECOVERED BIRATIONALLY as the image $U \subset W$ via μ of the curves of the congruence contained in X;
2. HENCE the map μ restricted to a general X should factor through the blow-up of $\mathbb{P}^4$ (or of some rational variety) along U, exactly as in the case $e = 1$ (Fano's examples for $d = 14$);
3. let $L \subset \mathbb{P}^5$ be a trisecant line to $S \subset \mathbb{P}^5$ and let $D \in |H^0(\mathcal{I}_S^e(3e-1))|$, then

   $$D \cdot L = (3e-1) - 3e = -1,$$

   that is trisecant lines to S are also contained in the base locus of

   $$\mu : \mathbb{P}^5 \dashrightarrow W \subseteq |H^0(\mathcal{I}_S^e(3e-1))|;$$

4. trisecant lines to S contained in a general $X \in |H^0(\mathcal{I}_S(3))|$ belong to the base locus B of the restriction of μ to X. HENCE B has at least two irreducible components: S and the trisecant lines to S contained in X.

Congruences & $\mu : X \dashrightarrow W \subseteq |H^0(\mathcal{I}_S^e(3e-1))|$

	d	e	$S \subset X \subset \mathbb{P}^5$	W	$U \subset W$	Y
0	42	3	Rational surface of degree 9 and sectional genus 2 with 5 nodes, which is a special projection of the image of $\mathbb{P}^2$ in $\mathbb{P}^8$ via the linear system of quartic curves with 3 simple points and one double point $(*)$	$\mathbb{G}(1,4) \cap \mathbb{P}^7 \subset \mathbb{P}^7$	Non-minimal K3 surface of degree 21 and sectional genus 18, cut out in $\mathbb{P}^7$ by 5 quadrics and 8 cubics	4-fold of degree 14 in $\mathbb{P}^7$ cut out by 7 cubics
i	14	2	Isomorphic projection of a smooth surface in $\mathbb{P}^6$ of degree 8 and sectional genus 3, obtained as the image of $\mathbb{P}^2$ via the linear system of quartic curves with 8 general base points	$\mathbb{P}^4$	Singular $K3$ surface of degree 10 and sectional genus 7, cut out by 12 quintics and having 8 singular points	4-fold of degree 28 in $\mathbb{P}^{11}$ cut out by 16 quadrics
ii	26	2	Rational scroll of degree 7 with 3 nodes	$\mathbb{P}^4$	Singular K3 surface of degree 10 and sectional genus 8, cut out by 12 quintics and one sextic, and having 3 singular points	4-fold of degree 29 in $\mathbb{P}^{11}$ cut out by 15 quadrics
iii	38	2	Smooth surface of degree 10 and sectional genus 6, obtained as the image of $\mathbb{P}^2$ via the linear system of curves of degree 10 with 10 general triple points	$\mathbb{P}^4$	Smooth non-minimal K3 surface of degree 12 and sectional genus 14 cut out by 9 quintics	4-fold of degree 20 in $\mathbb{P}^8$ cut out by 16 cubics
iv	26	2	Projection of a smooth del Pezzo surface of degree 7 in $\mathbb{P}^7$ from a line intersecting the secant variety in one general point	$\mathbb{G}(1,4) \cap \mathbb{P}^7 \subset \mathbb{P}^7$	Non-minimal K3 surface of degree 17 and sectional genus 11, cut out in $\mathbb{P}^7$ by 5 quadrics and 13 cubics	4-fold of degree 34 in $\mathbb{P}^{12}$ cut out by 20 quadrics
v	38	3	Rational scroll of degree 8 with 6 nodes $(*)$	$\mathbb{G}(1,5) \cap \mathbb{P}^{10} \subset \mathbb{P}^{10}$	Smooth non-minimal K3 surface of degree 22 and sectional genus 14, cut out in $\mathbb{P}^{10}$ by 24 quadrics	4-fold of degree 17 in $\mathbb{P}^8$ cut out by 3 quadrics and 4 cubics
vi	14	3	Projection from 3 general internal points of a minimal $K3$ surface of degree 14 and sectional genus 8	Cubic fourfold	Projection from 3 general internal points of a minimal $K3$ surface of degree 14 and sectional genus 8	Complete intersection in $\mathbb{P}^7$ of 2 quadrics and one cubic
vii	14	3	Projection of a $K3$ surface of degree 10 and sectional genus 6 in $\mathbb{P}^6$ from a general point on its secant variety	Gushel-Mukai fourfold in $\mathbb{P}^8$	Smooth minimal K3 surface of degree 14 and sectional genus 8	Hypercubic section of a hyperplane section of $\mathbb{G}(1,4)$
viii	14	5	General hyperplane section of a conic bundle in $\mathbb{P}^6$ of degree 13 and sectional genus 12 $(*)$	Complete intersection of three quadrics in $\mathbb{P}^7$	Smooth non-minimal K3 surface of degree 13 and sectional genus 8, cut out by 9 quadrics	Hypersurface of degree 5 in $\mathbb{P}^5$
ix	14	5	General hyperplane section of a pfaffian threefold in $\mathbb{P}^6$ of degree 14 and sectional genus 15	$\mathbb{G}(1,6) \cap \mathbb{P}^{14} \subset \mathbb{P}^{14}$	Smooth minimal K3 surface of degree 14 and sectional genus 8 embedded in $\mathbb{P}^8 \subset \mathbb{P}^{14}$	Hypersurface of degree 5 in $\mathbb{P}^5$

