Seshadri and packing constants

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1 Introduction

In this note we consider a certain connection between the (local) positivity of line bundles on algebraic varieties and the symplectic packing of balls into symplectic manifolds.
Let X be a smooth complex projective variety with an ample line bundle L. Local positivity of the bundle may be measured by Seshadri constants (introduced by Demailly in [9] and then generalized by Xu, [25]). These constants measure, roughly speaking, how small can be the ratio between the degree of a curve and the sum of the curve's multiplicities in given points.
On the other hand, if X is a smooth projective variety, it is also a symplectic manifold, with the symplectic form given by $c_1(L)$. We may then consider a symplectic packing of X, ie a symplectic embedding of a disjoint union of standard balls into X. The amount of the volume of X which may be filled by the images of the symplecticly embedded balls is measured by so called packing constants - introduced and investigated by Gromov, McDuff, Polterovich and Biran, see [11], [15], [3].
In the first part of this note we collect some results concerning both kinds of constants and then we show a connection between them. In the second part of the paper we consider the special case of toric varieties, and we show that there is a formula connecting Seshadri constants in fixed points of a toric variety with packing constants, where packing in this case is symplectic and equivariant.

2 Seshadri constants

In this chapter we recall the definition and basic facts about Seshadri constants.

Let X be a smooth complex projective variety of dimension n, with an ample line bundle L and let $P_1, ..., P_N$ be N different points on X.

Definition 1. *Seshadri constant of L in $P_1, ..., P_N$ is defined as the number*

$$\varepsilon(X, L, P_1, ..., P_N) := inf \left\{ \frac{LC}{mult_{P_1} C + ... + mult_{P_N} C} \right\},$$

where infimum is taken over the set of all curves on X passing through at least one P_i.
Equivalently

$$\varepsilon(X, L, P_1, ..., P_N) := \ sup \ \{\varepsilon \mid \pi^* L - \varepsilon(E_1 + ... + E_N) \textit{ is numerically effective}\},$$

where $\pi : \tilde{X} \longrightarrow X$ is the blow up of X in $P_1, ..., P_N$, with exceptional divisors $E_1, ..., E_N$.

If the points $P_1, ..., P_N$ are very general on X (ie they are outside a countable sum of algebraic subsets of $Hilb_N(X)$) we will write $\varepsilon(X, L, N)$ instead of $\varepsilon(X, L, P_1, ..., P_N)$. (See [14] Example 5.1.11 for the fact that $\varepsilon(X, L, P_1, ..., P_N) \geq \varepsilon(X, L, Q_1, ..., Q_N)$ if $P_1, ..., P_N$ are very general on X and $Q_1, ..., Q_N \in X$).

Remark 2. *From Seshadri criterion of ampleness (see eg [14], Theorem 1.4.13) it follows that for an ample line bundle L on X we have*

$$0 < \varepsilon(X, L, P_1, ..., P_N) \leq \sqrt[n]{\frac{L^n}{N}}.$$

Finding the exact value of Seshadri constants is in most cases a difficult problem. For $X = \mathbb{P}^2$ with $L = \mathcal{O}_{\mathbb{P}^2}(1)$ the exact values of $\varepsilon(X, L, N)$ are known only if $N \leq 9$ or $N = k^2, k \in \mathbb{N}$. Namely, for $N = 1, ..., 9$ we have $\varepsilon(X, L, N) = 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{5}, \frac{2}{5}, \frac{3}{8}, \frac{6}{17}, \frac{1}{3}$ respectively; for $N = k^2$, $\varepsilon(X, L, N) = \frac{1}{k}$.
Also, for $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the line bundle of type $(1, 1)$, we know the values of $\varepsilon(X, L, N)$ only for $N \leq 8$ or $N = 2k^2, k \in \mathbb{N}$: $\varepsilon(X, L, N) = 1, 1, \frac{2}{3}, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{8}{15}, \frac{1}{2}$ for $N = 1, ...8$ respectively and $\varepsilon(X, L, N) = \frac{1}{k}$ for $N = 2k^2$.
The famous conjecture of Nagata states that $\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), N) = \sqrt{\frac{1}{N}}$ (so it is maximal possible) for all integers $N \geq 10$ (cf [12], [16]).
We do not know so far a single example of Seshadri constant with an irrational value. The main obstacle in proving that the constant is irrational is that at present we are able to compute the constant only when either $\sqrt[n]{\frac{L^n}{N}}$ is rational or when we can find a curve C such that

$$\varepsilon(X, L, P_1, ..., P_N) = \frac{LC}{\mathrm{mult}_{P_1} C + ... + \mathrm{mult}_{P_N} C}.$$

