On stratified vector bundles in characteristic p

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- char. k=0: stratified vector bundles are the same as vector bundles endowed with an algebraic connection, with vanishing curvature. Thus if $k = \mathbb{C}$ and X is proper, there is a (Riemann-Hilbert) correspondence

$$\{\text{stratified vector bundles of rank } n \text{ on } X\} \leftrightarrow \{\text{representations } \pi_1^{top}(X_{an}) \to \mathrm{GL}_n(\mathbb{C})\}$$

- char.$k = p > 0$: N. Katz obtained an equivalence

$$\{\text{stratified vector bundles of rank } r \text{ on } X\}$$
$$\updownarrow$$
$$\{\text{sequences } (\mathcal{E}_n, \sigma_n)_{n\geq 0}, \text{ where each } \mathcal{E}_n \text{ is a vector bundle of rank } r, \text{ and } \sigma_n : F^*\mathcal{E}_{n+1} \xrightarrow{\cong} \mathcal{E}_n \text{ is an isomorphism}\}$$

Here $F : X \to X$ is the (absolute) Frobenius morphism.

In one direction, the Katz correspondence is given by

$$(\mathcal{E}_n, \sigma_n)_{n\geq 0} \mapsto \{\mathcal{E}_0, \text{ endowed with its 'tautological' } \mathcal{D}_X\text{-structure.}\}$$

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The 'tautological' $\mathcal{D}_X$-structure is given along similar lines.

Basic Ref: Gieseker, D., "Flat vector bundles and the fundamental group in non-zero characteristics",
Ann. Sc. Norm. Pisa (1975).

Finite stratified bundles

If $\rho : \pi_1^{et}(X) \to GL_n(k)$ is a continuous representation (i.e. ρ has finite image), and $f : Y \to X$ is the Galois étale covering associated to $\ker \rho$, then ρ defines descent data on $\mathcal{O}_Y^{\oplus n}$, and so determines a vector bundle $\mathcal{E}$ on X with a $\pi_1^{et}(X)$-equivariant isomorphism

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Such a stratified bundle $\mathcal{E}$ is called a *finite stratified bundle*.

Finite stratified bundles

Theorem

(Lange-Stuhler, 1977) If k is alg. closed of characteristic $p > 0$, then a vector bundle $\mathcal{E}$ satisfying $(F^N)^*\mathcal{E} \cong \mathcal{E}$, for some $N \geq 1$, is a finite stratified bundle. The converse holds with $k = \overline{\mathbb{F}}_p$, or if $\mathcal{E}$ is stable.

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In one direction, if E corresponds to a representation ρ which is conjugate to a representation into $\mathrm{GL}_n(\mathbb{F}_q)$, with $q = p^N$, then $(F^N)^*\mathcal{E} \cong \mathcal{E}$. (This condition on ρ may not hold unless $k = \overline{\mathbb{F}}_p$, or ρ is irreducible.)

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The converse is a sort of “non-abelian Artin-Schreier” argument, using the Lang torsor.

In characteristic 0, it is easy to see that if X is proper, and $\pi_1^{et}(X) = 0$, then all stratified vector bundles are trivial (that is, isomorphic to $\mathcal{O}_X^{\oplus n}$ as a stratified bundle, for some n).

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A similar result holds even for non-proper X, if we restrict ourselves to stratified bundles with *regular singularities at infinity* (i.e., admitting a coherent extension to a good compactification as a 'logarithmic connection'). This is a consequence of results of Deligne.

Gieseker's conjectures

Gieseker conjectured the following.

Conjecture

Let k be an alg. closed field k of char. $p > 0$.
(i) If X is smooth and projective over k, with $\pi_1^{et}(X) = 0$, then all stratified bundles on X are trivial.

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(i) If X is smooth and projective over k, with $\pi_1^{et}(X) = 0$, then all stratified bundles on X are trivial.
(ii) If $X = \overline{X} \setminus D$ where $\overline{X}$ is smooth and projective over k, D an SNCD, and if $\pi_1^{tame}(X) = 0$, then all regular singular stratified bundles on X are trivial.

Gieseker gave a suitable definition of "regular singular stratified bundles" in the setup of (ii).

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Subsequently, Esnault and I used related techniques to get some further results, which I discuss next.