Other examples

	d	e	$S \subset X \subset \mathbb{P}^5$	W	$U \subset W$	Y
x	38	5	Smooth surface of degree 11 and sectional genus 7, obtained as the image of $\mathbb{P}^2$ via the linear system of curves of degree 12 with one general simple point, 4 general triple points, and 6 general quadruple points $(*)$	$\mathbb{G}(1,5) \cap \mathbb{P}^{10} \subset \mathbb{P}^{10}$	Smooth non-minimal K3 surface of degree 25 and sectional genus 17, cut out in $\mathbb{P}^{10}$ by 21 quadrics	Hypersurface of degree 7 in $\mathbb{P}^5$
xi	38	3	Projection of an octic del Pezzo surface isomorphic to $\mathbb{F}_1$ from a plane intersecting the secant variety in 3 general points (cut out by 10 cubics)	$\mathbb{G}(1,3) \subset \mathbb{P}^5$	Non-minimal $K3$ surface of degree 13 and sectional genus 10, cut out in $\mathbb{P}^5$ by one quadric, 9 quartics, and 3 quintics	4-fold of degree 17 in $\mathbb{P}^8$ cut out by 3 quadrics and 4 cubics
xii	38	3	Projection of an octic del Pezzo surface isomorphic to $\mathbb{F}_0$ from a plane intersecting the secant variety in 3 general points (cut out by 10 cubics and one quartic)	$\mathrm{LG}_3(\mathbb{C}^6) \cap \mathbb{P}^{11} \subset \mathbb{P}^{11}$	Non-minimal $K3$ surface of degree 26 and sectional genus 17, cut out in $\mathbb{P}^{11}$ by 30 quadrics	4-fold of degree 18 in $\mathbb{P}^8$ cut out by 2 quadrics and 8 cubics
xiii	14	3	Isomorphic projection of a smooth surface in $\mathbb{P}^7$ of degree 8 and sectional genus 2, obtained as the image of $\mathbb{P}^2$ via the linear system of quartic curves with 4 simple base points and one double point (cut out by 10 cubics and 3 quartics)	Complete intersection of 2 quadrics in $\mathbb{P}^6$	Singular $K3$ surface of degree 14 and sectional genus 8, cut out in $\mathbb{P}^6$ by 2 quadrics and 9 cubics, and having one singular point	Complete intersection of 4 quadrics in $\mathbb{P}^8$
xiv	26	5	Rational scroll of degree 8 with 4 nodes (cut out by 8 cubics and 3 quartics)	$\mathbb{G}(1,3) \subset \mathbb{P}^5$	Non-minimal $K3$ surface of degree 14 and sectional genus 11, cut out in $\mathbb{P}^5$ by one quadric, 7 quartics, and 2 quintics	Complete intersection in $\mathbb{P}^6$ of a quadric and a quartic

