ON ENTROPY OF DYNAMICAL SYSTEMS WITH ALMOST SPECIFICATION

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Abstract. We construct a family of shift spaces with almost specification and multiple measures of maximal entropy. This answers a question from Climenhaga and Thompson [*Israel J. Math.* **192** (2012), no. 2, 785–817]. Elaborating on our examples we prove that sufficient conditions for every shift factor of a shift space to be intrinsically ergodic given by Climenhaga and Thompson are in some sense best possible, moreover, the weak specification property neither implies intrinsic ergodicity, nor follows from almost specification. We also construct a dynamical system with the weak specification property, which does not have the almost specification property. We prove that the minimal points are dense in the support of any invariant measure of a system with the almost specification property. Furthermore, if a system with almost specification has an invariant measure with non-trivial support, then it also has uniform positive entropy over the support of any invariant measure and can not be minimal.

We study dynamical systems with weaker forms of the specification property. Throughout this paper a *dynamical system* is always a continuous self-map of a compact metric space. We focus on the topological entropy and the problem of uniqueness of a measure of maximal entropy for systems with the almost specification or weak specification property (for definitions, see Section 2). We also prove that these two specification-like properties are non-equivalent — neither of them implies the other. Recall that dynamical systems with a unique measure of maximal entropy are known as *intrinsically ergodic*. The problem of intrinsic ergodicity of shift spaces with almost specification was mentioned in [5, p. 798], where another approach was developed in order to prove that certain classes of symbolic systems and their factors are intrinsically ergodic. We answer the problem in the negative and provide examples of shift spaces with the weak (almost) specification property and many measures of maximal entropy[1]. Our approach is based on the construction of a special family of shift spaces which allows us also to prove that the sufficient condition for the inheritance of intrinsic ergodicity by factors from the Climenhaga-Thompson paper [5] cannot be removed. We restate the Climenhaga-Thompson condition in Section 4.4.

Theorem 4.1. *There exist one-sided shift spaces X_1, X_2, X_3, X_4 such that*

(i) X_1 has the almost specification property and multiple measures of maximal entropy;
(ii) X_2 has the weak specification property and multiple measures of maximal entropy;
(iii) X_3 has the almost specification property but not the weak specification property;

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[1]When we communicated this result to Dan Thompson, he kindly informed us that Ronnie Pavlov had also solved the same problem (see [20]). Both solutions were independently discovered in July 2014. We would like to thank Ronnie Pavlov for sharing his work with us.

(iv) X_4 *has a shift factor* Y *such that*
  (a) the languages of X_4 *and* Y *have Climenhaga-Thompson decompositions* $\mathcal{B}(X_4) = \mathcal{C}^{\mathrm{p}}_{\mathrm{X}} \cdot \mathcal{G}_X \cdot \mathcal{C}^{\mathrm{s}}_{\mathrm{X}}$ *and* $\mathcal{B}(Y) = \mathcal{C}^{\mathrm{p}}_{\mathrm{Y}} \cdot \mathcal{G}_Y \cdot \mathcal{C}^{\mathrm{s}}_{\mathrm{Y}}$*,*
  (b) $h(\mathcal{G}_X) > h(\mathcal{C}^{\mathrm{p}}_{\mathrm{X}} \cup \mathcal{C}^{\mathrm{s}}_{\mathrm{X}})$ *and* $h(\mathcal{G}_Y) \le h(\mathcal{C}^{\mathrm{p}}_{\mathrm{Y}} \cup \mathcal{C}^{\mathrm{s}}_{\mathrm{Y}})$*,*
  (c) X_4 *is intrinsically ergodic,*
  (d) Y *has multiple measures of maximal entropy.*

Note that the examples constructed in the theorem above can be easily turned into two-sided shift spaces with the same properties (see [5, Sec. 2.1] for more details).

We also prove that nontrivial dynamical systems with the almost specification property and a full invariant measure have uniform positive entropy and horseshoes (subsystems which are extensions of the full shift over a finite alphabet).

Theorem 5.3. *Let* (X, T) *be a dynamical system with the almost specification property. Then the restriction of* (X, T) *to the measure center is a topological K system. If the measure center is non-trivial, then* (X, T) *contains a horseshoe.*

It follows that minimal points are dense in the measure center (the smallest closed invariant subset of the phase space which contains the support of *every* invariant measure) of a system with almost specification and that these systems cannot be minimal if they are nontrivial.

As the last result of the paper we prove the following.

Theorem 6.1. *There exists a dynamical system* $(\mathbf{X}, S)$ *with the periodic weak specification property, for which the almost specification property fails.*

1. Basic definitions and notation

We write $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

A *dynamical system* consists of a compact metric space X together with a continuous map $T\colon X \to X$. By ρ we denote a metric on X compatible with the topology. A *subsystem* of a dynamical system (X, T) is a nonempty closed subset K of X such that $T(K) \subset K$. A *minimal set* for (X, T) is a subsystem, which is minimal with respect to inclusion.

A dynamical system (X, T) is

- *(topologically) transitive* if for every non-empty open sets $U, V \subset X$ there is $n \in \mathbb{N}$ such that $U \cap T^{-n}(V) \neq \emptyset$;
- *(topologically) weakly mixing* if the product system $(X \times X, T \times T)$ is topologically transitive;
- *(topologically) mixing* if for every non-empty open sets $U, V \subset X$ there is $N \in \mathbb{N}$ such that for all $n \ge N$ we have $U \cap T^{-n}(V) \neq \emptyset$.