Then $\varepsilon(X, L, P_1, ..., P_N)$ is rational (see [17]). If it is less than $\sqrt[n]{\frac{L^n}{N}}$, we say that the constant is submaximal, and C is called a submaximal curve.
The problem is that we do not have (at the moment) many ways of proving the nonexistence of submaximal curves. This makes it difficult proving that the Seshadri constants are maximal.

3 Packing numbers

Let us now look at the symplectic side of the problem. Recall, that a symplectic manifold is a smooth real manifold X (of real dimension $2n$) with a closed nondegenerate differential $2-$form ω, so the volume form on X is given by $\frac{1}{n!}\omega^{\wedge n}$. A basic example is $\mathbb{R}^{2n}$ with the 2-form $\omega_0 := dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$. As mentioned in the Introduction, another example of a symplectic manifold is produced by a smooth complex projective variety X with an ample line bundle L. This variety may be treated as a real $2n$ dimensional manifold with the closed nondegenerate differential 2-form given by the first Chern class of L, $\omega_L = c_1(L)$. Then the volume of X equals $\mathrm{vol}X = \frac{1}{n!}L^n$.
For two symplectic manifolds, (X_1, ω_1) and (X_2, ω_2) we define a symplectic embedding of X_1 to X_2.

Definition 3. *We say that $f : (X_1, \omega_1) \longrightarrow (X_2, \omega_2)$ is a symplectic embedding if f is a $\mathcal{C}^\infty$-diffeomorphism onto the image and*

$$f^*\omega_2 = \omega_1.$$

Consider the symplectic packing problem: Given a symplectic manifold (X, ω) find the maximal radius R such that there exists a symplectic embedding of a disjoint union of N euclidean balls of radius R into a given symplectic manifold (X, ω),

$$f : \coprod_{i=1}^{N} (B^{2n}(R), \omega_0) \longrightarrow (X, \omega).$$

Let the volume of X be finite. Then there is an obvious upper bound on R, $N\mathrm{vol}(B^{2n}(R)) \leq \mathrm{vol}(X)$. However, it may happen, that the volume bound is not the only obstacle for packing the balls into X, and even if the volume of X is infinite, there may be obstructions for packing balls into X. Let us recall here the Gromov Nonsqueezing Theorem (see [11]), which says that if there exists a symplectic embedding of a ball $(B^{2n}(R), \omega_0)$ into $(B^2(\epsilon) \times \mathbb{R}^{2n-2}, \omega_0)$, then $R \leq \epsilon$.

Let (X, ω) be a symplectic manifold and assume that vol$X$ is finite. Packing constants (or packing numbers) measure how much of the volume of (X, ω) may be filled with the symplectic images of euclidean balls (see [3],[15]).

Definition 4. *Let (X, ω) be a symplectic manifold and let N be a natural number. A symplectic packing constant is defined as*

$$v_N(X, \omega) := \sup \left\{ \frac{N vol(B^{2n}(R))}{vol(X)} \right\},$$

where the supremum is taken over all R, such that there exists a symplectic packing $f : \coprod_{i=1}^{N} (B^{2n}(R), \omega_0) \longrightarrow (X, \omega)$.
If $v_N(X, \omega) = 1$ we say that full packing exists.
By $v_N(X, \omega, P_1, ..., P_N)$ we will denote the analogously defined packing constant, with the images of the centers of the balls in $P_1, ..., P_N$.
If (X, ω) are clear from the context, we will write v_N instead of $v_N(X, \omega)$.

