THE ENTROPY OF LYAPUNOV-OPTIMIZING MEASURES OF SOME MATRIX COCYCLES

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Abstract. We consider one-step cocycles of 2×2 matrices, and we are interested in their Lyapunov-optimizing measures, i.e., invariant probability measures that maximize or minimize a Lyapunov exponent. If the cocycle is dominated, that is, the two Lyapunov exponents are uniformly separated along all orbits, then Lyapunov-optimizing measures always exist, and are characterized by their support. Under an additional hypothesis of nonoverlapping between the cones that characterize domination, we prove that the Lyapunov-optimizing measures have zero entropy. This conclusion certainly fails without the domination assumption, even for typical one-step $\mathrm{SL}(2,\mathbb{R})$-cocycles; indeed we show that in the latter case there are measures of positive entropy with zero Lyapunov exponent.

L'entropie des mesures Lyapunov-optimisantes pour quelques cocycles de matrices

Résumé. Nous considérons des cocycles à un pas de matrices 2×2 et nous nous intéressons à leurs mesures Lyapunov-optimisantes, i.e. aux mesures de probabilité invariantes qui soit maximisent soit minimisent un exposant de Lyapunov. Si le cocycle est dominé, i.e. si les deux exposants de Lyapunov sont uniformément séparés le long toutes les orbites, alors des mesures Lyapunov-optimisantes existent toujours et elles sont characterisées par leurs supports. Sous l'hypothèse supplementaire de non-chévauchement des cônes qui characterisent la domination, nous démontrons que les mesures Lyapunov-optimisantes sont d'entropie nulle. Sans l'hypotèse de domination ce résultat n'est plus vrai, même pour des cocycles à un pas à valeurs dans $\mathrm{SL}(2,\mathbb{R})$; en effet, dans ce cas-là nous démontrons qu'il y a des mesures d'entropie positive dont les exposants de Lyapunov sont nuls.

1. Introduction

Ergodic Optimization is concerned with the maximization or minimization of Birkhoff averages of a given function (called the *potential*) over a given dynamical system: see [Je'06]. A paradigm of this subject is that for sufficiently hyperbolic base dynamics and for typical potentials, optimizing orbits should have low dynamical complexity. This is confirmed in by a recent result by Contreras [C], who showed that the optimizing orbits with respect to generic Lipschitz potentials over an expanding base are periodic. An important component of Contreras' proof is the fact previously shown by Morris [Mo'08] that in this generic situation, optimizing orbits have subexponential complexity (i.e., zero entropy).

In this paper we are interested in Ergodic Optimization in a noncommutative setting. We will replace Birkhoff sums by matrix products, and the quantities we want to maximize or minimize are the associated Lyapunov exponents. We would

Date: January 4, 2014.

like to know whether the low complexity phenomena mentioned above is also typical in this noncommutative setting.

A natural starting point is to consider *one-step* matrix cocycles. In this case, the optimization problems above can be restated in more elementary terms: we are given finitely many square matrices, and we want to find sequences of products of them attaining the maximum or minimum growth rate. Actually these maximization and minimization problems were first considered by Rota and Strang [RS'60] and by Gurvits [G'95], respectively. The associated growth rates are called *joint spectral radius* and *joint spectral subradius*, respectively; they play an important role in Control Theory and there is a large body of literature about them (especially about the former): see the monograph [Ju'09] and references therein. An important contribution to this field was made by Bousch and Mairesse [BMa'02] who showed that the maximizing products are not always periodic, thus disproving the so called *Finiteness Conjecture*. (Of course, for one-step cocycles the parameter space is finite-dimensional and thus perturbative arguments are more difficult.)

In this paper we deal with 2×2 one-step cocycles. We give explicit open conditions that ensure that the Lyapunov-optimizing orbits form a set of low complexity, more precisely of zero topological entropy. These conditions are related to hyperbolicity on the projective space, and are satisfied in some of the counterexamples to the Finiteness Conjecture exhibited in the literature.

In order to appreciate the importance of the hyperbolicity hypotheses, we show that for typical non-hyperbolic one-step $\mathrm{SL}(2,\mathbb{R})$-cocycles, the set of minimizing orbits has positive topological entropy.

Let us proceed with the precise definitions and results.

1.1. **Extremal Lyapunov exponents for 2×2 matrix cocycles.** Let Ω be a compact metric space and let $T\colon \Omega \to \Omega$ be a continuous transformation. Let $A\colon \Omega \to \mathrm{GL}(2,\mathbb{R})$ be a continuous map. The pair (T, A) is a called a 2×2 *matrix cocycle*. We are interested in the following products, which play the role of Birkhoff sums in our noncommmutative setting:

$$A^{(n)}(\omega) := A(T^{n-1}\omega) \cdots A(\omega), \quad \omega \in \Omega,\ n \geqslant 0. \tag{1.1}$$

The *Lyapunov exponents* of the cocycle at a point $\omega \in \Omega$, when they exist, are the limits:

$$\lambda_1(A,\omega) := \lim_{n\to+\infty} \frac{1}{n} \log \|A^{(n)}(\omega)\| , \quad \lambda_2(A,\omega) := \lim_{n\to+\infty} \frac{1}{n} \log \mathfrak{m}(A^{(n)}(\omega)) .$$

where, for definiteness, $\|L\|$ is the Euclidian operator norm of a matrix L, and $\mathfrak{m}(L) := \|L^{-1}\|^{-1}$ is its *mininorm*.

Let $i \in \{1, 2\}$. If μ is a T-invariant probability measure, then $\lambda_i(A,\omega)$ exists for μ-almost every ω, and we denote $\lambda_i(A,\mu) = \int \lambda_i(A,\omega)\, d\mu(\omega)$. If μ is ergodic then $\lambda_i(A,\omega) = \lambda_i(A,\mu)$ for μ-almost every ω.

The *maximal* (or *top*) and *minimal* (or *bottom*) Lyapunov exponents are defined respectively as:

$$\lambda_i^{\top}(A) := \sup_{\mu\in\mathcal{M}_T} \lambda_i(A,\mu) , \quad \lambda_i^{\perp}(A) := \inf_{\mu\in\mathcal{M}_T} \lambda_i(A,\mu) , \tag{1.2}$$

where $\mathcal{M}_T$ denotes the set of all T-invariant Borel probability measures. These four numbers are called the *extremal Lyapunov exponents* of the cocycle.

A basic question is whether the sup's and inf's that appear in (1.2) are attained. The answer is "yes" in the cases of $\lambda_1^\top$ and $\lambda_2^\perp$, and "not necessarily" in the cases of $\lambda_2^\top$ and $\lambda_1^\perp$; see subsection A.3. However, under the assumption of domination (that we will explain next), all sup's and inf's in (1.2) are attained.

1.2. **Domination.** Consider a 2×2 matrix cocycle (T, A) where T is a homeomorphism. Suppose that for each $\omega \in \Omega$ it is given a splitting of $\mathbb{R}^2$ as the sum of two one-dimensional subspaces $e_1(\omega)$, $e_2(\omega)$. We say that this is a *dominated splitting* with respect to the cocycle (T, A) if the following conditions hold:

- equivariance: $A(\omega)(e_i(\omega)) = e_i(T\omega)$ for each $\omega \in \Omega$, $i \in \{1, 2\}$;
- dominance: there are constants $c > 0$ and $\delta > 0$ such that

$$\frac{\|A^{(n)}(\omega)|e_1(\omega)\|}{\|A^{(n)}(\omega)|e_2(\omega)\|} \geqslant ce^{\delta n} \quad \text{for all } \omega \in \Omega \text{ and } n \geqslant 1. \tag{1.3}$$

An important property of dominated splittings is that they are always continuous, that is, e_1 and e_2, viewed as maps from Ω to the projective space $\mathbb{P}^1$, are automatically continuous (see e.g. [BDV'05, § B.1]).

We say that a cocycle is *dominated* if admits a dominated splitting. Some authors say that the cocycle is *exponentially separated*, which is perhaps a better terminology. Domination is also sometimes called *projective hyperbolicity*, because it can be expressed in terms of uniform contraction and expansion on the projective space.

As shown in [Y'04, BG'09], a 2×2 cocycle (T, A) is dominated if and only if there are constants $c > 0$ and $\delta > 0$ such that

$$\frac{\|A^{(n)}(\omega)\|}{\mathfrak{m}(A^{(n)}(\omega))} \geqslant ce^{\delta n} \quad \text{for all } \omega \in \Omega \text{ and } n \geqslant 1. \tag{1.4}$$

Notice that the LHS is a measure of "non-conformality" of the matrix $A^{(n)}(\omega)$.

If a cocycle is dominated then the Lyapunov exponents are always distinct; actually $\delta > 0$ as in (1.4) is a uniform lower bound for the gap between them. Moreover (see subsection 2.1 for details) the associated Oseledets directions coincide wherever they exist (thus μ-almost everywhere) with the directions e_1 and e_2 forming the dominated splitting. In particular, the Lyapunov exponents are given by integrals:

$$\lambda_i(A, \mu) = \int \varphi_i \, d\mu \,, \quad \text{where } \varphi_i(\omega) := \log \|A(\omega)|e_i(\omega)\| \,, \quad (i = 1, 2). \tag{1.5}$$

As a consequence of these formulas, the problem of maximizing or minimizing Lyapunov exponents for dominated cocycles is equivalent to the optimization of Birkhoff averages of the continuous functions φ_i, and so many standard results apply (see [Je'06]).[1] In particular, one can easily show that:

$$\lambda_i^\top(A) = \lim_{n\to\infty} \frac{1}{n} \sup_{\omega\in\Omega} \log \|A^{(n)}(\omega)|e_i(\omega)\| \,,$$
$$\lambda_i^\perp(A) = \lim_{n\to\infty} \frac{1}{n} \inf_{\omega\in\Omega} \log \|A^{(n)}(\omega)|e_i(\omega)\| \,.$$

[1]In order to avoid complications, our definition (1.2) of the extremal Lyapunov exponents only considers regular points, as the alert reader will have noticed. On the other hand, non-regular points have no effect in the optimization of Birkhoff averages (see [Je'06]). Therefore, for dominated cocycles at least, non-regular points have no effect in the optimization of Lyapunov exponents.

Another consequence is that all sup's and inf's that appear in (1.2) are attained in this case.

1.3. **One-step cocycles.** Fix an integer $k \geqslant 2$. Let $\Omega = \{1, \dots, k\}^{\mathbb{Z}}$ be the space of bi-infinite words on k symbols. With some abuse of notation, we denote this set by $k^{\mathbb{Z}}$. Let $T \colon k^{\mathbb{Z}} \to k^{\mathbb{Z}}$ be the shift transformation.

Given a k-tuple of matrices $\mathsf{A} = (A_1, \dots, A_k) \in \mathrm{GL}(2, \mathbb{R})^k$, we associate the locally constant map $A \colon k^{\mathbb{Z}} \to \mathrm{GL}(d, \mathbb{R})$ given by $A(\omega) = A_{\omega_0}$. In this case, (T, A) is called a *one-step cocycle*, and the products (1.1) are simply

$$A^{(n)}(\omega) = A_{\omega_{n-1}} \cdots A_{\omega_0} .$$

The k-tuple of matrices A is called the *generator* of the cocycle. We denote $\lambda_i^{\top}(\mathsf{A}) = \lambda_i^{\top}(A)$, $\lambda_i^{\perp}(\mathsf{A}) = \lambda_i^{\perp}(A)$.

We remark that for one-step cocycles, the values $\lambda_1^{\top}(\mathsf{A})$ and $\lambda_1^{\perp}(\mathsf{A})$ can be alternatively defined in a more elementary way (without speaking of measures) as:

$$\lambda_1^{\top}(\mathsf{A}) = \lim_{n \to \infty} \frac{1}{n} \log \sup_{i_1, \dots, i_n} \| A_{i_n} \dots A_{i_1} \| , \tag{1.6}$$

$$\lambda_1^{\perp}(\mathsf{A}) = \lim_{n \to \infty} \frac{1}{n} \log \inf_{i_1, \dots, i_n} \| A_{i_n} \dots A_{i_1} \| . \tag{1.7}$$

(see subsection A.2).

The numbers $\varrho^{\top}(\mathsf{A}) := e^{\lambda_1^{\top}(\mathsf{A})}$ and $\varrho^{\perp}(\mathsf{A}) := e^{\lambda_1^{\perp}(\mathsf{A})}$ are called *joint spectral radius* and *joint spectral subradius* and constitute an active topic of research: see [Ju'09].