Our results-1

Theorem

(E-VS,2014) Let X be smooth quasi-projective over $\overline{\mathbb{F}}_p$, with $\pi_1^{et}(X) = 0$. Assume there exists a normal projective variety $\overline{X}$ containing X as a dense open, such that $\operatorname{codim}_{\overline{X}}(\overline{X} \setminus X) \geq 2$. Then all stratified bundles on X are trivial.

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We note that $\pi_1^{et}(X) = 0$ implies $\Gamma(X, \mathcal{O}_X) = k$ (since $\pi_1^{et}(\mathbb{A}^1_k) \neq 0$); the existence of a compactification with "small" boundary is a similar, but stronger, condition.

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We note that $\pi_1^{et}(X) = 0$ implies $\Gamma(X, \mathcal{O}_X) = k$ (since $\pi_1^{et}(\mathbb{A}_k^1) \neq 0$); the existence of a compactification with "small" boundary is a similar, but stronger, condition.

On the other hand, the result is for $\pi_1^{et} = 0$, rather than π_1^{tame}, and regular singularities are not imposed.

Our results-2

Theorem

(E-VS,2017) Let $f : Y \to X$ be a morphism of smooth projective varieties over an alg. closed field k of char. $p > 0$ such that $f_* : \pi_1^{et}(Y) \to \pi_1^{et}(X)$ is trivial. Then any stratified bundle on X has trivial pullback to Y, as a stratified bundle.

This is a relative version of the Gieseker Conjecture (i). There is related work of X. Sun, giving another approach.

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This is a relative version of the Gieseker Conjecture (i). There is related work of X. Sun, giving another approach.
But it is not the only possible such statement:

Question

If X, Y are projective smooth over an alg. closed k of char. $p > 0$, such that $f : Y \to X$ induces an iso. on π_1^{et}, does f^* induce an equivalence on stratified bundles?

I believe this is still an open question.

An example

Let Y be a nonsingular proper variety over an alg. closed k of char. $p > 0$. Let $\mathcal{L} \in \mathrm{Pic}\,(Y)$ be an invertible sheaf, such that
(i) $\mathcal{L} \not\cong \mathcal{O}_Y$
(ii) there exists a non-zero $s \in \Gamma(Y, \mathcal{L})$. Define

$$X = \mathbf{Spec}_Y \oplus_{n \geq 0} \mathcal{L}^{-n}.$$

Then the structure morphism $f : X \to Y$ is an $\mathbb{A}^1$-bundle, and X is in particular non-proper; also $\Gamma(X, \mathcal{O}_X) = k$.

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Claim: $f_* : \pi_1^{et}(X) \to \pi_1^{et}(Y)$ is an isomorphism.

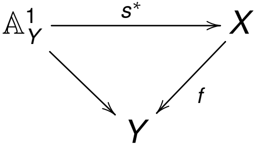
In particular, taking Y to be (say) a smooth complete intersection of dimension ≥ 2, we get plenty of examples of simply connected X.

An example

To prove the Claim, consider the inclusion $\mathcal{L}^{-1} \hookrightarrow \mathcal{O}_Y$ determined by s. This gives a graded inclusion

$$\oplus_{n\geq 0}\mathcal{L}^{-n} \hookrightarrow \mathcal{O}_Y[t], \;\; s^n : \mathcal{L}^{-n} \hookrightarrow \mathcal{O}_Y t^n$$

and hence on applying $\mathbf{Spec_Y}$, we obtain a commutative diagram of $\mathbb{A}^1$-fibrations



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(ii) if we fix a base point $y \in Y$, then the pointed inclusions

$$(Y, y) \to (\mathbb{A}^1_Y, (0, y)) \text{ (the zero section)},$$

$$(\mathbb{A}^1_k, 0) \to (\mathbb{A}^1_Y, (0, y)) \text{ (fibre over } y)$$

induce an *isomorphism*

$$\pi_1^{et}(Y, y) \times \pi_1^{et}(\mathbb{A}^1_k, 0) \to \pi_1^{et}(\mathbb{A}^1_Y, (0, y)) = \pi_1^{et}((\mathbb{A}^1_k, 0) \times (Y, 0)).$$

Here (ii) holds even though $\mathbb{A}^1_k$ is non-proper, since the other factor Y is proper.