Following Lazarsfeld in [13] we define similar constants for embeddings being both symplectic and holomorphic:

Definition 5. *Let (X, ω) be a symplectic and holomorphic manifold and let N be a natural number. A symplectic and holomorphic packing constant is defined as*

$$v_N^h(X, \omega) := \sup \left\{ \frac{N vol(B^{2n}(R))}{vol(X)} \right\},$$

where the supremum is taken over all R, such that there exists a symplectic and holomorphic packing $f : \coprod_{i=1}^{N} (B^{2n}(R), \omega_0) \longrightarrow (X, \omega)$.

There are many interesting results about the constants $v_N(X, \omega)$, cf eg [3], [4], [15]. In his famous paper [4], Biran proved the following theorem (here quoted in the version restricted to algebraic surfaces with the symplectic form ω_L):

Theorem 6. *Let (X, L) be a smooth projective algebraic surface, treated as a four dimensional symplectic manifold with the symplectic form ω_L. Then there exists a number N_0, such that for any $N \geq N_0$ there exists full packing, ie $v_N(X, \omega) = 1$. Moreover, this N_0 can be taken equal $k_0^2 L^2$ where k_0 is such, that the linear system $|k_0 L|$ contains a curve C of genus at least one.*

4 Connection

It seems that there exists a close connection between Seshadri constants and packing numbers. The possibility that such a connection exists was first observed in [15] and then in [3], [4], [13] and others.
Consider the following examples:

Example 7. Let $X = \mathbb{P}^2$ with $L = \mathcal{O}_{\mathbb{P}^2}(1)$. For $N = 1, ..., 9$ we have $\varepsilon(X, L, N) = 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{5}, \frac{2}{5}, \frac{3}{8}, \frac{6}{17}, \frac{1}{3}$ respectively. In the same range of N, we have (see [3]): $v_N = 1, \frac{1}{2}, \frac{3}{4}, 1, \frac{20}{25}, \frac{24}{25}, \frac{63}{64}, \frac{288}{289}, 1$, so $\varepsilon(X, L, N) = \sqrt{\frac{L^2 v_N}{N}}$ here.

Example 8. For $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the line bundle of type $(1, 1)$, we know $\varepsilon(X, L, N)$ for $N \leq 8$: $\varepsilon(X, L, N) = 1, 1, \frac{2}{3}, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{8}{15}, \frac{1}{2}$. From [3] we know that $v_N = \frac{1}{2}, 1, \frac{2}{3}, \frac{8}{9}, \frac{9}{10}, \frac{48}{49}, \frac{224}{225}$, for $N < 9$ and 1 for all $N \geq 9$. Thus here $\varepsilon(X, L, N) = \sqrt{\frac{L^2 v_N}{N}}$ for $N < 9$.

The next set of examples is given by some surfaces with Picard number $\rho = 1$. Recently Szemberg in [23] proved that for a surface X with Picard number one and with $\sqrt{\frac{L^2}{a}} \in \mathbb{N}$ (where a is a positive integer and L is the ample generator of the Picard group of X) the constant $\varepsilon(X, L, a)$ is the maximal possible. Roé and Ross in [20] proved the following result.

Theorem 9. *Let X be a projective variety of dimension n with an ample line bundle L. Let r, s be integers. Then*

$$\varepsilon(X, L, sr) \geq \varepsilon(X, L, s) \cdot \varepsilon(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), r).$$

Let now X be a projective surface with $\rho = 1$ and let L be the ample generator of the Picard group of X. Let $\sqrt{\frac{L^2}{N}} = \frac{p}{q}$ (where $(p, q) = 1$). Then $L^2 = ap^2$ and $N = aq^2$ for a positive integer a. We know that $\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), q^2) = \frac{1}{q}$. Szemberg's result gives that $\varepsilon(X, L, a) = p$. Thus, Theorem 9 implies that

$$\sqrt{\frac{L^2}{N}} \geq \varepsilon(X, L, N) \geq \varepsilon(X, L, a)\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), q^2) = \frac{p}{q},$$

so

$$\varepsilon(X, L, N) = \frac{p}{q}.$$

This gives us the following example.