1.4. **Domination for one-step** $\mathrm{GL}(2, \mathbb{R})$ **cocycles.** An one-step cocycle (T, A) is dominated if and only if the number

$$(\lambda_1 - \lambda_2)^{\perp}(A) := \inf_{\mu \in \mathcal{M}_T} (\lambda_1(A, \mu) - \lambda_2(A, \mu)) \tag{1.8}$$

is positive; see subsection A.2 for the (easy) proof. Let us see still another characterization of domination for one-step cocycles.

The *standard positive cone* in $\mathbb{R}^2_* := \mathbb{R}^2 \smallsetminus \{0\}$ is

$$C_+ := \{(x, y) \in \mathbb{R}^2_*;\ xy \geqslant 0\}$$

A *cone* in $\mathbb{R}^2_*$ is the image of C_+ by a linear isomorphism. A *multicone* in $\mathbb{R}^2_*$ is a disjoint union of finitely many cones.

We say that a multicone $M \subset \mathbb{R}^2_*$ is *forward-invariant* with respect to $\mathsf{A} = (A_1, \dots, A_k)$ if the image multicone $\bigcup_i A_i(M)$ is contained in the interior of M.

For example, if A_i's has positive entries then the standard positive cone C_+ is a forward-invariant multicone for $(A_1, \dots, A_k)$. For more complicate examples, see [ABY'10].

It was proved in [ABY'10, BG'09] that *the one-step cocycle generated by* A *is dominated if and only if* A *has a forward-invariant multicone.*

If M is a multicone, its *complementary multicone* M_{co} is defined as the closure (relative to $\mathbb{R}^2_*$) of $\mathbb{R}^2_* \smallsetminus M$. Notice that if M is forward-invariant with respect to $(A_1, \dots, A_k)$ then M_{co} is *backwards-invariant*, that is, forward-invariant with respect to $(A_1^{-1}, \dots, A_k^{-1})$.

1.5. **Mather sets.** Under the assumptions above, the extremal Lyapunov exponents "live" in certain invariant sets:

Theorem 1.1. *Suppose that the one-step cocycle generated by* $\mathsf{A} \in \mathrm{GL}(2,\mathbb{R})^k$ *is dominated. For each* $\star \in \{\top, \bot\}$*, let* $K^\star$ *be the union of all supports of measures* $\mu \in \mathcal{M}_T$ *such that* $\lambda(\mu) = \lambda_1^\star$*. Then:*

- $K^\star$ *is a compact, nonempty,* T*-invariant set;*
- *any measure* $\mu \in \mathcal{M}_T$ *supported in* $K^\star$ *satisfies* $\lambda(\mu) = \lambda^\star$*.*

An obvious consequence of the theorem is the existence of λ_1-optimizing measures.

We call $K^\top$ and $K^\bot$ *upper* and *lower Mather sets*, respectively. (Our upper Mather set corresponds to what Morris [Mo'13] calls a Mather set.) The terminology is coherent with Lagrangian Dynamics, where Mather sets were first studied in [Ma'91].

Actually the existence of the upper Mather set is guaranteed for 1-step cocycles (in any dimension) without assumptions of domination: see [Mo'13].

The existence of both Mather sets in Theorem 1.1 can be deduced from Hölder continuity of the Oseledets directions using the usual (commutative) ergodic optimization theory. However, the proof of Theorem 1.1 that we will present is self-contained and gives extra information which will be useful in the proof of our major result, Theorem 1.3 below.

1.6. **Zero entropy.** We say that $\mathsf{A} = (A_1, \dots, A_k)$ satisfies the *forward NOC (non-overlapping condition)* if it has a forward-invariant multicone $M \subset \mathbb{R}^2_*$ such that

$$A_i(M) \cap A_j(M) = \varnothing \quad \text{whenever } i \neq j\,.$$

We say that $\mathsf{A} = (A_1, \dots, A_k)$ satisfies the *backwards NOC* if $(A_1^{-1}, \dots, A_k^{-1})$ satisfies the forward NOC. We say that $(A_1, \dots, A_k)$ satisfies the *NOC* if it satisfies both the forward and the backwards NOC.

Remark 1.2. The forward and the backwards NOC are not equivalent: for example, if

$$A_1 := \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}, \quad A_2 := \begin{pmatrix} \beta^{-1} & 0 \\ 1 & \beta \end{pmatrix}, \quad \text{with } \alpha > 0,\ \beta > 0,\ \alpha^2 + \beta^2 < 1$$

then (A_1, A_2) satisfies the forward NOC, but not the backwards NOC, as one can easily check. $\lhd$

The main result of this paper is the following:

Theorem 1.3. *For every* k*, for every* $\mathsf{A} \in \mathrm{GL}(2,\mathbb{R})^k$*, if the one-step cocycle generated by* A *is dominated and satisfies the NOC then the restriction of the shift map to either Mather set* $K^\top$ *or* $K^\bot$ *has zero topological entropy.*

The conclusion of the theorem means that for each $\star \in \{\top, \bot\}$, the number $w^\star(\ell)$ of words of length ℓ in the alphabet $\{1, \dots, k\}$ that can be extended to an bi-infinite word in the Mather set $K^\star$ is a subexponential function of ℓ, that is,

$$\lim_{\ell \to \infty} \frac{1}{\ell} \log w^\star(\ell) = 0. \tag{1.9}$$

(see [P'89, p. 265–266]).

There are examples where Theorem 1.3 applies and $K^{\top}$ is non-discrete: In the family of examples given in [BMa'02] where a maximizing measure is Sturmian non-periodic, the NOC condition holds for some choices of the parameters.

There are also examples where Theorem 1.3 applies and either $K^{\top}$ or $K^{\perp}$ is not uniquely ergodic: see subsection A.4.

1.7. **Positive entropy.** As a counterpoint to Theorem 1.3, we will see next non-trivial situations where λ_1-minimizing measures with *positive* entropy exist.

A cocycle (T, A) is called *uniformly hyperbolic* if it has an equivariant splitting into two bundles, one being uniformly expanding and the other being uniformly contracting. Any uniformly hyperbolic cocycle is dominated, and the converse holds for $\mathrm{SL}(2,\mathbb{R})$-cocycles.

Theorem 1.4. *Fix $k \geqslant 2$ and let T be the full shift in k symbols. There exists an open and dense subset $\mathcal{U}$ of $\mathrm{SL}(2,\mathbb{R})^k$ such that for every $\mathsf{A} \in \mathcal{U}$,*

i) *either the one-step cocycle over T generated by A is uniformly hyperbolic;*
ii) *or there exists a compact T-invariant set $K \subset k^{\mathbb{Z}}$ of positive topological entropy and such that the norms $\|A^{(n)}(\omega)\|$ are uniformly bounded over $(\omega, n) \in K \times \mathbb{Z}$.*

Notice that in the first case we have $\lambda_1^{\downarrow}(\mathsf{A}) > 0$, while in the second case by the entropy variational principle (see [P'89, p. 269]), there exists a measure $\mu \in \mathcal{M}_T$ such that $h(T, \mu) > 0$ and $\lambda_1(\mathsf{A}, \mu) = 0$.

For a nonlinear version of Theorem 1.4, see [BBD, Theorem 2].

1.8. **Organization of the paper and overview of the proofs.** In section 2 we collect basic fact about dominated cocycles.

A standard procedure to solve ergodic optimization problems is to look for a change of variables under which the optimizing orbits become evident, or "revealed". Following this idea, in section 3 we construct what we call "Barabanov functions" (in analogy to the Barabanov norms from joint spectral radius theory), and immediately use them to prove the existence of the Mather sets (Theorem 1.1).

In section 4 we use the Barabanov functions to prove that the directions of the dominated splitting for points on the Mather sets must obey severe geometrical obstructions, which in turn imply that one direction uniquely determines the other, with an at most countable number of exceptions. Using this property, we prove Theorem 1.3 in section 5.

The simpler proof of Theorem 1.4 is given in section 6, and is independent of the previous sections.

In Appendix A we present complementary information, including counterexamples showing the limits of our results and alternative definitions for some of the concepts we have discussed. We also pose a few problems and suggest some directions for future research.

2. Preliminaries: Basic facts about 2×2 dominated cocycles

In this section we collect some simple facts about dominated cocycles that will be needed in the sequel.

2.1. **General cocycles.**

Proposition 2.1. *Let $T\colon \Omega \to \Omega$ be a homeomorphism, and let $A\colon \Omega \to \mathrm{GL}(2,\mathbb{R})$ be continuous. Assume that the cocycle (T, A) has a dominated splitting into directions e_1, e_2. Then there exists $C > 1$ such that*

$$C^{-1}\|A^{(n)}(\omega)|e_2(\omega)\| \leqslant \mathfrak{m}(A^{(n)}(\omega)) \leqslant \|A^{(n)}(\omega)\| \leqslant C\|A^{(n)}(\omega)|e_1(\omega)\| \tag{2.1}$$

for any $\omega \in \Omega$ and $n \geqslant 1$.

Proof. Denote $a_i^{(n)}(\omega) := \|A^{(n)}(\omega)|e_i(\omega)\|$. By dominance, there exists n_0 such that $a_1^{(n)}(\omega) > a_2^{(n)}(\omega)$ for every $n \geqslant n_0$. Let (u_1, u_2) be the canonical basis of $\mathbb{R}^2$. For each $\omega \in \Omega$, let $M(\omega)$ be a 2×2 matrix such that such that $M(\omega)u_i$ is a unit vector in the direction $e_i(\omega)$. (The map M may be discontinuous.) Consider $D(\omega) := M(T(\omega))^{-1}A(\omega)M(\omega)$. Then $D^{(n)}(\omega)$ is a diagonal matrix with entries $\pm a_1^{(n)}(\omega)$ and $\pm a_2^{(n)}(\omega)$. On the other hand, since the angle between e_1 and e_2 is uniformly bounded from below, there exists $c > 1$ such that $\|M(x)^{\pm 1}\| \leqslant c$. In particular, we obtain

$$c^{-2}\mathfrak{m}(D^{(n)}(\omega)) \leqslant \mathfrak{m}(A^{(n)}(\omega)) \leqslant \|A^{(n)}(\omega)\| \leqslant c^2\|D^{(n)}(\omega)\| \,.$$

So inequalities (2.1) hold with $C = c^2$ for $n \geqslant n_0$. Increasing C if necessary, we ensure that these inequalities hold for every $n \geqslant 1$. ☐

Corollary 2.2. *If the cocycle (T, A) is dominated then, for any $i \in \{1, 2\}$,*

$$\lambda_i(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(\omega)|e_i(\omega)\| \tag{2.2}$$

for every $\omega \in \Omega$ such that at least one of these quantities is well-defined.

Proof. Use Proposition 2.1 together with the obvious estimates:

$$\|A^{(n)}(\omega)|e_2(\omega)\| \geqslant \mathfrak{m}(A^{(n)}(\omega)) \quad \text{and} \quad \|A^{(n)}(\omega)|e_1(\omega)\| \leqslant \|A^{(n)}(\omega)\| \,.$$ ☐

Notice that the RHS of (2.2) is a limit of Birkhoff averages, so the integral formulas (1.5) follow.

Remark 2.3. Actually Proposition 2.1 implies that the dominated splitting coincides with the Oseledets splitting whenever the latter is defined. The properties alluded in Corollary 2.2 hold in general for Oseledets splittings. ◅

2.2. **One-step cocycles.** Let us fix some notation. The projective space of $\mathbb{R}^2$ is denoted by $\mathbb{P}^1$. Given $x \in \mathbb{R}^2_*$, let x' denote the unique line in $\mathbb{P}^1$ containing x. Given a linear isomorphism L of $\mathbb{R}^2$, let L' the self-map of $\mathbb{P}^1$ defined by $L'(u') = (L(u))'$. If $M \subset \mathbb{R}^2_*$ is a multicone then let $M' := \{x' \in \mathbb{P}^1;\ x \in M\}$.