Table: Further examples of maps $\mu : X \dashrightarrow W$. Here, $X \in \mathcal{C}_d$ but is not necessarily a general element, and $S \subset \mathbb{P}^5$ is not necessarily cut out by cubics.

General (simplified) assumptions

The trisecant locus

Let us fix a smooth irreducible non-degenerate surface $S \subset \mathbb{P}^5$ cut out scheme-theoretically by cubics which define a birational map

$$\Phi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^N.$$

For simplicity, we can assume that the Koszul syzygies of the cubics defining S are generated by the linear ones (*Vermeire's* $\mathcal{K}_3$ *condition*).

This leads to the simplification that the exceptional locus of Φ coincides set-theoretically with the trisecant locus of $S \subset \mathbb{P}^5$, $\mathrm{Trisec}(S) \subset \mathbb{P}^5$, which is defined as the closure of the union of all the trisecant lines to S.

Expected trisecant behaviour

Let $\mathrm{Al}^3 S$ be the Hilbert scheme of length 3 aligned subschemes of $S \subset \mathbb{P}^5$.

If $\mathrm{Al}^3 S \neq \emptyset$ then every irreducible component of $\mathrm{Al}^3 S$ has dimension at least 2 with expected dimension 2. The smoothness of $\mathrm{Al}^3 S$ is related to the tangential behaviour of S at the points of intersection of a general trisecant line and to the dimension of the locus of the trisesant lines in $\mathrm{Al}^3 S$. More precisely we have:

Proposition (Gruson and Peskine, 2013)

Let $L \subset \mathbb{P}^5$ be a proper trisecant line to $S \subset \mathbb{P}^5$, with $[L \cap S]$ belonging to a 2-dimensional irreducible component A of $\mathrm{Al}^3 S$. Then $\mathrm{Al}^3 S$ is smooth at $[L \cap S]$ if and only if the tangent planes to S at the points in $L \cap S$ are in general linear position.

In this case, A is generically smooth, and the irreducible component of $\mathrm{Trisec}(S)$ corresponding to A has dimension 3.

We say that $S \subset \mathbb{P}^5$ has the expected trisecant behaviour if $\mathrm{Al}^3 S$ is of pure dimension two and every irreducible component is generically smooth.

Induced flop small contractions

Let $X \subset \mathbb{P}^5$ be a general cubic fourfold containing S. The restriction to X of the map $\Phi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^N$ **defined by cubics through S (assumption)** is a birational map

$$\varphi : X \dashrightarrow Y \subset \mathbb{P}^{N-1},$$

where Y is the corresponding hyperplane section of Z.

Let $\lambda : X' = \mathrm{Bl}_S X \to X$ be the blow-up of X along S. Then the induced morphism

$$\tilde{\varphi} : X' \to Y$$

is defined by the linear system

$$|3\,\lambda^*(H) - E| = |\lambda^*(-K_X) - E| = |-K_{X'}|.$$

If S has the expected trisecant behaviour, then $\tilde{\varphi}$ is a flop small contraction. Indeed its exceptional locus consists of the strict transforms $L' \subset X'$ of the trisecant lines L to S contained in X (which is at most 2-dimensional). We have

$$K_{X'} \cdot L' = (E - 3\lambda^*(H)) \cdot L' = 3 - 3 = 0.$$

Consequence of the expected trisecant behaviour

Let us list some consequences of the expected trisecant behaviour:

- if $T \subset X$ is an irreducible component of the two dimensional locus of trisecant lines to S contained in X and if $T' \subset X' = \mathrm{Bl}_S X$ is its strict transform, then

$$N_{T'/X'|L'} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

- The morphism $\tilde{\varphi} : T' \to C'$ has as general fiber the strict transform of a trisecant line L';
- if $\sigma : E' \to T'$ is the exceptional divisor of $\sigma : \mathrm{Bl}_{T'} X' \to X'$, then $\sigma^{-1}(L') \simeq \mathbb{P}^1 \times \mathbb{P}^1$;
- the smooth 3-fold E' has another contraction $\omega : E' \to R$ with R a smooth ruled surface;
- the contraction ω extends to a birational morphism $\omega : \mathrm{Bl}_{T'} X' \to W'$ which is the inverse of the blow-up of $R \subset W'$.
- there exists a birational morphism $\tilde{\psi} : W' \to Y$, specular to $\tilde{\phi} : X' \to Y$.

Existence of the trisecant flop

Theorem

Under the above assumptions, if the exceptional locus of $\tilde{\varphi}$ is a smooth irreducible surface T, then there exists a smooth projective fourfold W', a smooth surface $R \subset W'$, and a small contraction $\tilde{\psi} : W' \to Y$ defined by $|-K_{W'}|$, such that we have a diagram:

$$\begin{tikzcd}
 & \mathrm{Bl}_T X' = \mathrm{Bl}_R W' \arrow[dl, "\sigma"'] \arrow[dr, "\omega"] & \\
X' \arrow[rr, dashed, "\tau"'] \arrow[d, "\lambda"'] \arrow[dr, "\tilde{\varphi}"'] & & W' \arrow[dl, "\tilde{\psi}"] \\
X & Y &
\end{tikzcd}$$

where $\sigma^*(K_{X'}) = \omega^*(K_{W'})$.

The birational map $\tau : X' \dashrightarrow W'$ is the Trisecant Flop of T' and it is an isomorphism in codimension one.

Extremal contraction of the congruence

Theorem

In the previous hypothesis, assume also that $S \subset \mathbb{P}^5$ admits a congruence of $(3e-1)$-secant rational curves of degree $e \geq 2$. Then there exists a $\mathbb{Q}$-factorial Fano variety W with $\operatorname{Pic}(W) \simeq \mathbb{Z}$ and index $i(W)$, and a birational morphism $\nu : W' \dashrightarrow W$ which is generically the blow-up of an irreducible surface $U \subset W$, such that the previous diagram can be completed as follows:

$$\begin{tikzcd}
 & \mathrm{Bl}_T X' = \mathrm{Bl}_R W' \arrow[dl, "\sigma"'] \arrow[dr, "\omega"] & \\
X' \arrow[rr, dashed, "\tau"] \arrow[d, "\lambda"'] \arrow[dr, "\tilde{\varphi}"] & & W' \arrow[d, "\nu"] \arrow[dl, "\tilde{\psi}"'] \\
X \arrow[r, dashed, "\varphi"] \arrow[rr, dashed, bend right, "\mu"'] & Y & W \arrow[l, dashed, "\psi"'] \arrow[ll, dashed, bend right, "\mu^{-1}"']
\end{tikzcd}$$

$$\psi : |H^0(\mathcal{I}_U(i(W)))|$$

$$\mu : |H^0(\mathcal{I}_S^e(3e-1))|$$

$$\mu^{-1} : |H^0(\mathcal{I}_U^e(i(W)e-1))|.$$

Idea of the proof

- The strict transform $C' \subset X'$ of a proper $3e$–secant curve $C \subset X$ to S of degree $e \geq 1$ satisfies $[C'] = [eL']$ and $K_{X'} \cdot C' = 0$.
- $\tilde{C}'$ strict transform of $[\tilde{C}] \in \mathcal{H}$. Then