An example

Now take $y \in Y$ to be a point with $s(y) = 0$ (such a point y always exists, since $\mathcal{L}$ is a *nontrivial* line bundle). We then see that the composition

$$(\mathbb{A}^1_k, 0) \to (\mathbb{A}^1_Y, (0, y)) \to (X, s^*(0, y))$$

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Hence the compostion

$$\pi_1^{et}(Y, y) \times \pi_1^{et}(\mathbb{A}^1_k, 0) \xrightarrow{\cong} \pi_1^{et}(\mathbb{A}^1_Y, (0, y)) \twoheadrightarrow \pi_1^{et}(X, s^*(0, y))$$

is trivial on the second factor, i.e. the inclusion $Y \to X$, as the 0-section of $f : X \to Y$, induces a surjection on π_1^{et}. This easily gives the Claim, that f_* induces an iso. on π_1^{et}.

The argument of Esnault-Mehta

Fix a very ample $\mathcal{O}_X(1)$, so that there are associated notions of Hilbert polynomials, slopes μ, stability, Harder-Narasimhan filtrations, moduli etc.

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Observe that under Frobenius pullbacks, we have

$$\mu((F^n)^*\mathcal{G}) = p^n\mu(\mathcal{G}), \quad \mu_{max}((F^n)^*\mathcal{G}) \geq p^n\mu_{max}(\mathcal{G}),$$

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Hence if $\mathcal{E} = (F^n)^*\mathcal{G}$ with $p^n > (\mu_{max}(\mathcal{E}) - \mu_{min}(\mathcal{E}))(\operatorname{rank}\mathcal{E})$, then *$\mathcal{G}$ must be μ-semistable*.

For stratified bundles $\mathbb{E} = (\mathcal{E}_n, \sigma_n)_{n\geq 0}$ this implies the following.

- The Hilbert polynomials $\chi(\mathcal{E}_n(m)) \in \mathbb{Q}[m]$ all coincide with $(\operatorname{rank} \mathbb{E})\chi(\mathcal{O}_X(m))$
- Any $\mathbb{E}$ has a Jordan-Holder filtration by stratified subbundles

$$\mathbb{E} \supset F^1\mathbb{E} \supset \cdots \supset F^s\mathbb{E} \supset F^{s+1}\mathbb{E} = 0$$

  such that, for some $n_0 = n_0(\mathbb{E}) \geq 0$, the vector bundles $(F^i\mathbb{E}/F^{i+1}\mathbb{E})_n$ occuring in the quotient stratifications are all μ-stable, for all $n \geq n_0$.

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So it suffices to prove that (i) all *irreducible* stratified bundles are trivial, and (ii) all *unipotent* stratified bundles are trivial.

Here (ii) follows easily from Artin-Schreier theory, so (i) is the main point to prove.

We'll use a result of Hrushovski, originally obtained from a Model Theory perspective:

Theorem

(Hrushovski) Let $\Phi : Z \dashrightarrow Z$ be a dominant rational map of $\overline{\mathbb{F}}_p$-schemes which is defined over $\mathbb{F}_q$. Then the graph $\Gamma_\Phi \subset Z \times Z$ has a Zariski dense set of points $(x, F_q^s(x))$ where x is a closed point at which Φ is defined, F_q is the geometric Frobenius (associated to the $\mathbb{F}_q$ structure), and s is some positive integer (which may depend on x).

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Hrushovski's results are now obtainable using "standard" arithmetic geometry methods (Y. Varshavsky, and K. V. Shuddhodan).

Esnault-Mehta (continued)

If X has an irreducible stratified $\mathbb{E}$ of rank r, where we may also assume $\mathcal{E}_n$ are all μ-stable, and if $\mathcal{M}^r_X$ is the moduli of μ-stable rank r bundles $\mathcal{E}$ with Hilbert polynomial $\chi(\mathcal{E}(m)) = r\chi(\mathcal{O}_X(m))$, then each $\mathcal{E}_n$ in $\mathbb{E}$ gives a point $x_n \in \mathcal{M}^r_X(k)$. Using this, one can construct a Zariski closed $Z \subset \mathcal{M}^r_X$, together with a dominant rational map $\Phi : Z \dashrightarrow Z$ such that $\Phi([\mathcal{E}]) = [F^*\mathcal{E}]$.