Let $\mathbb{Z}_-$ (resp. $\mathbb{Z}_+$) be the set of negative (resp. nonnegative) integers. Define projections

$$\pi_-\colon k^{\mathbb{Z}} \to k^{\mathbb{Z}_-}\,, \qquad \pi_-(\omega) = (\dots, \omega_{-2}, \omega_{-1})\,, \tag{2.3}$$

$$\pi_+\colon k^{\mathbb{Z}} \to k^{\mathbb{Z}_+}\,, \qquad \pi_+(\omega) = (\omega_0, \omega_1, \dots)\,. \tag{2.4}$$

Proposition 2.4 ([ABY'10, BG'09])**.** *Assume that $(A_1, \dots, A_k)$ generate a dominated one-step cocycle. Let e_1, $e_2\colon k^{\mathbb{Z}} \to \mathbb{P}^1$ be the invariant directions forming the* *dominated splitting, and let* $M \subset \mathbb{R}^2_*$ *be a forward-invariant multicone. Then for any* $\omega \in k^{\mathbb{Z}}$ *we have*

$$\{e_1(\omega)\} = \bigcap_{n=1}^{\infty} A'_{\omega_{-n}} \cdots A'_{\omega_{-1}}(M') \quad \textit{and} \quad \{e_2(\omega)\} = \bigcap_{n=1}^{\infty} A'_{\omega_{n-1}} \cdots A'_{\omega_0}(M'_{\mathrm{co}}) \, .$$

In particular the directions $e_1(\omega)$ *and* $e_2(\omega)$ *depend only on* $\pi_-(\omega)$ *and* $\pi_+(\omega)$*, respectively, and so there are continuous maps* $\tilde{e}_1$*,* $\tilde{e}_2$ *such that the following diagrams commute:*

$$\begin{tikzcd} k^{\mathbb{Z}} \arrow[r, "e_1"] \arrow[d, "\pi_-"'] & \mathbb{P}^1 \\ k^{\mathbb{Z}_-} \arrow[ur, "\tilde{e}_1"'] & \end{tikzcd} \qquad \begin{tikzcd} k^{\mathbb{Z}} \arrow[r, "e_2"] \arrow[d, "\pi_+"'] & \mathbb{P}^1 \\ k^{\mathbb{Z}_+} \arrow[ur, "\tilde{e}_2"'] & \end{tikzcd}$$

Corollary 2.5. *Let* $\tilde{e}_1$*,* $\tilde{e}_2$ *be as in Proposition 2.4. Then:*

- *If the forward NOC is satisfied then* $\tilde{e}_1$ *is one-to-one; in particular* $e_1(k^{\mathbb{Z}})$ *is a Cantor set contained in* M*.*
- *If the backwards NOC is satisfied then* $\tilde{e}_2$ *is one-to-one; in particular* $e_2(k^{\mathbb{Z}})$ *is a Cantor set contained in* M_{co}*.*

3. Barabanov functions and Mather sets

3.1. **Statements.** A *Barabanov norm* for a compact set A of $d \times d$ matrices is a norm $|||\cdot|||$ on $\mathbb{R}^d$ such that

$$\max_{A \in \mathsf{A}} |||Ax||| = \varrho^{\top}(\mathsf{A})\, |||x||| \quad \text{for all } x \in \mathbb{R}^d,$$

where $\varrho^{\top}(\mathsf{A}) = e^{\lambda_1^{\top}(\mathsf{A})}$ is the joint spectral radius of A. It is known that a Barabanov norm exists whenever A is irreducible (i.e., has no nontrivial invariant subspace): see [B'88, W'02].

For definiteness, let us consider finite sets $\mathsf{A} \subset \mathrm{GL}(2,\mathbb{R})$. One may wonder about the existence of a version of the Barabanov for the joint spectral subradius $\varrho^{\perp}(\mathsf{A}) = e^{\lambda_1^{\perp}(\mathsf{A})}$, that is, a norm $|||\cdot|||$ such that

$$\min_{A \in \mathsf{A}} |||Ax||| = \varrho^{\perp}(\mathsf{A})\, |||x||| \quad \text{for all } x \in \mathbb{R}^2. \tag{3.1}$$

Unfortunately, no such norm can in general exist, even assuming irreducibility of A. For example, if the cocycle is such that $\lambda_2^{\top}(\mathsf{A}) < \lambda_1^{\perp}(\mathsf{A})$ then applying relation (3.1) to the orbit of a nonzero vector in the second Oseledets bundle e_2 we reach a contradiction.

This example shows that if such a "minimizer Barabanov norm" exists, relation (3.1) cannot hold for all vectors, but only for vectors away from the e_2-directions. In general, the set of e_2-directions can be large or even the whole $\mathbb{P}^1$, but for dominated cocycles it is a proper compact subset of $\mathbb{P}^1$.

As we show in this section, under the assumption of domination it is indeed possible to construct an object that retains the most useful properties of (the logarithm of) a "minimizer Barabanov norm". For convenience, we simultaneously consider both the maximizer and minimizer cases:

Theorem 3.1. *Let* $(A_1, \ldots, A_k)$ *be generators of a dominated one-step cocycle, and let* $M \subset \mathbb{R}^2$ *be a forward-invariant multicone. Then there exist functions*

$$p^{\top} \colon M \to \mathbb{R} \quad \textit{and} \quad p^{\perp} \colon M \to \mathbb{R}$$

with the following properties:

- *extremality: for all* $x \in M$,
$$\max_{i\in\{1,\dots,k\}} p^{\top}(A_i x) = p^{\top}(x) + \lambda_1^{\top}\,, \tag{3.2}$$
$$\min_{i\in\{1,\dots,k\}} p^{\perp}(A_i x) = p^{\perp}(x) + \lambda_1^{\perp}\,; \tag{3.3}$$
- log-*homogeneity: for all* $\star \in \{\top, \perp\}$, $x \in M$, *and* $t \in \mathbb{R} \smallsetminus \{0\}$,
$$p^{\star}(tx) = p^{\star}(x) + \log|t|\,; \tag{3.4}$$
- *regularity: there exists* $c_0 > 0$ *such that for all* $\star \in \{\top, \perp\}$ *and* x, $y \in M$,
$$|p^{\star}(x) - p^{\star}(y)| \leqslant c_0 \measuredangle(x,y) + |\log\|x\| - \log\|y\||\,. \tag{3.5}$$

Related functions were used by Bousch and Mairesse [BMa'02, § 2.1]. Our construction combines their techniques with properties of multicones and the Hilbert metric. A higher-dimensional version of our construction was obtained in [BMo].

Let us also mention that similar constructions also play an important role on ergodic optimization, action minimization in Lagrangian dynamics, and optimal control: see [BMa'02] and references therein.

3.2. **Proofs.** In the rest of this section we prove Theorems 3.1 and 1.1.

The first step is the construction of an "adapted metric". As in section 2, we use a prime to denote projectivization.

Lemma 3.2. *Let* $(A_1, \dots, A_k)$ *be generators of a dominated one-step cocycle, and let* $M \subset \mathbb{R}^2$ *be a forward-invariant multicone. There exist a metric* d *on the projectivization* M' *and constants* $c_2 > 1$ *and* $0 < \tau < 1$ *such that for all* x, $y \in M$, *we have*
$$d\left(A_i' x', A_i' y'\right) \leqslant \tau d\left(x', y'\right) \quad \text{for all } i \in \{1, \dots, k\}, \tag{3.6}$$
$$c_1^{-1} \measuredangle(x,y) \leqslant d\left(x', y'\right) \leqslant c_1 \measuredangle(x,y)\,. \tag{3.7}$$

Proof. By a compactness argument, there exists an open neighborhood U of M' in $\mathbb{P}^1$ such that $A_i'(U) \subset M'$ for all $i \in \{1, \dots, k\}$. We can assume that each connected component of U contains exactly one connected component of M'.

Endow each connected component of U with its Hilbert metric, and restrict it to the corresponding connected component of M'. We use the same letter d to denote all those metrics. Rescaling if necessary, we can assume that $d \leqslant 1/2$ whenever defined. Moreover, there are constants $c_1 > 1$ and $0 < \tau < 1$ such that properties (3.6) and (3.7) hold whenever x' and y' are in the same connected component of M'.

Given x', $y' \in M'$, define $\ell(x', y')$ as the least integer $n \geqslant 0$ with the property that for all $\omega \in k^{\mathbb{Z}}$, the directions $A^{(n)}(\omega)'x'$ and $A^{(n)}(\omega)'y'$ belong to the same connected component of M'. The function ℓ is uniformly bounded, has the following property:
$$\ell\left(A_i' x', A_i' y'\right) \leqslant \max\left(\ell\left(x', y'\right) - 1, 0\right), \quad \text{for all } i \in \{1, \dots, k\},$$
and satisfies an ultrametric inequality:
$$\ell\left(x', y'\right) \leqslant \max\left(\ell\left(x', z'\right), \ell\left(y', z'\right)\right).$$
We now extend d by setting $d(x', y') := \ell(x', y')$ if x' and y' are in different connected components of M'. Then d is a distance function. Moreover, increasing c_1 and τ if necessary, properties (3.6) and (3.7) are satisfied. ☐

In the following proof of Theorem 3.1, we will also establish some facts that are necessary for the subsequent proof of Theorem 1.1.

Proof of Theorem 3.1. For each $i \in \{1, \dots, k\}$, define $h_i \colon \mathbb{P}^1 \to \mathbb{R}$ by

$$h_i(x') := \log \frac{\|A_i x\|}{\|x\|} \,,$$

where $\|\cdot\|$ is the Euclidian metric, as usual. Fix a constant $c_2 > 0$ such that

$$\left|h_i(x') - h_i(y')\right| \leqslant c_2 \measuredangle(x, y) \quad \text{for all } x,\, y \in \mathbb{R}^2_*.$$

Let M be a forward-invariant multicone for $(A_1, \dots, A_k)$, and let d me the metric on the projectivization M' given by Lemma 3.2. Let $\mathbb{B}$ be the vector space of continuous functions from M' to $\mathbb{R}$, endowed with the uniform (supremum) distance $|\cdot|_\infty$. Let $c_3 := c_1 c_2/(1-\tau)$ and let $\mathbb{K} \subset \mathbb{B}$ be the set of functions that are c_3-Lipschitz with respect to d.

For each function $f \in \mathbb{K}$, define two functions $T^\star f \colon M' \to \mathbb{R}$ (where $\star \in \{\top, \bot\}$) by

$$\begin{aligned}
(T^\top f)(x') &:= \max_{i \in \{1,\dots,k\}} \left[f\left(A_i' x'\right) + h_i(x')\right] \,, \\
(T^\bot f)(x') &:= \min_{i \in \{1,\dots,k\}} \left[f\left(A_i' x'\right) + h_i(x')\right] \,.
\end{aligned}$$

We claim that $T^\star f \in \mathbb{K}$. Indeed, for all x', $y' \in M'$, we have

$$\begin{aligned}
\left|(T^\star f)(x') - (T^\star f)(y')\right| &\leqslant \max_i \left| \left[f\left(A_i' x'\right) + h_i(x')\right] - \left[f\left(A_i' y'\right) + h_i(y')\right] \right| \\
&\leqslant \max_i \left|f\left(A_i' x'\right) - f\left(A_i' x'\right)\right| + \max_i \left|h_i(x') - h_i(y')\right| \\
&\leqslant c_3 \max_i d\left(A_i' x', A_i' y'\right) + c_2 \measuredangle(x, y) \\
&\leqslant c_3 \tau d(x', y') + c_1 c_2 d(x', y') \\
&= c_3 d(x', y') \,.
\end{aligned}$$

Thus we have defined maps $T^\star \colon \mathbb{K} \to \mathbb{K}$. Next, we claim that these maps are continuous. Indeed, for all f, $g \in \mathbb{K}$, we have

$$\begin{aligned}
|T^\star f - T^\star g|_\infty &= \sup_{x' \in M'} \left|(T^\star f)(x') - (T^\star g)(x')\right| \\
&\leqslant \sup_{x' \in M'} \max_i \left|f\left(A_i' x'\right) - g\left(A_i' x'\right)\right| \\
&\leqslant |f - g|_\infty \,.
\end{aligned}$$

Let $\hat{\mathbb{B}}$ be the quotient of the space $\mathbb{B}$ by the subspace of constant functions; it is a Banach space endowed with the quotient norm $|\hat{f}|_\infty := \inf\{|f|_\infty ;\ \pi(f) = \hat{f}\}$, where $\pi \colon \mathbb{B} \to \hat{\mathbb{B}}$ denotes the quotient projection. By the Arzelà–Ascoli theorem, the convex set $\hat{\mathbb{K}} := \pi(\mathbb{K})$ is compact. Since $T^\star$ commutes with the addition of a constant, there exists a map $\hat{T}^\star \colon \hat{\mathbb{K}} \to \hat{\mathbb{K}}$ such that $\pi \circ T^\star = \hat{T}^\star \circ \pi$. The map $\hat{T}^\star$ is continuous, as it is easy to check; in particular, by the Schauder theorem, it has a fixed point $\hat{f}_0^\star$. This means that there exist $f_0^\star \in \mathbb{K}$ and $\beta^\star \in \mathbb{R}$ such that $T^\star f_0^\star = f_0^\star + \beta^\star$. Define

$$p^\star(x) := f_0^\star(x') + \log \|x\| \quad \text{for all } x \in M.$$

Note that for every $x \in M$, the following properties hold: property (3.4),

$$\max_{i\in\{1,\dots,k\}} p^{\top}(A_i x) = p^{\top}(x) + \beta^{\top}\,, \tag{3.8}$$

$$\min_{i\in\{1,\dots,k\}} p^{\perp}(A_i x) = p^{\perp}(x) + \beta^{\perp}\,, \tag{3.9}$$

and

$$\Big| p^{\star}(x) - \log\|x\| \Big| \leqslant c_4 \tag{3.10}$$

where $c_4 := \max\big(|f_0^{\top}|_\infty\,, |f_0^{\perp}|_\infty\big)$.