$$K_{X'} \cdot \tilde{C}' = (E - 3H') \cdot \tilde{C}' = 3e - 1 - 3e = -1.$$

- If $[\tilde{C}'] = [C_1 + C_2]$, then

$$1 = -K_{X'} \cdot \tilde{C}' = -K_{X'} \cdot (C_1 + C_2),$$

  with $-K_{X'} \cdot C_1 \geq 0$ and with $-K_{X'} \cdot C_2 \geq 0$.
- Either $[C_2] = [e_2 L']$ or $[C_1] = [e_1 L']$.
- $\overline{C'} \subset W'$ strict transform of $\tilde{C}' \subset X'$. Then $[\overline{C'}]$ generates an extremal ray because $\tilde{\phi}$ has contracted all the rational curves in $\mathbb{R}[L']$.
- The locus of the extremal ray $[\overline{C'}]$ will determine the type of the associated elementary Mori contraction from W' onto a $\mathbb{Q}$-factorial Fano variety W. For a congruence the locus on W' is a divisor and hence the associated contraction is birational.

Now we specialize the previous setting to the smooth rational surface $S_{38} \subset \mathbb{P}^5$ described before. The surface S is cut out by 10 cubics satisfying the $\mathcal{K}_3$ condition, and the map

$$\Phi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^9,$$

defined by $|H^0(\mathcal{I}_S(3))|$, is birational onto its image Z.

The exceptional locus of Φ, *i.e.* $\mathrm{Trisec}(S)$, consists of

- ten planes (contracted to ten points), and
- an irreducible threefold B (contracted to a Veronese surface), which is a degeneration of the Bordiga scroll of degree 6 and sectional genus 3.

A trisecant flop for the rationality of the cubics in $\mathcal{C}_{38}$

Flop small contraction and congruence of 5-secant conics for a Coble surface

Let $X \subset \mathbb{P}^5$ be a general cubic fourfold containing a surface $S_{38} \subset \mathbb{P}^5$ as above, and let $\varphi : X \dashrightarrow Y = Z \cap \mathbb{P}^8 \subset \mathbb{P}^8$ be the restriction of $\Phi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^9$ to X.

- X is a general cubic fourfold in $\mathcal{C}_{38}$; in particular, it cannot contain any of the ten planes in the exceptional locus of Φ.
- The exceptional locus of φ is a rational scroll surface of degree 8 with 6 nodes.
- The exceptional locus of the induced morphism

$$\tilde{\varphi} : X' = \mathrm{Bl}_{S_{38}} X \dashrightarrow Y \subset \mathbb{P}^8$$

  is a smooth rational surface, which is contracted to a rational normal quartic curve.

In particular, $\tilde{\varphi}$ is a flop small contraction and all the hypothesis for the existence of the Trisecant Flop are satisfied.

Associated K3 surfaces for the cubics in $\mathcal{C}_{38}$

In conclusion, we have a commutative diagram:

$$\begin{tikzcd}
 & & \mathrm{Bl}_T X' = \mathrm{Bl}_R W' \arrow[dll, "\sigma"'] \arrow[drr, "\omega"] & & \\
X' \arrow[rrrr, dashed, "\tau"] \arrow[d, "\lambda"'] \arrow[drr, "\tilde{\varphi}"] & & & & W' \arrow[d, "\nu"] \arrow[dll, "\tilde{\psi}"'] \\
X \arrow[rr, dashed, "\varphi"] \arrow[rrrr, dashed, bend right, "\mu"'] & & Y & & \mathbb{P}^4 \arrow[ll, dashed, "\psi"'] \arrow[llll, dashed, bend right, "\mu^{-1}"']
\end{tikzcd}$$

$\psi : |H^0(\mathcal{I}_U(5))|$

$\mu : |H^0(\mathcal{I}_S^2(5))|$

$\mu^{-1} : |H^0(\mathcal{I}_U^2(9))|.$

It turns out that the surface $U \subset \mathbb{P}^4$ is a smooth surface of degree 12 and sectional genus 14 cut out scheme-theoretically by 9 quintics.