Example 10. Let X be a surface with the Picard number $\rho = 1$. Let L be the ample generator of the Picard group of X. Assume also that $L^2 + LK_X \geq 0$ - this implies that $|L|$ contains a curve of genus at least one. Thus, from Theorem 6 it follows that for these surfaces $v_N(X, \omega_L) = 1$, for any $N \geq L^2$.
So, we have that if $N \geq L^2$ and $\sqrt{\frac{L^2}{N}} \in \mathbb{Q}$, then

$$\varepsilon(X, L, N) = \sqrt{v_N(X, \omega_L)\frac{L^2}{N}} = \sqrt{\frac{L^2}{N}}.$$

Remark 11. We know by the results of Biran, [4, 3], that if $N \geq 9$ for $\mathbb{P}^2$ with $L = \mathcal{O}_{\mathbb{P}^2}(1)$ or $N \geq 8$ for $\mathbb{P}^1 \times \mathbb{P}^1$ with L of type $(1,1)$, then $v_N = 1$.

All the above speak in favour of the following Biran-Nagata-Szemberg Conjecture (see eg [22]):

Conjecture 12. *For any algebraic surface X, with an ample line bundle L there exists a number N_0, such that for any $N \geq N_0$ we have $\varepsilon(X, L, N) = \sqrt{\frac{L^2}{N}}$.*

We may consider also the following problem:

Problem 13. *For which complex projective surfaces X with an ample line bundle L and the symplectic form given by $\omega_L = c_1(L)$ and for which natural N*

$$\varepsilon(X, L, N) = \sqrt{v_N(X, \omega_L)\frac{L^2}{N}}? \tag{1}$$

We have seen that for $X = \mathbb{P}^2, L = \mathcal{O}_{\mathbb{P}^2}(1)$ and $N \leq 9$ the equality holds. It holds also for $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the line bundle of type $(1,1)$ and $N \leq 8$ or for the surfaces as in Example 10 (and $N = 1$), but it is too much to expect it holds for any X, L and N. Consider the following example.

Example 14. All abelian surfaces (X, L) with line bundle L of type $(1,1)$ are symplectomorphic, so $v_1(X, \omega_L)$ is the same for them (however, unknown so far). On the other hand, if $X = E \times E$, where E is an elliptic curve, then $\varepsilon(X, L, 1) = 1$ and for a generic X we have $\varepsilon(X, L, 1) = \frac{4}{3}$, see [21].

Anyway, it would be very interesting to understand when and why the equality in Problem 13 does or does not hold.
There are also known results giving bounds for Seshadri constants by means of packing constants. In [5] Biran and Cieliebak proved that there is always an inequality in (1).

Theorem 15. *Let X be a projective n-dimensional manifold with an ample line bundle L. Then*

$$\sqrt[n]{v_N(X, \omega_L)\frac{L^n}{N}} \geq \varepsilon(X, L, N).$$

On the other hand, holomorphic and symplectic packing constants give the lower bound. Lazarsfeld in [13] proved the following result.

Theorem 16. *For a projective n-dimensional manifold X with an ample line bundle L we have*

$$\varepsilon(X, L, N) \geq \sqrt[n]{v_N^h(X, \omega_L)\frac{L^n}{N}}.$$

Remark 17. Lazarsfeld's proof of this result is based on the construction of symplectic blowing up, cf [15]. The theorem in [13] is actually stated for $N = 1$, but it can be generalized for $N \geq 1$. For another proof of the result see [24].

5 Special case: toric manifolds

In this part of the paper we consider the special case of toric varieties. On the one hand we have Seshadri constants in a fixed point of a toric variety, on the other hand we may consider a symplectic and equivariant embedding of a ball into the manifold and for such an embedding analogously define the packing constant v_1. It turns out that in this special case the equality (1) holds.

5.1 Seshadri constants on toric manifolds

This chapter is written on the base of Chapter 4 form [2]; for more about toric varieties see eg [10] or [7]. First, let us recall some basic facts about toric manifolds. Let X be a nonsingular compact toric variety, ie an n-dimensional smooth compact complex manifold with an action of a torus $(\mathbb{C}^*)^n$, such that $(\mathbb{C}^*)^n$ is a Zariski open subset of X and the action of $(\mathbb{C}^*)^n$ on itself extends to the action of $(\mathbb{C}^*)^n$ on X.
In what follows we assume that X is smooth and projective. We assume also that the standard volume form ω_0 is normalized in such a way, that the area of the unit disc is one. This implies that $\mathrm{vol}(B^{2n}(R)) = \frac{R^{2n}}{n!}$.
X may be described by means of a fan, $\Delta \subset M$, where $M \cong \mathbb{Z}^{2n}$ is a lattice. In particular, prime torus invariant divisors correspond bijectively to 1-dimensional cones in Δ. The toric variety X with an ample line bundle L may be described by a certain lattice polytope. Every line bundle on X may be written as $L = \mathcal{O}_X(\sum_{i=1}^{\rho} a_i D_i)$, where ρ is the rank of Pic$X$ and D_i are prime action invariant divisors on X. Denote by n_i the lattice generators of the cones corresponding to D_i. We define a polytope of (X, L) as