Taking $c_0 = c_1 c_3$, we see that property (3.5) holds when x' and y' are in the same connected component of M'. Since the angle between directions in different components is uniformly bounded from below, we can increase c_0 if necessary so that property (3.5) fully holds.

To complete the proof of Theorem 3.1 we need to show that the numbers $\beta^{\top}$ and $\beta^{\perp}$ that appear in (3.8) and (3.9) are respectively equal to the numbers $\lambda_1^{\top}$ and $\lambda_1^{\perp}$ that appear in (3.2) and (3.3). As we prove these equalities, we will also establish some facts that will be useful in the forthcoming proof Theorem 1.1.

Take any $\omega \in k^{\mathbb{Z}}$ and $x \in e_1(\omega) \smallsetminus \{0\}$. Recall from Proposition 2.4 that $x \in M$, and so consider

$$\psi^{\star}(\omega) := p^{\star}(A_{\omega_0} x) - p^{\star}(x)\,.$$

By (3.4) this value does not depend on the choice of x in $e_1(\omega) \smallsetminus \{0\}$; in this way we define a continuous function $\psi^{\star}\colon k^{\mathbb{Z}} \to \mathbb{R}$.

By equivariance of the e_1 direction, for every $\omega \in k^{\mathbb{Z}}$, $x \in e_1(\omega) \smallsetminus \{0\}$, and $n \geqslant 1$ we have

$$p^{\star}(A^{(n)}(\omega)x) - p^{\star}(x) = \sum_{j=0}^{n-1} \psi^{\star}(T^j\omega).$$

Letting $\varphi_1(\omega) := \log\|A(\omega)|e_1(\omega)\|$, it follows from (3.10) that

$$-2c_4 \leqslant \sum_{j=0}^{n-1} \psi^{\star}(T^j\omega) - \sum_{j=0}^{n-1} \varphi_1^{\star}(T^j\omega) \leqslant 2c_4\,.$$

Integrating with respect to some $\mu \in \mathcal{M}_T$, dividing by n, and making $n \to \infty$, we conclude that $\int \psi^{\star}\,d\mu = \int \varphi_1\,d\mu$. Recalling the integral formula (1.5) (proved in subsection 2.1), we conclude that

$$\lambda_1(\mu) = \int \psi^{\star}\,d\mu \quad \text{for any } \mu \in \mathcal{M}_T.$$

On the other hand, by (3.8) and (3.9), we have

$$\psi^{\top} \leqslant \beta^{\top} \quad \text{and} \quad \psi^{\perp} \geqslant \beta^{\perp}\,,$$

which in particular implies that

$$\beta^{\perp} \leqslant \lambda_1^{\perp} \leqslant \lambda_1^{\top} \leqslant \beta^{\top}\,. \tag{3.11}$$

Moreover, for any $\mu \in \mathcal{M}_T$, we have $\lambda_1(\mu) = \beta^{\star}$ if and only if $\psi^{\star} = \beta^{\star}$ μ-almost everywhere, or equivalently, if the T-invariant set

$$L^{\star} := \{\omega \in k^{\mathbb{Z}};\ \psi^{\star}(T^n\omega) = \beta^{\star}\ \forall\, n \in \mathbb{Z}\} \tag{3.12}$$

has total μ-measure.

We will show that $L^{\star}$ is compact and nonempty. We begin showing the following:

Claim 3.3. For any $\omega_- \in k^{\mathbb{Z}_-}$ there exists $\omega_+ \in k^{\mathbb{Z}_+}$ such that if $\omega = \omega_-\omega_+$ is concatenation of ω_- and ω_+ then $\psi^\star(T^n\omega) = \beta^\star$ for all $n \geqslant 0$.

Proof of the claim. Recall from Proposition 2.4 that given a semi-infinite word $\omega_- = (\dots, \omega_{-2}, \omega_{-1})$, the direction $\tilde{e}_1(\omega_-)$ is determined, and by (3.2) or (3.3) there exists a letter ω_0 such that $\psi^\star(\omega)$ (which is well-defined even if ω_1, ω_2, ... are still undefined) equals $\beta^\star$. Next we consider the shifted word $(\dots, \omega_{-1}, \omega_0)$, and repeat the reasoning above to find ω_1 such that $\psi^\star(T\omega) = \beta^\star$. Continuing by induction, we find the desired ω^+, thus proving the claim. $\square$

Let $L_+^\star$ be the set of $\omega \in k^{\mathbb{Z}}$ such that $\psi^\star(T^n\omega) = \beta^\star$ for all $n \geqslant 0$, which by Claim 3.3 is nonempty. Since $L_+^\star$ is compact and contains $T(L_+^\star)$, the set $L_\star = \bigcap_{n\geqslant 0} T^n(L_+^\star)$ is compact and nonempty, as announced. In particular, there exists at least one T-invariant probability measure $\mu^\star$ supported on $L^\star$, and so with $\lambda_1(\mu^\star) = \beta^\star$. Together with (3.11) this implies that $\beta^\star = \lambda_1^\star$. So (3.2) and (3.3) respectively follow from (3.8) and (3.9) and the proof of Theorem 3.1 is complete. $\square$

Proof of Theorem 1.1. For each $\star \in \{\top, \bot\}$, let $\mathcal{M}_T^\star$ be the set of measures $\mu \in \mathcal{M}_T$ such that $\lambda(\mu) = \lambda_1^\star$. We have seen in the proof of Theorem 3.1 that there exists a nonempty compact T-invariant set $L^\star$ such that $\mu \in \mathcal{M}_T^\star$ if and only if $\operatorname{supp}\mu \subset L^\star$.

Define the Mather set $K^\star$ as the union of the supports of all measures μ in $\mathcal{M}_T^\star$. To show that this is a compact set, we follow an argument from [Mo'13]. The set of all Borel probabilities on $k^{\mathbb{Z}}$ with the usual weak-star topology is metrizable and compact, and $\mathcal{M}_T$ is a compact subset. Since $L^\star$ is compact, using Urysohn's lemma we see that the set $\mathcal{M}_T^\star$ is also compact. In particular, it has a countable dense sequence $(\nu_n^\star)$. Consider $\nu^\star := \sum 2^{-n}\nu_n^\star$, which is an element of $\mathcal{M}_T^\star$. It is then easy to show that $\operatorname{supp}\nu^\star = K^\star$, which in particular shows that $K^\star$ is compact.

The remaining assertions in Theorem 1.1 are now obvious, and the proof is complete. $\square$

4. Properties of Lyapunov-optimal orbits

In this section we explore consequences of Theorem 3.1. Let us remark that none of the results of this section requires the nonoverlapping conditions.

Fix generators $(A_1, \dots, A_k)$ of a dominated one-step cocycle, a forward-invariant multicone M, and Barabanov functions $p^\top$, $p^\bot$ on M.

4.1. Geometrical obstructions.

In this subsection, we will show that the invariant directions of points on the Mather sets must obey certain geometrical obstructions.

We begin considering certain sets of optimal future trajectories. For each $\star \in \{\top, \bot\}$, let

$$J^\star := \left\{(\omega^+, x) \in k^{\mathbb{Z}_+} \times M;\ p^\star(A^{(n)}(\omega^+)x) = p^\star(x) + n\lambda_1^\star\ \forall n \geqslant 0\right\}.$$

Since the functions $p^\star$ are continuous, these sets are closed. Also notice that, as a consequence of (3.2) and (3.3),

$$\forall\, x \in M\ \exists\, \omega^+ \in k^{\mathbb{Z}_+} \text{ such that } (\omega^+, x) \in J^\star.$$

Lemma 4.1. *If $(\omega^+, x) \in J^\star$ and $y \in M$ are such that $x - y \in \tilde{e}_2(\omega^+)$ then*

$$p^\top(x) \leqslant p^\top(y) \quad \text{if } \star = \top,$$
$$p^\perp(x) \geqslant p^\perp(y) \quad \text{if } \star = \perp.$$

Proof. Let $\omega^+ \in k^{\mathbb{Z}^+}$ and x, $y \in M$ be such that $x - y \in \tilde{e}_2(\omega^+)$. Let $x_n := A^{(n)}(\omega^+)x$ and $y_n := A^{(n)}(\omega^+)y$, for $n \geqslant 0$. First, we will show that

$$\lim_{n\to\infty} \left[p^\star(y_n) - p^\star(x_n)\right] = 0\,. \tag{4.1}$$

Indeed, by property (3.5),

$$\left|p^\star(y_n) - p^\star(x_n)\right| \leqslant c_0 \measuredangle(y_n, x_n) + \left|\log\|y_n\| - \log\|x_n\|\right|.$$

Since $x - y \in \tilde{e}_2(\omega^+)$ and $x \notin \tilde{e}_2(\omega^+)$, domination implies that $\measuredangle(y_n, x_n)$ tends to zero as $n \to \infty$. On the other hand,

$$\left|\log\|y_n\| - \log\|x_n\|\right| \leqslant \max\left(\frac{\|y_n\|}{\|x_n\|} - 1, \frac{\|x_n\|}{\|y_n\|} - 1\right) \leqslant \frac{\|x_n - y_n\|}{\min\left(\|x_n\|, \|y_n\|\right)},$$

which, by domination again, tends to zero as $n \to \infty$. This proves (4.1).

Next, assume $(\omega^+, x) \in J^\star$. So, for all $n \geqslant 0$,

$$p^\star(x_n) = p^\star(x) + n\lambda_1^\star\,.$$

By properties (3.2) and (3.3) we have

$$p^\top(y_n) \leqslant p^\top(y) + n\lambda_1^\top \quad \text{if } \star = \top,$$
$$p^\perp(y_n) \geqslant p^\perp(y) + n\lambda_1^\perp \quad \text{if } \star = \perp.$$

In particular,

$$p^\top(y_n) - p^\top(x_n) \leqslant p^\top(y) - p^\top(x) \quad \text{if } \star = \top,$$
$$p^\perp(y_n) - p^\perp(x_n) \geqslant p^\perp(y) - p^\perp(x) \quad \text{if } \star = \perp.$$

Passing to a limit $n \to \infty$ and using (4.1) we obtain the lemma. □

Given vectors x_1, y_1, x_2, $y_2 \in \mathbb{R}^2_*$, no three of them collinear, we define their *cross-ratio*

$$[x_1, y_1; x_2, y_2] := \frac{x_1 \times x_2}{x_1 \times y_2} \cdot \frac{y_1 \times y_2}{y_1 \times x_2} \in \mathbb{R} \cup \{\infty\}\,,$$

where $\times$ denotes cross-product in $\mathbb{R}^2$, i.e. determinant. The cross-ratio actually depends only on the directions defined by the four vectors, which allows us to apply the same definition 4-tuples in $(\mathbb{P}^1)^4$ without three coinciding points. Also, the cross-ration is invariant under linear transformations.

We now use Lemma 4.1 to prove the following important Lemma 4.2, which a character similar to Proposition 2.6 from [BMa'02]:

Lemma 4.2. *For all (ξ, x_1), $(\eta, y_1) \in J^\star$ and nonzero vectors $x_2 \in \tilde{e}_2(\xi)$, $y_2 \in \tilde{e}_2(\eta)$ we have*

$$\left|[x_1, y_1; x_2, y_2]\right| \geqslant 1 \quad \text{if } \star = \top,$$
$$\left|[x_1, y_1; x_2, y_2]\right| \leqslant 1 \quad \text{if } \star = \perp.$$

Proof. Let us consider the case of $J^\top$; the other case is analogous.