The surface U had been also constucted by Decker, Ein, and Schreyer (1993), who showed that it is the blow-up at eleven points of a minimal K3 surface of degree 38 and genus 20 in $\mathbb{P}^{20}$.

Section 6

Rationality of the cubics in $\mathcal{C}_{42}$

A new divisor of rational GM fourfolds

- In [Hoff and Staglianò 2019], using MACAULAY2 the authors constructed a smooth rational surface $S \subset \mathbb{P}^8$ of degree 9 and sectional genus 2 contained in a del Pezzo fivefold $Y = \mathbb{G}(1,4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$ with class $9\sigma_{3,1} + 3\sigma_{2,2}$.
- They showed that inside Y there is a 25-dimensional family of such surfaces, from which they deduced that the closure of the family of GM fourfolds containing such a surface coincides with $\mathfrak{p}^{-1}(\mathcal{D}_{20})$.
- The surface S inside Y admits a congruence of 3-secant conics, so that the rationality of the general GM fourfold X with $S \subset X \subset Y$ follows.
- Moreover, there exists a commutative diagram

$$\begin{tikzcd} & M \subset \mathbb{P}^{12} & \\ X \arrow[ur, dashed, "m_1"] \arrow[rr, dashed, "|H^0(\mathcal{I}_S^2(3))|"] & & \mathbb{P}^4 \arrow[ul, dashed, "m_2"'] \end{tikzcd}$$

where m_1 and m_2 are birational maps defined, respectively, by the linear system of quadrics through S, and by the linear system of quintics through the projection in $\mathbb{P}^4$ of a minimal K3 surface of degree 20 in $\mathbb{P}^{11}$.

Birational representations in $\mathbb{P}^5$ of the surface of degree 9 and sectional genus 2

Recall that inside a del Pezzo fivefold $Y \subset \mathbb{G}(1,4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$ there are two types of planes: a 3-dimensional family of planes with class $\sigma_{2,2}$, and a 4-dimensional family of planes with class $\sigma_{3,1}$.

- By projecting from a plane of the first type, we get a birational map $p_{\sigma_{2,2}} : Y \dashrightarrow \mathbb{P}^5$ whose inverse is defined by the quadrics through a cubic scroll in $\mathbb{P}^4 \subset \mathbb{P}^5$.
- By projecting from a plane of the second type, we get a dominant rational map $p_{\sigma_{3,1}} : Y \dashrightarrow Q \subset \mathbb{P}^5$ onto a smooth quadric hypersurface $Q \subset \mathbb{P}^5$ whose general fibre is a line.

Let $S \subset Y$ be a rational surface of degree 9 and sectional genus 2 as before.

- A general projection of the first type sends S into a surface $S_{42} \subset \mathbb{P}^5$ of degree 9 and sectional genus 2 cut out 9 by cubics and having 5 non-normal nodes.
- A general projection of the second type sends S into a surface of $S_{48} \subset \mathbb{P}^5$ of degree 9 and sectional genus 2 cut out by 4 cubics and one quadric, and having 6 non-normal nodes.

Description of the divisors $\mathcal{C}_{42}$ and $\mathcal{C}_{48}$

From the number of non-normal nodes of the projected surfaces in $\mathbb{P}^5$ and from a standard parameter count (see [Russo and Staglianò, 2019b]), it follows that:

$$\mathcal{C}_{42} = \overline{\{[X] \in \mathcal{C} : X \text{ contains a } S_{42} = p_{\sigma_{2,2}}(S) \text{ of a general } S \subset Y\}}$$

$$\mathcal{C}_{48} = \overline{\{[X] \in \mathcal{C} : X \text{ contains a } S_{48} = p_{\sigma_{3,1}}(S) \text{ of a general } S \subset Y\}}$$

In particular, we have that $\mathcal{C}_{42}$ and $\mathcal{C}_{48}$ are uniruled. ($\mathcal{C}_{42}$ is actually unirational by a very recent result of Farkas and Verra (2019).)