$$P(X, L) := \{m \in \mathcal{M} | \langle m, n_i \rangle \geq -a_i, \text{ for any } n_i \in M\},$$

where $\mathcal{M}$ denotes the lattice dual to M.
A polytope of (X, L) is called a *Delzant polytope* if there are exactly n edges from each vertex and for each vertex, the first integer points on the edges (originating from the vertex) form the basis of the lattice.
If X is nonsingular, then its polytope is Delzant, and in [8] it is proved that X as above is uniquely determined by its Delzant polytope.
Let us denote by $P(k)$ the set of faces of $P(X, L)$ of dimension k. The elements of $P(k)$ correspond bijectively to the invariant (with respect to the action of

$(\mathbb{C}^*)^n$) subvarieties of X of dimension k; so to each vertex of the polytope corresponds one fixed point of X, each edge corresponds to an invariant curve etc.
For any edge $e \in P(1)$ by $l(e)$ denote the length of e, ie the number of lattice points on e minus one. For a vertex $w \in P(0)$ define

$$s(P(X,L),w) := \min\{l(e)|w \in e\}.$$

In [2] the following theorem is proved:

Theorem 18. *Let x be a fixed point of X, corresponding to the vertex $w \in P(0)$. Then*

$$\varepsilon(X,L,x) = s(P(X,L),w).$$

5.2 Packing one ball into a toric manifold

This chapter is based on the results proved by Pelayo and Schmidt in [18], [19]. Let (X,L) be a smooth projective toric variety, as described in the previous subsection. Then X has a symplectic form $\omega_L = c_1(L)$. It may be seen that the restriction of the $(\mathbb{C}^*)^n$ action to its real subgroup $\mathbb{T}^n = (S^1)^n$ is effective and Hamiltonian, see [6]. Thus, X is a symplectic toric manifold (ie a compact connected symplectic manifold of real dimension $2n$ with an effective and Hamiltonian torus action of $\mathbb{T}^n$, see [6]). For such X we may consider a symplectic and equivariant packing of balls.

Definition 19. *Let (X,L) be a symplectic toric manifold, with the torus action $\psi : \mathbb{T}^n \times X \longrightarrow X$. Let $\Lambda \in Aut(\mathbb{T}^n)$. A subset $B \subset X$ is a Λ-equivariantly embedded ball (of radius r) if there exists a symplectic embedding $f : B(r) \longrightarrow B = f(B(r)) \subset X$, such that*

$$\begin{array}{ccc} \mathbb{T}^n \times B(r) & \xrightarrow{\Lambda\times f} & \mathbb{T}^n \times X \\ \downarrow\ Rot & \odot & \downarrow \psi \\ B(r) & \xrightarrow{f} & X \end{array}$$

where the action Rot: $\mathbb{T}^n \times B(r) \longrightarrow B(r)$ is given by $(\theta_1, ..., \theta_n)(z_1, ..., z_n) \longrightarrow (\theta_1 z_1, ..., \theta_n z_n)$.
B is called an equivariantly embedded ball if there exists $\Lambda \in Aut(\mathbb{T}^n)$, such that B is a Λ-equivariantly embedded ball.

Let $e_1, ..., e_n$ be the standard basis of $\mathbb{R}^n$.