Recall from Proposition 2.4 that any e_1 direction is different from any e_2 direction. So neither x_1 nor y_1 can be collinear to x_2 or y_2. Hence the cross-ratio is well defined. Moreover, we can write:

$$x_1 = \alpha x_2 + \beta y_1 \quad \text{and} \quad y_1 = \gamma y_2 + \delta x_1.$$

By Lemma 4.1,

$$p^\top(x_1) \leqslant p^\top(\beta y_1) \leqslant p^\top(\beta\delta x_1) = p^\top(x_1) + \log|\beta\delta|.$$

Hence, $|\beta\delta| \geqslant 1$. Substituting

$$\beta = \frac{x_1 \times x_2}{y_1 \times x_2} \quad \text{and} \quad \delta = \frac{y_1 \times y_2}{x_1 \times y_2}$$

we get the assertion. ☐

From Lemma 4.2 we immediately obtain:

Lemma 4.3. *If ξ, $\eta \in K^\star$ then*

$$\big|[e_1(\xi), e_1(\eta); e_2(\xi), e_2(\eta)]\big| \geqslant 1 \quad \text{if } \star = \top,$$
$$\big|[e_1(\xi), e_1(\eta); e_2(\xi), e_2(\eta)]\big| \leqslant 1 \quad \text{if } \star = \perp.$$

Let $(x_1, y_1; x_2, y_2)$ be a 4-tuple of distinct points in $\mathbb{P}^1$. Then one and only one of the following possibilities hold:

- *antiparallel configuration*: $x_1 < y_2 < y_1 < x_2 < x_1$ for some cyclic order $<$ on $\mathbb{P}^1$ (see Fig. 1);
- *coparallel configuration*: $x_1 < y_1 < y_2 < x_2 < x_1$ for some cyclic order $<$ on $\mathbb{P}^1$ (see Fig. 2);
- *crossing configuration*: $x_1 < y_1 < x_2 < y_2 < x_1$ for some cyclic order $<$ on $\mathbb{P}^1$ (see Fig. 3).

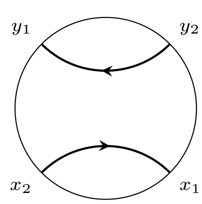

FIG. 1. Antiparallel configuration

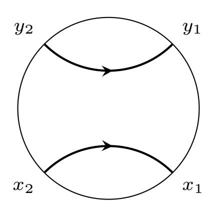

FIG. 2. Coparallel configuration

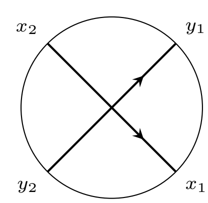

FIG. 3. Crossing configuration

The configuration is expressed in terms of the cross-ratio as follows:

Proposition 4.4. *Consider a 4-tuple $(x_1, y_1; x_2, y_2)$ of distinct points in $\mathbb{P}^1$. Then:*

- *the configuration is* antiparallel *iff* $[x_1, y_1; x_2, y_2] < 0$;
- *the configuration is* coparallel *iff* $0 < [x_1, y_1; x_2, y_2] < 1$;
- *the configuration is* crossing *iff* $[x_1, y_1; x_2, y_2] > 1$.

Proof. With a linear change of coordinates, we can assume that the directions y_1, x_2, y_2 contain the vectors $(1, 1)$, $(1, 0)$, $(0, 1)$, respectively. Let (a, b) be a nonzero vector in the x_1 direction. Then $[x_1, y_1; x_2, y_2] = b/a$. The proposition follows by inspection. ☐

Define the following compact subsets of the torus $\mathbb{P}^1 \times \mathbb{P}^1$:

$$G^\star := \left\{(e_1(\omega), e_2(\omega));\ \omega \in K^\star\right\}. \tag{4.2}$$

As a consequence of Lemma 4.3 and Proposition 4.4, we have:

Corollary 4.5. *Let (x_1, x_2), $(y_1, y_2) \in G^\star$. Then:*

- *if $\star = \top$ then $(x_1, y_1; x_2, y_2)$ cannot be in coparallel configuration;*
- *if $\star = \perp$ then $(x_1, y_1; x_2, y_2)$ cannot be in crossing configuration.*

4.2. **Each invariant direction essentially determines the other.** Now we will use show that for points ω on the Mather sets, each invariant direction $e_1(\omega)$ or $e_2(\omega)$ uniquely determines the other, except for a countable number of bad directions. This fact (stated precisely in Lemma 4.6 below) is actually a simple consequence of Corollary 4.5, and forms the core the proof of Theorem 1.3.

Consider the set $G^\star$ defined (4.2); we decompose it into fibers in two different ways:

$$G^\star = \bigcup_{x_1 \in e_1(K^\star)} \{x_1\} \times G_2^\star(x_1) = \bigcup_{x_2 \in e_2(K^\star)} G_1^\star(x_2) \times \{x_2\}.$$

Define also

$$N_1^\star := \{x_1 \in e_1(K^\star);\ G_2^\star(x_1) \text{ has more than one element}\}, \tag{4.3}$$

$$N_2^\star := \{x_2 \in e_2(K^\star);\ G_1^\star(x_2) \text{ has more than one element}\}. \tag{4.4}$$

So the following implication holds:

$$\left.\begin{array}{l} \xi, \eta \in K^\star \\ e_i(\xi) = e_i(\eta) \notin N_i^\star \end{array}\right\} \Rightarrow \left\{\begin{array}{l} e_1(\xi) = e_1(\eta) \\ e_2(\xi) = e_2(\eta) \end{array}\right. \tag{4.5}$$

Lemma 4.6. *For each $\star \in \{\top, \perp\}$ and $i \in \{1, 2\}$, the set $N_i^\star$ is countable.*

For the proof of the lemma it is convenient to consider the unit disk $\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\}$ endowed with the Poincaré hyperbolic metric. Given two different points x_1, x_2 in the unit circle $\partial\mathbb{D}$, let $\overrightarrow{x_2 x_1}$ denote the oriented hyperbolic geodesic from x_2 to x_1. We identify $\partial\mathbb{D}$ with the projective space $\mathbb{P}^1$ as follows:

$$e^{2\theta i} \in \partial\mathbb{D} \ \leftrightarrow\ (\cos\theta, \sin\theta)' \in \mathbb{P}^1.$$

Under this identification, we say that two geodesics $\overrightarrow{x_2 x_1}$ and $\overrightarrow{y_2 y_1}$ with distinct endpoints are *antiparallel*, *coparallel*, or *crossing* according to the configuration of the 4-tuple $(x_1, y_1; x_2, y_2)$.

Proof of Lemma 4.6. We will consider the case $i = 1$; the case $i = 2$ is entirely analogous.

For each $x \in N_1^\star$, let $I^\star(x)$ be the least closed subinterval of $\mathbb{P}^1 \smallsetminus \{x\}$ containing $G_2^\star(x)$.

We begin with the case of $N_1^\top$.

Claim 4.7. If x, $y \in N_1^\top$ are distinct then $I^\top(x)$ and $I^\top(y)$ have disjoint interiors in the circle $\mathbb{P}^1$. (See Fig. 4.)

Proof of the claim. Let v and w be the endpoints of the interval $I^\top(x)$ and take any point z in its interior. Then the geodesic $\overrightarrow{zy}$ is coparallel to one of the two geodesics $\overrightarrow{vx}$ or $\overrightarrow{wx}$. Since (x, v) and (x, w) belong to $G^\top$, by Corollary 4.5 we conclude that (y, z) does not. This shows that $G_2^\top(y) \cap \operatorname{int} I^\top(x) = \varnothing$, and, in particular,

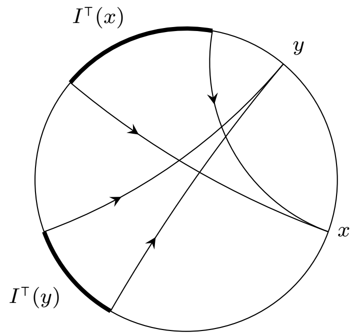

FIG. 4. $x \neq y \in N_1^\top$; the intervals $I^\top(x)$ and $I^\top(y)$ have disjoint interiors.

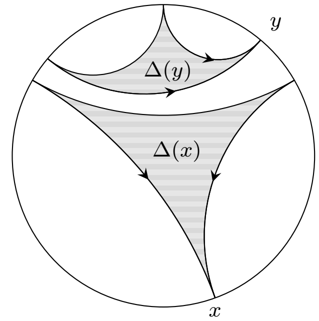

FIG. 5. $x \neq y \in N_1^\perp$; the triangles $\Delta(x)$ and $\Delta(y)$ have disjoint interiors.

$\partial I^\top(y) \cap \operatorname{int} I^\top(x) = \varnothing$. An analogous argument gives $\partial I^\top(x) \cap \operatorname{int} I^\top(y) = \varnothing$. It follows that $\operatorname{int} I^\top(x) \cap \operatorname{int} I^\top(y) = \varnothing$. $\square$

It follows from separability of the circle that $N_1^\top$ is countable.

Now let us consider the case of $N_1^\perp$. For each $x \in N_1^\perp$, let $\Delta(x)$ be the ideal triangle whose vertices are x and the two endpoints of the interval $I^\perp(x)$.

Claim 4.8. If x, $y \in N_1^\perp$ are distinct then $\Delta(x)$ and $\Delta(y)$ have disjoint interiors in the disk $\mathbb{D}$. (See Fig. 5.)

Proof of the claim. Let v and w be the endpoints of the interval $I^\top(x)$. Since these points belong to $e_2(K^\perp)$, which is disjoint from $e_1(K^\perp)$, none of them can be equal to y. Let C be the connected component of $\mathbb{D} \smallsetminus \operatorname{int} \Delta(x)$ whose closure at infinity contains y. Let $z \in G_2^\perp(y)$. By Corollary 4.5, the geodesic $\overrightarrow{zy}$ does not cross $\overrightarrow{vx}$ nor $\overrightarrow{wx}$. It follows that $\overrightarrow{zy}$ is disjoint from $\operatorname{int} \Delta(x)$, and so it is contained in C. Since C is geodesically convex, it follows that $\Delta(y) \subset C$. This proves the claim. $\square$

It follows from separability of the disc that $N_1^\perp$ is also countable, thus completing the proof of Lemma 4.6. $\square$

5. Obtaining zero entropy

In this section we prove Theorem 1.3. The basic idea is as follows: Given a bi-infinite word $\omega \in k^{\mathbb{Z}}$, write it as a concatenation $\omega_-\omega_+$ of its "past" ω_- and its "future" ω_+ (i.e., $\omega_\pm = \pi_\pm(\omega)$ in notation (2.3)–(2.4)). Due to the NOC, there is a bijection between possible pasts (resp. futures) and e_1 (resp. e_2) directions, as we have seen in Corollary 2.5. Using Lemma 4.6, we will show that $e_1(\omega)$ uniquely determines $e_2(\omega)$ and vice versa for almost every point with respect to any probability measure supported on a Mather set. So, with respect to those measures, the past and the future almost surely determine each other and therefore the entropy is zero. A precise proof follows.

5.1. **Generalities about entropy.** Let $\mathcal{C} := \{[1], \dots, [k]\}$ be the partition of $k^{\mathbb{Z}}$ into the time-0 cylinders $[j] := \{\omega \in k^{\mathbb{Z}};\ \omega_0 = j\}$. If $K \subset k^{\mathbb{Z}}$ is a T-invariant compact set, define the partition $\mathcal{C}(K) := \{K \cap [1], \dots, K \cap [k]\}$.

Let $\mathcal{B}(K)$ be the Borel σ-algebra of K, and consider the following sub-σ-algebra:

$$\mathcal{C}_{-\infty}^{-1}(K) := \bigvee_{n<0} T^{-n}(\mathcal{C}(K)),$$

Fix a measure $\mu \in \mathcal{M}_T$ supported on K. Recall (see [P'89, p. 244–245]) that a sufficient (and actually also necessary) condition for the vanishing of the entropy $h(T,\mu)$ is that the partition $\mathcal{C}(K)$ is a *one-sided generator* modulo zero sets, that is,

$$\overline{\mathcal{C}_{-\infty}^{-1}(K)} = \overline{\mathcal{B}(K)},$$

where the bar denotes taking the completion of a σ-algebra with respect to the measure μ. We will find it useful to give an alternative description of the σ-algebra on the LHS.

Let π_- be as in (2.3). Define

$$\mathcal{S}(K) := \left\{D \in \mathcal{B}(K);\ K \cap \pi_-^{-1}(\pi_-(D)) = D\right\}. \tag{5.1}$$

The elements of $\mathcal{S}(K)$ are called *saturated* sets.