We are not aware of other explicit descriptions of $\mathcal{C}_d$ with $d > 48$ via irreducible surfaces $S_d \subset \mathbb{P}^5$, and of any alternative description of $\mathcal{C}_{48}$.

Rationality for $\mathcal{C}_{42}$ using the surface $S_{42} = p_{\sigma_{2,2}}(S)$

Let $S_{42} = p_{\sigma_{2,2}}(S) \subset \mathbb{P}^5$ be a general projection of a general surface $S \subset Y$. If $p \in \mathbb{P}^5$ is a general point, we have

- $9 = \#\{2$–secant lines to S_{42} passing through $p\}$;
- $7 = \#\{5$–secant conics to S_{42} passing through $p\}$;
- $1 = \#\{8$-secant twisted cubic to S_{42} passing through $p\}$.

In particular, the surface S_{42} admits a congruence of 8-secant twisted cubics, from which we deduce the rationality for the general cubic fourfold $[X] \in \mathcal{C}_{42}$.

Moreover, the linear system $|H^0(\mathcal{I}^3_{S_{42}}(8))|$ of hypersurfaces of degree 8 with points of multiplicity 3 along S_{42} gives a dominant rational map

$$\mu : \mathbb{P}^5 \dashrightarrow W = \mathbb{G}(1,4) \cap \mathbb{P}^7 \subset \mathbb{P}^7.$$

The restriction of μ to a general cubic $X \supset S_{42}$ is a birational map $X \dashrightarrow W$, whose inverse is defined by the restriction to W of the linear system of hypersurfaces of degree 8 with points of multiplicity 3 along a special projection of a minimal K3 surface of degree 42 and genus 22.

Section 7

Considerations on the further admissible cases

For $d \geq 86$ admissible the divisor $\mathcal{C}_d$ is not uniruled

- The framework of the Trisecant Flops allowed us to give uniform proofs of the rationality of a general cubic fourfold in $\mathcal{C}_d$ for $d = 14, 26, 38, 42$.
- It emerged that the existence of the associated $K3$ surface is essentially equivalent to the existence of a congruence of $(3e-1)$–secant curves of degree e to a surface $S_d \subset \mathbb{P}^5$. We always used in an essential way that $|H^0(\mathcal{I}_{S_d}(3))|$ defined a birational map onto the image.
- $\dim |H^0(\mathcal{I}_{S_d}(3))| \geq 1$ for general S_d implies $\mathcal{C}_d$ uniruled.
- For $d \geq 86$ admissible the divisor $\mathcal{C}_d$ is not uniruled.
- For $d \geq 86$ admissible, one has might try to construct directly a surface U and a rational fourfold W containing U with a birational map to a general cubic in $\mathcal{C}_d$ of the kind described above.
- For $d \geq 86$ one might construct surfaces $S_d \subset \mathbb{P}^5$ representing $\mathcal{C}_d$. On $X' = \mathrm{Bl}_{S_d} X$ the divisor $-K_{X'}$ is nef but not big. By Shokurov Theorem $|-mK_{X'}|$ is generated by global sections and big. There should exists a Trisecant Flop $\tau : X' \dashrightarrow W'$, if S_d is the *right surface*. Then, one might recover $U \subset W$ with W rational.
- The difficulty of the Conjecture for $d \geq 86$ seems to be closely related with $X \subset \mathbb{P}^5$ and less with the global geometry of *a right* $S_d \subset \mathbb{P}^5$.

Section 8

Some references

Some references

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