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If $k = \overline{\mathbb{F}}_p$, then Hrushovski's theorem yields Φ-period points of Z, which by Lange-Stuhler, are finite stratified bundles, lying in $\mathcal{M}^r_X$. Hence X has nontrivial étale fundamental group, unless $r = 1$, and $\mathbb{E}$ is trivial.

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In general, one makes a (slightly tricky) specialization argument, using that a specialization of a proper, simply connected variety is itself simply connected.

Comments on E-VS I

I recall the statement:

Theorem

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We need the condition that $k = \overline{\mathbb{F}}_p$ because we do not know an answer to the following:

Question

Let $\overline{X}$ be a normal projective k-variety with simply connected smooth locus. Do "most" specializations of $\overline{X}$ to $\overline{\mathbb{F}}_p$ have the same property?

We also use a (corollary of a) Lefschetz theorem of J-B. Bost:

Theorem

Let $\overline{X}$ be a normal projective variety, X its smooth locus, and $C \subset \overline{X}$ a complete intersection curve which is contained in X. Then $\pi_1^{strat}(C) \to \pi_1^{strat}(X)$ is surjective.

Here π_1^{strat} denotes the (Tannakian) group scheme determined by stratified bundles; π_1^{et} turns out to be the analogue of the "group of connected components", in the smooth projective case.

Comments on E-VS 1 (continued)

Let $\overline{X} \supset X$ be the compactification with “small” boundary, on which we fix a very ample $\mathcal{O}_{\overline{X}}(1)$, and let $j : X \hookrightarrow \overline{X}$ the inclusion.

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$$F^* j_* \mathcal{E}_{n+1} \xrightarrow{\theta_{n+1}} j_* F^* \mathcal{E}_{n+1} \xrightarrow[\cong]{j_* \circ \sigma_n} j_* \mathcal{E}_n$$

which may not be isomorphisms.

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which may not be isomorphisms.
The Mehta-Esnault argument can be adapted, after making the following key observation:

Proposition

There are a finite number of polynomials $p_1(t), \ldots, p_m(t) \in \mathbb{Q}[t]$ such that for any stratififed bundle $\mathbb{E} = (\mathcal{E}_n, \sigma_n)$ on X of rank r, there exists $n_0 = n_0(\mathbb{E})$ so that if $n \geq n_0$, the Hilbert polynomial of $j_*\mathcal{E}_n$ on $\overline{X}$ is one of the $p_j(t)$.

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The first is the Boundedness Theorem for a family of μ-semistable sheaves whose first three Hilbert coefficients are bounded, in char. $p > 0$. This reduces the Proposition to the surface case.

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In that case, arguments from an older work of Langer's on "c_2" for sheaves on a normal (complex) surface, adapted here, shows that if $\pi : \widetilde{X} \to \overline{X}$ is a resolution of singularities, then

$$\deg_{\overline{X}} \left[ch_2 \left((\pi^* j_* \mathcal{E}_n)^{\vee\vee} \right) \right]$$

is bounded by a constant depending only on r and the geometry of $\pi : \widetilde{X} \to \overline{X}$, for all $n \geq n_0 = n_0(\mathbb{E})$.

Comments on E-VS 2

I restate our second result:

Theorem

(E-VS,2017) Let $f : Y \to X$ be a morphism of smooth projective varieties over an alg. closed field k of char. $p > 0$ such that $f_* : \pi_1^e(Y) \to \pi_1^{et}(X)$ is trivial. Then any stratified bundle on X has trivial pullback to Y, as a stratified bundle.

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Following the reasoning of Mehta-Esnault, one sees readily that any *irreducible* stratified $\mathbb{E}$ has trivial pullback.
The new issue is of non-trivial extensions; the Jordan-Holder property implies any stratified $\mathbb{E} = (\mathcal{E}_n, \sigma_n)$ has *unipotent* pullback.

If in addition one has that each $\mathcal{E}_n$ is "F-nilpotent" (ie, has trivial pullback under some power of Frobenius), then we are able to show $f^*\mathbb{E}$ is in fact trivial (by a sort of Artin-Schreier argument).

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THANK YOU FOR YOUR ATTENTION!