Lemma 5.1. *$\mathcal{S}(K)$ is a σ-algebra and $\overline{\mathcal{S}(K)} = \overline{\mathcal{C}_{-\infty}^{-1}(K)}$.*

Proof. It is easily checked that saturated sets form a σ-algebra of subsets of K. It is also clear that $\bigvee_{i=-n}^{-1} T^{-i}(\mathcal{C}(K)) \subset \mathcal{S}(K)$ for each $n > 0$, and it follows that $\mathcal{C}_{-\infty}^{-1}(K) \subset \mathcal{S}(K)$. To conclude the proof of the lemma we need to prove that

$$\mathcal{S}(K) \subset \overline{\mathcal{C}_{-\infty}^{-1}(K)}. \tag{5.2}$$

Let us first consider sets $U \in \mathcal{S}(K)$ that are relatively open in K. Then there is a sequence of cylinders C_n on $k^{\mathbb{Z}}$ such that

$$U = \bigcup_n K \cap C_n.$$

For each n, the set $\hat{C}_n := \pi_-^{-1}(\pi_-(C_n))$ is also a cylinder. Then each set $K \cap \hat{C}_n$ is a subset of the saturated set U and an element of $\mathcal{C}_{-\infty}^{-1}(K)$; therefore $U = \bigcup_n K \cap \hat{C}_n$ is also an element of $\mathcal{C}_{-\infty}^{-1}(K)$. We have shown that $\mathcal{C}_{-\infty}^{-1}(K)$ contains all saturated sets that are relatively open in K, and so it also contains all compact saturated sets.

This proves that all elements of $\mathcal{S}(K)$ that are relatively open belong to $\mathcal{C}_{-\infty}^{-1}(K)$. Consequently, the same holds for compact subsets.

Now consider an arbitrary $D \in \mathcal{S}(K)$. By regularity of the measure, there exist a sequence of compact sets E_n and a set $Z \in \mathcal{B}(K)$ with $\mu(Z) = 0$ such that

$$D = Z \cup \bigcup_n E_n.$$

For each n, let $\hat{E}_n := K \cap \pi_-^{-1}(\pi_-(E_n))$. Since each $\hat{E}_n$ is a subset of D, we have $D = Z \cup \bigcup_n \hat{E}_n$. Each set $\hat{E}_n$ is compact and is an element of $\mathcal{S}(K)$, and it follows from what was proved previously that $\hat{E}_n \in \mathcal{C}_{-\infty}^{-1}(K)$. In particular, $D \in \overline{\mathcal{C}_{-\infty}^{-1}(K)}$, therefore completing the proof of (5.2) and the lemma. □

5.2. **Proof of Theorem 1.3.**

Proof. Fix $\star \in \{\top, \bot\}$. By the entropy variational principle (see [P'89, p. 269]), in order to prove that $T|K^\star$ has zero topological entropy, it is sufficient to prove that $h(T, \mu) = 0$ for every ergodic probability measure μ supported on $K^\star$. Fix any such measure μ. Let us assume that μ is non-atomic, because otherwise there is nothing to prove. Recall the definitions (4.3)–(4.4) of the sets $N_i^\star$.

Claim 5.2. $\mu\left(e_i^{-1}(N_i^\star)\right) = 0$ for each $i \in \{1, 2\}$.

Proof of the claim. By Lemma 4.6, the set $N_1^\star \subset \mathbb{P}^1$ is countable. Since A has the forward NOC, by Corollary 2.5 the set $e_1^{-1}(N_1^\star) \subset k^{\mathbb{Z}}$ is a countable union of sets of the form $\{\omega_-\} \times k^{\mathbb{Z}_+}$. Assume for a contradiction that $\mu\left(e_1^{-1}(N_1^\star)\right) > 0$. Then there exists $\omega_- \in k^{\mathbb{Z}_-}$ such that $F := \{\omega_-\} \times k^{\mathbb{Z}_+}$ has measure $\mu(F) > 0$.

By Poincaré recurrence, there exists $p \geqslant 1$ such that $T^{-p}(F) \cap F \neq \varnothing$. It follows that the infinite word ω_- is periodic with period p, which in turn implies that $T^{-p}(F) \subset F$. By invariance, $\mu\left(F \smallsetminus T^{-p}(F)\right) = 0$ and

$$\begin{aligned}\mu\left(\bigcap_{n\geqslant 0} T^{-np}(F)\right) &= \mu(F) - \mu\left(F \smallsetminus T^{-p}(F)\right) - \mu\left(T^{-p}(F) \smallsetminus T^{-2p}(F)\right) - \cdots \\ &= \mu(F) > 0.\end{aligned}$$

But the set $\bigcap_{n\geqslant 0} T^{-np}(F)$ is a singleton, thus contradicting the assumption that μ is non-atomic.

We have proved the claim when $i = 1$. The case $i = 2$ is analogous, using instead the backwards NOC. □

Recalling notation (5.1), our next step is to show the following:

Claim 5.3. $\overline{\mathcal{S}(K^\star)} = \overline{\mathcal{B}(K^\star)}$.

Proof. We need to show that $\mathcal{B}(K^\star) \subset \overline{\mathcal{S}(K^\star)}$. For that, it is sufficient to prove that the σ-algebra $\overline{\mathcal{S}(K^\star)}$ contains all the compact subsets of $K^\star$. So fix an arbitrary compact set $C \subset K^\star$, and define

$$D := K^\star \cap e_2^{-1}(e_2(C)) \smallsetminus e_1^{-1}(N_1^\star),$$

The set $e_2(C)$ is compact and, in particular, Borel; so D is a Borel subset of $K^\star$. Let us show that

$$D \in \mathcal{S}(K^\star). \tag{5.3}$$

Take $\eta \in K^\star \cap \pi_-^{-1}(\pi_-(D))$. Then there exists $\xi \in D$ such that $\pi_-(\xi) = \pi_-(\eta)$, that is, ξ and η have the same past. So $e_1(\eta)$ equals $e_1(\xi)$, which, by definition of D, does not belong to $N_1^\star$. Using (4.5) we obtain $e_2(\eta) = e_2(\xi) \in e_2(C)$, thus proving that $\eta \in D$. We have shown that $K^\star \cap \pi_-^{-1}(\pi_-(D)) \subset D$, which implies (5.3).

Next, let us show that

$$\mu\left(D \vartriangle C\right) = 0. \tag{5.4}$$

Take $\xi \in D \smallsetminus C$; then there exists $\eta \in C$ such that $e_2(\eta) = e_2(\xi)$. Since $\eta \neq \xi$, by Corollary 2.5 we deduce that $e_1(\eta) \neq e_1(\xi)$, and so $e_2(\xi) \in N_2^\star$. This shows that $D \smallsetminus C \subset e_2^{-1}(N_2^\star)$. On the other hand, it is immediate that $C \smallsetminus D \subset e_1^{-1}(N_1^\star)$. Using Claim 5.2 we obtain (5.4).

Facts (5.3) and (5.4) put together imply that $C \in \overline{\mathcal{S}(K^\star)}$, as we wanted to prove. □

Using Lemma 5.1 and Claim 5.3 we conclude that the partition $\mathcal{C}(K^\star)$ is a one-sided generator up to sets of zero μ-measure. It follows that $h(T, \mu) = 0$, completing the proof of the theorem. □

6. Obtaining positive entropy

In this section we prove Theorem 1.4.

6.1. Sufficient conditions for the existence of many bounded products.

Lemma 6.1. *Given a sequence B_0, B_1, …of matrices in $\mathrm{SL}(2,\mathbb{R})$, let $P_i := B_{i-1} \cdots B_0$ and let u_i, v_i be unit vectors in $\mathbb{R}^2$ such that $P_i u_i = \|P_i\| v_i$. Suppose that there are constants $0 < \kappa < 1 < C$ such that*

$$\|B_i\| \leqslant C \quad \textit{and} \quad \|B_i v_i\| \leqslant \kappa \quad \textit{for every } i.$$

Then

$$\|P_i\| \leqslant \frac{\sqrt{2}\, C}{\sqrt{1-\kappa^2}} \quad \textit{for every } i.$$

Proof. Recall that the Hilbert–Schmidt norm of a matrix A is defined as $\|A\|_{\mathrm{HS}} := \sqrt{\operatorname{tr} A^* A}$. If $A \in \mathrm{SL}(2,\mathbb{R})$ then $\|A\|_{\mathrm{HS}}^2 = \|A\|^2 + \|A\|^{-2}$.

Let B_i, P_i, u_i, v_i, C and κ be as in the statement of the lemma. Let $v_i^\perp$ be a unit vector orthogonal to v_i. With respect to the basis $\{v_i, v_i^\perp\}$ we can write

$$P_i P_i^* = \begin{pmatrix} \rho_i^2 & 0 \\ 0 & \rho_i^{-2} \end{pmatrix} \quad \text{and} \quad B_i^* B_i = \begin{pmatrix} \alpha_i & \beta_i \\ \beta_i & \gamma_i \end{pmatrix},$$

where $\rho_i = \|P_i\|$ and $\alpha_i = \langle B_i^* B_i v_i, v_i \rangle = \|B_i v_i\|^2$. So

$$\begin{aligned} \|P_{i+1}\|_{\mathrm{HS}}^2 &= \operatorname{tr} B_i^* B_i P_i P_i^* \\ &= \alpha_i \rho_i^2 + \gamma_i \rho_i^{-2} \\ &= \|B_i v_i\|^2 \|P_i\|^2 + \left(\|B_i\|_{\mathrm{HS}}^2 - \|B_i v_i\|^2\right) \|P_i\|^{-2} \\ &\leqslant \|B_i v_i\|^2 \|P_i\|_{\mathrm{HS}}^2 + \|B_i\|_{\mathrm{HS}}^2 \\ &\leqslant \kappa^2 \|P_i\|_{\mathrm{HS}}^2 + 2C^2 . \end{aligned}$$

It follows by induction that $\|P_i\|_{\mathrm{HS}}^2 \leqslant 2C^2/(1-\kappa^2)$ for every i, which implies the lemma. □

Given $\mathsf{A} = (A_1, \ldots, A_k) \in \mathrm{SL}(2,\mathbb{R})^k$, let $\langle \mathsf{A} \rangle$ be the semigroup generated by A, that is, the set of all products of the form $A_{i_n} \ldots A_{i_1}$ (where $n \geqslant 1$).

Let $\mathcal{C}$ be the set of $\mathsf{A} \in \mathrm{SL}(2,\mathbb{R})^k$ such that for every $v \in \mathbb{R}^2$ and $\varepsilon > 0$ there exists $P \in \langle \mathsf{A} \rangle$ such that $\|Pv\| < \varepsilon$. It is easily seen that $\mathsf{A} \in \mathcal{C}$ if and only if for every unit vector $v \in S^1$ there exists $P = A_{i_{n(v)}} \cdots A_{i_1} \in \langle \mathsf{A} \rangle$ such that $\|Pv\| < 1$. It follows from compactness of the unit circle that the lengths $n(v)$ can be chosen uniformly bounded, and that $\mathcal{C}$ is open.

Lemma 6.2. *Every $\mathsf{A} \in \mathcal{C}$ satisfies the second alternative in Theorem 1.4.*

Proof. Fix $\mathsf{A} \in \mathcal{C}$. Let $C := \max \|A_i\|$. It is an easy exercise to show that there exist $\kappa \in (0,1)$ and an integer $\ell \geqslant 2$ such that for every unit vector $v \in \mathbb{R}^2$ there exists a product $P \in \langle \mathsf{A} \rangle$ of length $\ell - 1$ such that $\|Pv\| < C^{-1}\kappa$, and in particular $\|A_i P v\| < \kappa$ for every $i = 1, \dots, k$. Let

$$L := \left\{\omega \in k^{\mathbb{Z}};\ \|A^{\ell n}(\omega)\| \leqslant C_1\ \forall n \in \mathbb{Z}\right\}, \quad \text{where} \quad C_1 := \frac{\sqrt{2}\, C^{\ell}}{\sqrt{1-\kappa^2}}$$

It follows from Lemma 6.1 that $L \neq \varnothing$; actually given any bi-infinite sequence of symbols $\dots, \omega_{-\ell}, \omega_0, \omega_\ell, \omega_{2\ell}, \dots$ in the alphabet $\{1, \dots, k\}$, we can choose the remaining symbols to form a word ω in L.

Let

$$K := \left\{\omega \in k^{\mathbb{Z}};\ \|A^n(T^m\omega)\| \leqslant C^{2\ell} C_1^2\ \forall n, m \in \mathbb{Z}\right\}.$$

By definition, this set is compact and T-invariant, and it is easy to see that it contains L. It follows from the previous observations about L that the topological entropy of K is at least $\ell^{-1} \log k$, and thus positive as required. □

6.2. **Checking denseness.** Let $\mathcal{H}$ be the set of $\mathsf{A} \in \mathrm{SL}(2,\mathbb{R})^k$ such that the one-step cocycle generated by A is uniformly hyperbolic). Consider the open set $\mathcal{U} := \mathcal{H} \cup \mathcal{C}$. By Lemma 6.2, every element of $\mathcal{U}$ satisfies one of the alternatives of Theorem 1.4. Therefore, to prove the theorem, it is sufficient to show that $\mathcal{U}$ is dense.