Definition 20. *1. By $\Delta(r)$ we denote the set of points in $\mathbb{R}^n$, belonging to the convex hull of $\{0, r^2e_1, ..., r^2e_n\}$ but not to the convex hull of $\{r^2e_1, ..., r^2e_n\}$.*
2. Let Δ be a Delzant polytope. A subset Δ_w of Δ is called an admissible simplex of radius r with center at the vertex w, if Δ_w is the image of $\Delta(\sqrt{r})$ by an element of $AGL(n,\mathbb{Z})$ which takes the origin to w and the edges of $\Delta(\sqrt{r})$ to the edges of Δ_w meeting w. ($AGL(n,\mathbb{Z})$ is the special affine group of $\mathbb{R}^n$ with integer coefficients).
We define

$$r_w := \max\{r > 0 : \text{ there exists an admissible simplex } \Delta_w \text{ of radius } r\}.$$

Lemma 21. *([19, Lemma 2.10])*

$$r_w = \min\{l(e) | w \in e\}.$$

Thus, keeping the notation of the previous subsection, $r_w = s(P(X, L), w)$.

Remark 22. $\Delta(r)$ is the image of $B(r) \subset \mathbb{C}^n$ by the momentum map (for the action $\mathbb{T}^n \times B(r) \longrightarrow B(r)$ given by $(\theta_1, ..., \theta_n) \cdot (z_1, ..., z_n) = (\theta_1 z_1, ..., \theta_n z_n)$).

We have the following facts.

Theorem 23. *([19, Lemma 2.10]). Let (X, L) be a symplectic toric manifold with Delzant polytope $P(X, L)$. Let $w \in P(0)$ be a vertex of $P(X, L)$ Then, there is an admissible simplex $\Delta_w \subset P(X, L)$ of radius r if and only if $0 \leq r \leq r_w$.*

Theorem 24. *([19, Lemma 2.13]) Let (X, L) be as above, let $\psi : \mathbb{T}^n \times X \longrightarrow X$ and let the momentum map of ψ be $\mu : X \longrightarrow \mathbb{R}^n$. Let $B \subset X$ be a symplecticly and equivariantly embedded ball of radius r and the center mapped to a fixed point $x \in X$. Then $\mu(B)$ is an admissible simplex of radius r^2, with center $\mu(x)$. Conversely, having an admissible simplex Δ_w of radius r, there exists a symplecticaly and equivariantly embedded ball $B \in X$ of radius $\sqrt{r}$, with $\mu(B) = \Delta_w$.*

Summarizing the above, we see that for any symplecticly and equivariantly embedded ball (with the image of the center in a fixed point x of X, corresponding to the vertex w) we have an admissible simplex (and vice versa).
Let $x \in X$ be a fixed point and w the corresponding vertex. From Theorem 23, it follows that if we pack symplecticly and equivariantly a ball into X, so that x is the image of the center of the ball, then the radius of the ball must be such, that $r^2 \leq r_w$. Denote by

$$v_1(X, \omega_L, w) := \sup \frac{vol B^{2n}(r)}{vol X},$$

where supremum is taken over r such that there exists symplectic and equivariant embedding of $B^{2n}(r)$ into X, with the image of the center in x.

5.3 Problem 13 on toric manifolds

From Theorem 18 and from the discussion above we have the following result.

Proposition 25. *For a symplectic toric variety (X, L) with a fixed point x corresponding to the vertex $w \in P(0)$, we have*

$$\varepsilon(X, L, x) = \sqrt[n]{L^n v_1(X, \omega_L, w)}.$$

Proof. From Theorem 18 we know that

$$\varepsilon(X, L, x) = s(P(X, L), w).$$

Taking R being the supremum of radii of all balls, such that there exists a symplectic and equivariant embedding of $B^{2n}(r)$ into X, with the image of the center in w, we have

$$v_1(X, \omega_L, w) = \frac{\mathrm{vol} B^{2n}(R)}{L^n/n!}.$$

On the other hand, we know from Theorems 23 and 24 and from the observation above, that $R^2 = r_w = s(P(X, L), w)$. Thus,

$$v_1(X, \omega_L, w) = \frac{r_w^n/n!}{L^n/n!} = \frac{s(P(X, L), w)^n}{L^n}.$$

So,

$$\sqrt[n]{L^n v_1(X, \omega_L, w)} = s(P(X, L), w),$$

and finally

$$\sqrt[n]{L^n v_1(X, \omega_L, w)} = \varepsilon(X, L, w).$$

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