Let $\mathcal{E}$ be the set of $\mathsf{A} \in \mathrm{SL}(2,\mathbb{R})^k$ for which the semigroup $\langle \mathsf{A} \rangle$ contains an elliptic element R (that is, such that $|\mathrm{tr}\, R| < 2$). The sets $\mathcal{H}$ and $\mathcal{E}$ are open and pairwise disjoint. We recall the following result:

Theorem 6.3 ([Y'04, Prop. 6])**.** *$\mathcal{H} \cup \mathcal{E}$ is dense in* $\mathrm{SL}(2,\mathbb{R})^k$.

Therefore, to show that $\mathcal{U} := \mathcal{H} \cup \mathcal{C}$ is dense in $\mathrm{SL}(2,\mathbb{R})^k$, we need to show:

Lemma 6.4. *$\mathcal{C} \cap \mathcal{E}$ is dense in $\mathcal{E}$.*

Let $\mathcal{I}$ be the set of $\mathsf{A} \in \mathcal{E}$ such that $\langle \mathsf{A} \rangle$ contains a matrix conjugate to an irrational rotation.

Lemma 6.5. *$\mathcal{I}$ is dense in $\mathcal{E}$.*

Proof. Let $(A_1, \dots, A_k) \in \mathcal{E}$, and fix an elliptic product $A_{i_n} \dots A_{i_1}$. Let $P_\theta := R_\theta A_{i_n} \dots R_\theta A_{i_1}$, where R_θ denotes the rotation by angle θ. By [ABY'10, Lemma A.4], the function $\theta \mapsto \mathrm{tr}\, P_\theta$ has a nonzero derivative at $\theta = 0$. Therefore we can find θ_0 arbitrarily close to 0 such that P_{θ_0} is conjugate to an irrational rotation. Therefore $(R_{\theta_0} A_1, \dots, R_{\theta_0} A_k) \in \mathcal{I}$, proving the lemma. □

Proof of Lemma 6.4. Let $\mathcal{N}$ be the set of $\mathsf{A} = (A_1, \dots, A_k) \in \mathrm{SL}(2,\mathbb{R})^k$ such that not all A_i commute; then $\mathcal{N}$ is open and dense. We will show that

$$\mathcal{N} \cap \mathcal{I} \subset \mathcal{C}, \tag{6.1}$$

and so the desired result will follow from Lemma 6.5.

Take $\mathsf{A} = (A_1, \dots, A_k) \in \mathcal{N} \cap \mathcal{I}$. Let $R \in \langle \mathsf{A} \rangle$ be conjugate to an irrational rotation. Since the sets $\mathcal{N}$, $\mathcal{I}$ and $\mathcal{C}$ are invariant by conjugation, we can assume that R is an irrational rotation. Since $\mathsf{A} \in \mathcal{N}$, there exists a generator A_i that does not commute with R. Using the singular value decomposition of A_i, we see that there exist n, $m \geqslant 0$ such that $H := R^n A_i R^m$ is a hyperbolic matrix. Let s be the contracting eigendirection of H. Now, given any unit vector v and any $\varepsilon > 0$, we can find $j \geqslant 0$ such that the unit vector $R^j v$ is sufficiently close to s, and so there exists $\ell \geqslant 0$ such that $\|H^\ell R^j v\| < \varepsilon$. This shows that $\mathsf{A} \in \mathcal{C}$, thus proving (6.1) and the lemma. □

As explained before, Theorem 1.4 follows.

Comparing to the present paper, the proof of Theorem 2 in [BBD] uses similar but slightly simpler arguments to get zero exponents. It does not obtain bounded norms, however. The present construction, especially Lemma 6.1, is more related to strategy suggested on [BBD, Remark 11.3].

Appendix A. Complementary facts

A.1. **Optimization of other dynamical quantities.** The results we have proved up to this point concern the optimization (maximization or minimization) of the upper Lyapunov exponent λ_1. Let us discuss briefly how to obtain results for the lower Lyapunov exponent λ_2 and for the difference $\lambda_1 - \lambda_2$ (which is a measure of non-conformality).

Suppose $T\colon \Omega \to \Omega$ is a continuous transformation of a compact metric space and $A\colon X \to \mathrm{GL}(2,\mathbb{R})$ is a continuous map.

Define $B\colon X \to \mathrm{GL}(d,\mathbb{R})$ by

$$B(\omega) := A(T^{-1}\omega)^{-1}, \tag{A.1}$$

and consider it as a cocycle over T^{-1}. Then a point $\omega \in \Omega$ is Oseledets regular with respect to (T,A) iff if it is regular with respect to (T^{-1},B), and

$$\lambda_1(T^{-1},B,\omega) = -\lambda_2(T,A,\omega) \quad \text{and} \quad \lambda_2(T^{-1},B,\omega) = -\lambda_1(T,A,\omega).$$

In particular,

$$\lambda_2^{\top}(T,A) = -\lambda_1^{\perp}(T^{-1},B) \quad \text{and} \quad \lambda_2^{\perp}(T,A) = -\lambda_1^{\top}(T^{-1},B).$$

If (T,A) is an one-step cocycle then so is (T^{-1},B) (after taking an appropriate conjugation between T and T^{-1}), and a multicone for one of them is a complementary multicone for the other.

It is then obvious how to adapt Theorems 1.1, 1.3 and 1.4 to λ_2-optimization.

Now define another matrix-valued map

$$C(\omega) := |\det A(\omega)|^{-1/2} A(\omega). \tag{A.2}$$

Then for all ω in a full probability set,

$$\lambda_1(A,\omega) - \lambda_2(A,\omega) = 2\lambda_1(C,\omega) = -2\lambda_2(C,\omega)\,.$$

Also note that the cocycle (T,A) is dominated if and only if (T,C) is uniformly hyperbolic. If (T,A) is an one-step cocycle then so is (T,C), and a multicone for one of them is a multicone for the other.

It is then obvious how to adapt Theorems 1.1 and 1.3 to $(\lambda_1 - \lambda_2)$-optimization. In the converse direction, let us see $\mathrm{SL}(2,\mathbb{R})$-cocycles, can be adapted to cocycles taking values in $\mathrm{GL}_+(2,\mathbb{R})$ (the group of matrices with positive determinant) as follows:

Corollary A.1. *Fix $k \geqslant 2$ and let T be the full shift in k symbols. There exists an open and dense subset $\mathcal{V}$ of $\mathrm{GL}_+(2,\mathbb{R})^k$ such that for every $\mathsf{A} \in \mathcal{V}$,*

i) *either the one-step cocycle over T generated by A is dominated;*
ii) *or there exists a compact T-invariant set $K \subset k^{\mathbb{Z}}$ of positive topological entropy and such that the "non-conformalities" $\|A^{(n)}(\omega)\|/\mathfrak{m}(A^{(n)}(\omega))$. are uniformly bounded over $(\omega,n) \in K \times \mathbb{Z}$.*

Notice that in the first case we have $(\lambda_1 - \lambda_2)^\perp(\mathsf{A}) > 0$, while in the second case there exists a measure $\mu \in \mathcal{M}_T$ such that $(\lambda_1 - \lambda_2)(\mathsf{A}, \mu) = 0$ and moreover $h(T, \mu) > 0$.

Proof of Corollary A.1. Let $p\colon \mathrm{GL}_+(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ be the continuous open mapping $A \mapsto |\det A|^{-1/2} A$. Let $\mathcal{U}$ be given by Theorem 1.4, and define $\mathcal{V}$ as the preimage of $\mathcal{U}$ by p^k (the cartesian product of k copies of p). Then $\mathcal{V}$ has the stated properties. □

A.2. **Alternative characterizations of extremal exponents and of domination.** Some lesser assertions made at the Introduction were left unjustified, so let us deal with them now.

First, relation (1.6) actually holds in much greater generality: see Theorems 2.1 and A.3 in [Mo'13].

Next, let us prove relation (1.7). By subadditivity, its RHS equals

$$R := \inf_n \frac{1}{n} \log \inf_{i_1,\dots,i_n} \|A_{i_n} \dots A_{i_1}\| \, .$$

So $\lambda_1^\perp(\mathsf{A}) \geqslant R$ by definition. To check the converse inequality, fix $\varepsilon > 0$ and take symbols $i_1, \dots, i_n$ such that $\frac{1}{n} \log \|A_{i_n} \dots A_{i_1}\| < R + \varepsilon$. Consider the shift-invariant probability measure on $k^\mathbb{Z}$ supported on the periodic orbit $(i_1 \dots i_n)^\infty$. Then $\lambda_1^\perp(\mathsf{A}) \leqslant \lambda_1(A, \mu) < R + \varepsilon$. Since ε is arbitrary, we conclude that $\lambda_1^\perp(\mathsf{A}) = R$, so proving (1.7).

Finally, let us show that an one-step 2×2 cocycle is dominated if and only if the number $(\lambda_1 - \lambda_2)^\perp$ defined by (1.8) is positive. The "only if" part is evident, and actually does not require the one-step condition. To prove the "if" part, notice the equality

$$(\lambda_1 - \lambda_2)^\perp(\mathsf{A}) = \lim_{n\to\infty} \frac{1}{n} \log \inf_{i_1,\dots,i_n} \frac{\|A_{i_n} \dots A_{i_1}\|}{\mathfrak{m}(A_{i_n} \dots A_{i_1})} \, ,$$

which follows from (1.7) applied to the "normalized" one-step cocycle defined by (A.2). So if this number is positive then we can find positive constants c, δ such that (1.4) holds, and therefore the cocycle is dominated.

Let us remark that for general cocycles, $(\lambda_1 - \lambda_2)^\perp(A) > 0$ does not imply that the cocycle is dominated: for example T can be uniquely ergodic and the cocycle can have different Lyapunov exponents without being dominated: see e.g. [H'83, § 4].

A.3. **More on the existence of optimizing measures.** Given a cocycle (T, A), the numbers $\lambda_1(A, \mu)$ and $\lambda_2(A, \mu)$ respectively depend upper- and lower-semicontinuously on $\mu \in \mathcal{M}_T$, and therefore by compactness of $\mathcal{M}_T$, λ_1-maximizing and λ_2-minimizing measures always exist. For a similar reason, $(\lambda_1 - \lambda_2)$-maximizing measures always exist.

There are one-step cocycles where no λ_1-minimizing measure exists: see [BMo, Remark 1.7]; a simple example is $\mathsf{A} = (H, cR_\theta)$ where $H \in \mathrm{SL}(2,\mathbb{R})$ is hyperbolic, θ/π is irrational, and $c > 1$. Similarly, there are one-step cocycles where no λ_2-maximizing measure exists: the same example but with $c < 1$.

Let us give an example where no $(\lambda_1 - \lambda_2)$-minimizing measure exists. We will actually exhibit an example of an one-step $\mathrm{SL}(2,\mathbb{R})$-cocycle where no λ_1-minimizing measure exists.

Given a hyperbolic matrix L in $\mathrm{SL}(2,\mathbb{R})$, let u_L, $s_L \in \mathbb{P}^1$ denote its eigendirections, with u_L corresponding to an eigenvalue of modulus bigger than 1. For convenience, the action of L on $\mathbb{P}^1$ will also be denote by L.

Take A_1, A_2 hyperbolic matrices in $\mathrm{SL}(2,\mathbb{R})$ such that $\operatorname{tr} A_1$, $\operatorname{tr} A_2 > 2$ and $\operatorname{tr} A_1A_2 < -2$; then by [ABY'10, Prop. 3.4] there exists a cyclical order $<$ on $\mathbb{P}^1$ such that

$$u_{A_2} < u_{A_2A_1} < s_{A_2A_1} < s_{A_1} < u_{A_1} < u_{A_1A_2} < s_{A_1A_2} < s_{A_2} < u_{A_2}\,.$$

Now take a hyperbolic matrix $C \in \mathrm{SL}(2,\mathbb{R})$ such that (see Fig. 6):

$$u_{A_3} \in (s_{A_1}, u_{A_1}), \quad s_{A_3} \in (s_{A_2}, u_{A_2}), \quad \text{and} \quad A_3 u_{A_2} = s_{A_1}\,.$$

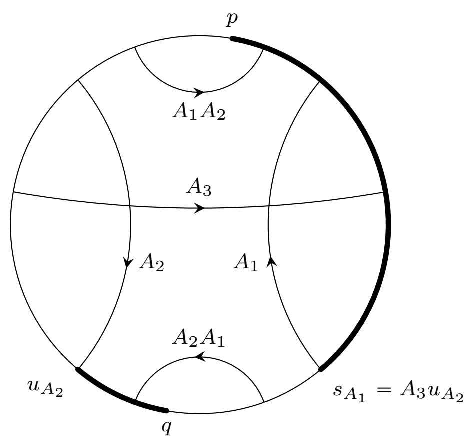

Fig. 6. The example of Proposition A.2. The thick part represents the "non-strict multicone" M. For each L, the arrow labelled L represents the hyperbolic geodesic from s_L to u_L.

Proposition A.2. *The one-step cocycle generated by* $\mathsf{A} := (A_1, A_2, A_3)$ *has no* λ_1*-minimizing measure.*

We note that the example is in the boundary of the hyperbolic component $H \subset \mathrm{SL}(2,\mathbb{R})^3$ described in [ABY'10, Prop. 4.16].

Before proving the proposition, let us describe a general geometrical construction. Consider a cocycle given by $T\colon \Omega \to \Omega$ and $A\colon \Omega \to \mathrm{GL}(2,\mathbb{R})$. Let S be the skew-product map on $\Omega \times \mathbb{P}^1$ induced by the cocycle. The derivative along the $\mathbb{P}^1$ fiber of the map S at a point $(\omega, x) \in \Omega \times \mathbb{P}^1$ is a linear map

$$L(\omega, x)\colon T_x\mathbb{P}^1 \to T_{A(\omega)x}\mathbb{P}^1\,. \tag{A.3}$$

Fix a rotation-invariant Riemannian metric on $\mathbb{P}^1$, and let $f(\omega, x)$ denote the operator norm of $L(\omega, x)$.

Now suppose that μ is an ergodic T-invariant measure and $\hat{\mu}$ is a S-invariant probability measure that projects to μ. Then we have the following fact (whose easy proof is left to the reader):

Lemma A.3. *If* $\lambda_1(A, \mu) = 0$ *then* $\int_{\Omega\times\mathbb{P}^1} \log f \, d\hat{\mu} = 0$.

Proof of Proposition A.2. Let A_1, A_2, A_3 be as above, and consider the one-step cocycle (T, A), where T is the shift on $\Omega := \{1, 2, 3\}^{\mathbb{Z}_+}$, and $A\colon \Omega \to \mathrm{SL}(2, \mathbb{R})$ is given by $A(\omega) = A_{\omega_0}$. Also let S be the induced skew-product map on $\Omega \times \mathbb{P}^1$.

Due to the "heteroclinic connection" $A_3 u_{A_2} = s_{A_1}$, the cocycle is not uniformly hyperbolic, and therefore $\lambda_1^{\perp}(\mathsf{A}) = 0$. To prove the proposition we will show that $\lambda_1(\mathsf{A}, \mu) > 0$ for every ergodic $\mu \in \mathcal{M}_T$.

Fix a point q in the interval $(u_{A_2A_1}, s_{A_2A_1})$, and then a point p in the interval $(A_1 q, A_2^{-1} q)$. Let $M := (s_{A_1}, p) \cup (u_{A_2}, q)$, as in Fig. 6. Then the set M is forward-invariant under the projection action of each matrix A_i.

Endow each connected component of M with its Riemannian Hilbert metric. Given a point $(\omega, x) \in \Omega \times M$, let $g(\omega, x)$ denote the operator norm of the linear map (A.3), where we take Hilbert metrics on both tangent spaces. Since the set $A(\omega)(M)$ is contained in M and none of its connected components coincided with a connected component of M, we have $g(\omega, x) < 1$.

Take any ergodic T-invariant measure μ, and lift it to a S-invariant measure $\hat{\mu}$ supported on the forward S-invariant compact set $\Omega \times \overline{M}$. We can assume that μ is neither δ_{1^∞} nor δ_{2^∞}, because otherwise $\lambda_1(\mathsf{A}, \mu) > 0$ trivially. It is then easy to see that $\hat{\mu}$ gives zero weight to the subset $\Omega \times \partial M$, and in particular the integral $I := \int \log g \, d\hat{\mu}$ is well-defined. It is immediate from the definitions that $\log g - \log f$ is coboundary with respect to S, and therefore $\int \log f \, d\hat{\mu} = I$. Since $g < 0$, we have $I < 0$ and so Lemma A.3 gives $\lambda_1(\mathsf{A}, \mu) \neq 0$, as we wanted to show. ☐

A.4. **Examples of non-uniqueness of optimizing measures.** Let us show that in the context of Theorem 1.3, the Mather sets $K^{\top}$ and $K^{\perp}$ are not necessarily uniquely ergodic. In other words, the λ_1-maximizing and λ_1-minimizing measures can fail to be unique.

Take a pair of matrices A_1 and A_2 in $\mathrm{GL}(2, \mathbb{R})^2$ with respective eigenvalues $\chi_1(A_1) > \chi_2(A_1)$ and $\chi_1(A_2) > \chi_2(A_2)$, all of them positive. Let $v_j(A_i) \in \mathbb{P}^1$ be the eigendirection of A_i corresponding to the eigenvalue $\chi_j(A_i)$. We can choose the pair $\mathsf{A} = (A_1, A_2)$ so that:

- the geodesics $\overrightarrow{v_2(A_1) v_1(A_1)}$ and $\overrightarrow{v_2(A_2) v_1(A_2)}$ cross;
- A has a forward invariant cone $M \subset \mathbb{P}^1$ with the forward nonoverlapping property;
- A has a backwards invariant cone $N \subset \mathbb{P}^1$ with the backards nonoverlapping property.

See Fig. 7.

Claim A.4. If ξ, $\eta \in K^{\top}$ are such that

$$\xi_{-1} = 1, \quad \xi_0 = 2, \quad \eta_{-1} = 2, \quad \eta_0 = 1. \tag{A.4}$$

then $\xi \notin K^{\top}$ or $\eta \notin K^{\top}$.

Proof of the claim. The four relations in (A.4) respectively imply:

$$e_1(\xi) \in A_1(M), \quad e_2(\xi) \in A_2^{-1}(N), \quad e_1(\eta) \in A_2(M), \quad e_2(\eta) \in A_1^{-1}(N).$$

It follows that the geodesics $\overrightarrow{e_2(\xi) e_1(\xi)}$ and $\overrightarrow{e_2(\eta) e_1(\eta)}$ are coparallel (see Fig. 7). The claim now follows from Corollary 4.5. ☐

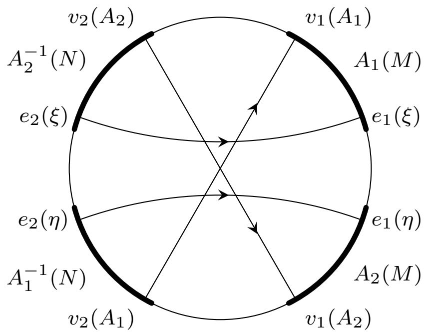

FIG. 7. An example with $K^\top = \{1^\infty, 2^\infty\}$.

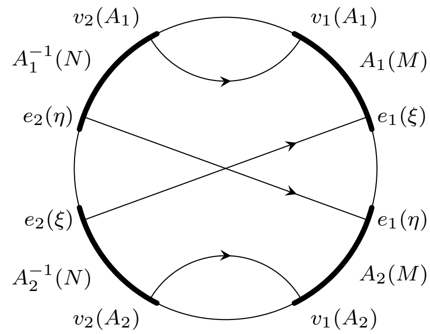

FIG. 8. An example with $K^\perp = \{1^\infty, 2^\infty\}$.

Let 1^∞ and $2^\infty \in \{1,2\}^{\mathbb{Z}}$ be the two fixed points of the shift, and let ζ^{12} and $\zeta^{21} \in k^{\mathbb{Z}}$ be the following "homoclinic points":

$$\zeta^{12}_n = \begin{cases} 1 & \text{if } n < 0, \\ 2 & \text{if } n \geqslant 0, \end{cases} \qquad \zeta^{21}_n = \begin{cases} 2 & \text{if } n < 0, \\ 1 & \text{if } n \geqslant 0. \end{cases}$$

It follows from Claim A.4 that $K^\top$ is contained in the closure of the orbit of either ζ^{12} or ζ^{21}. Since $K^\top$ equals the union of supports of the invariant probability measures that give full weight to $K^\top$ itself, it follows that $K^\top \subset \{1^\infty, 2^\infty\}$.

Of course we can choose A_1, A_2 such that additionally $\chi_1(A_1) = \chi_1(A_2)$; in this case $K^\top$ equals $\{1^\infty, 2^\infty\}$ and so it is not uniquely ergodic.

In a very similar way we produce an example where $K^\perp = \{1^\infty, 2^\infty\}$. The only difference is that $\mathsf{A} = (A_1, A_2)$ are chosen so that the geodesics $\overrightarrow{v_2(A_1)v_1(A_1)}$ and $\overrightarrow{v_2(A_2)v_1(A_2)}$ are coparallel, and so if the points ξ, η satisfy (A.4) then the geodesics $\overrightarrow{e_2(\xi)e_1(\xi)}$ and $\overrightarrow{e_2(\eta)e_1(\eta)}$ cross. (See Fig. 8.)

A.5. **Open questions and directions for future research.** There are several different directions along which one could try to extend the results of this paper.

Notice that the NOC is indeed necessary for the validity of Theorem 1.3; an example is given in Remark 1.2 for $\alpha = \beta$. However all the examples we know are very non-generic. So we ask whether the NOC can be replaced by a weaker condition, preferably one that is "typical" (open and dense) among k-tuples of matrices that generate dominated cocycles.

Regarding more general cocycles, we remark that there is also a notion of multicones for one-step cocycles over subshifts of finite type: see [ABY'10]. It seems to be straightforward to adapt the arguments given here to that more general situation (and thus also for n-step cocycles) with appropriate nonoverlapping conditions, but we have not checked the details.

Even more generally, we would like to have results about Lyapunov-optimizing measures for cocycles that are not locally constant. We believe that some of the construction of this paper should extend to cocycles admitting unstable and stable holonomies (over a hyperbolic base dynamics).

Let us return to one-step cocycles over the full shift. A possible strengthening of the conclusions of Theorem 1.3 would be to replace subexponential complexity (zero entropy) by linear complexity (as in [BMa'02]), or polynomial complexity (as in [HMS'13]). Perhaps under generic conditions we can even obtain bounded complexity (periodic orbits), in the style of [C].

Another line of study is to consider a relative Lyapunov-optimization problem for one-step cocycles where the frequencies of each matrix are fixed. The paper [JS'90] deals with a problem which can be reformulated in this terms. See [GL'07] for general results on relative optimization in the classical commutative setting. Let us also remark that this relative optimization setting is natural in the context of Lagrangian dynamics, where it corresponds to fixing the homology; see [Ma'91].

It should also be worthwhile to investigate the relations between Lyapunov-optimizing results as ours and the geometry of Riemann surfaces.

Regarding non-dominated one-step $\mathrm{SL}(2,\mathbb{R})$-cocycles, Theorem 1.4 says that we should not expect λ_1-minimizing measures to have zero entropy. However, it seems likely that λ_1-maximizing measures should have zero entropy. Notice that the corresponding Mather set (whose existence is given by [Mo'13]) is automatically uniformly hyperbolic.

Let us also remark that the only examples of k-tuples of matrices that do not satisfy the dichotomy of Theorem 1.4 (or Corollary A.1) are very particular ones (e.g., appropriate k-tuples with a common invariant direction). So we ask whether these counterexamples can be described explicitly, or at least whether they are contained in a finite union of submanifolds of positive codimension.

Of course most of the concepts and questions discussed in this paper make sense in higher dimension. In particular, we ask whether a higher-dimensional version of our zero entropy Theorem 1.3 (stated in terms of domination of index 1) holds true. As mentioned above, the construction of Barabanov functions can be adapted to this situation: see [BMo, § 2.2]. Lemma 4.2 should also be possible to extend: compare with [BMa'02, Prop. 2.6]. However, the rest of our proof relies on low-dimensional arguments.

Finally, we remark that the results obtained here can be considered as part of the multifractal analysis of Lyapunov exponents of linear cocycles, a broad field of study launched essentially by Feng [F'03].

Acknowledgements. We thank Artur Avila, Gonzalo Contreras, Oliver Jenkinson, Artur Lopes, Ian Morris, Mark Pollicott and Philippe Thieullen for enlightening discussions. The first named author acknowledges the hospitality of IMPAN and the support of CNPq and FAPERJ. The second author was hosted by PUC-Rio and supported by MNiSW grant N201 607640, Poland. Both authors were supported by EU BREUDS.

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