Let $\mathscr{U}$ and $\mathscr{V}$ be open covers of X. By $N(\mathscr{U})$ we denote the number of sets in a finite subcover of a $\mathscr{U}$ with smallest cardinality. By $T^{-i}\mathscr{U}$ $(i \in \mathbb{Z}_+)$ we mean the cover $\{T^{-i}(U) : U \in \mathscr{U}\}$ and $\mathscr{U} \vee \mathscr{V} = \{U \cap V : U \in \mathscr{U}, V \in \mathscr{V}\}$. The *topological entropy* $h(T, \mathscr{U})$ of an open cover $\mathscr{U}$ of X is defined (see [25]) as

$$\lim_{n\to\infty} \frac{1}{n} \log N\Big(\bigvee_{i=0}^{n-1} T^{-i}\mathscr{U}\Big).$$

The *topological entropy* of T is

$$h_{\mathrm{top}}(T) = \sup_{\mathscr{U}:\text{open cover of } X} h_{\mathrm{top}}(T, \mathscr{U}).$$

Let $\mathcal{M}_{\mathrm{T}}(X)$ be the space of T-invariant Borel probability measures on X. We denote the measure-theoretic entropy of $\mu \in \mathcal{M}_{\mathrm{T}}(X)$ by $h_\mu(T)$ (see [25]). The variational principle states that

$$h_{\text{top}}(T) = \sup_{\mu \in \mathcal{M}_{\text{T}}(X)} h_\mu(T).$$

A measure $\mu \in \mathcal{M}_{\text{T}}(X)$ that attains this supremum is a *measure of maximal entropy.* We say that a system (X, T) is *intrinsically ergodic* if it has a unique measure of maximal entropy.

Let $a, b \in \mathbb{Z}_+$, $a \leq b$. The *orbit segment* of $x \in X$ over $[a, b]$ is the sequence

$$T^{[a,b]}(x) = (T^a(x), T^{a+1}(x), \dots, T^b(x)).$$

We also write $T^{[a,b)}(x) = T^{[a,b-1]}(x)$. A *specification* is a family of orbit segments

$$\xi = \{T^{[a_j,b_j]}(x_j)\}_{j=1}^n$$

such that $n \in \mathbb{N}$ and $b_j < a_{j+1}$ for all $1 \leq j < n$.

The *Bowen distance between $x, y \in X$ along a finite set* $\Lambda \subset \mathbb{N}$ is

$$\rho_\Lambda^T(x, y) = \max\{\rho(T^j(x), T^j(y)) : j \in \Lambda\}.$$

By the *Bowen ball (of radius ε, centered at $x \in X$) along* Λ we mean the set

$$B_\Lambda(x, \varepsilon) = \{y \in X : \rho_\Lambda^T(x, y) < \varepsilon\}.$$

2. Specification and alikes

A dynamical system has the periodic specification property if one can approximate distinct pieces of orbits by single periodic orbits with a certain uniformity. Bowen introduced this property in [4] and showed that a basic set for an axiom A diffeomorphism T can be partitioned into a finite number of disjoint sets $\Lambda_1, \dots, \Lambda_k$ which are permuted by T and T^k restricted to Λ_j has the specification property for each $j = 1, \dots, k$. There are many generalizations of this notion. One of them is due to Dateyama, who introduced in [6] the *weak specification property* (Dateyama calls it "almost weak specification"). Dateyama's notion is a variant of a specification property used by Marcus in [17] (Marcus did not coined a name for the property he stated in [17, Lemma 2.1], we think that *periodic weak specification* is an appropriate name, see [15] for more details).

A dynamical system (X, T) has the *weak specification property* if for every $\varepsilon > 0$ there is a function $M_\varepsilon \colon \mathbb{N} \to \mathbb{N}$ with $\lim_{n\to\infty} M_\varepsilon(n)/n = 0$ such that for any specification $\{T^{[a_i,b_i]}(x_i)\}_{i=1}^k$ with $a_i - b_{i-1} \geq M_\varepsilon(b_i - a_i)$ for $i = 2, \dots, k$, we can find a point $x \in X$ such that for each $i = 1, \dots, k$ and $a_i \leq j \leq b_i$, we have

$$\rho(T^j(x), T^j(y_i)) \leq \varepsilon. \tag{1}$$

We say that M_ε is an ε-gap function for T.

Marcus proved in [17] that the periodic point measures are weakly dense in the space of invariant measures for ergodic toral automorphisms. Dateyama established that for an automorphism T of a compact metric abelian group the weak specification property is equivalent to ergodicity of T with respect to Haar measure [7].

Another interesting notion is the *almost specification property.* Pfister and Sullivan introduced the *g-almost product property* in [21]. Thompson [23] modified this notion slightly and renamed it the *almost specification property.* The primary examples of dynamical systems with the almost specification property are β-shifts (see [5, 21]). We follow Thompson's approach, hence the almost specification property presented below is a priori weaker (less restrictive) than the notion introduced by Pfister and Sullivan.

We say that $g\colon \mathbb{Z}_+ \times (0,\varepsilon_0) \to \mathbb{N}$, where $\varepsilon_0 > 0$ is a *mistake function* if for all $\varepsilon < \varepsilon_0$ and all $n \in \mathbb{Z}_+$ we have $g(n,\varepsilon) \le g(n+1,\varepsilon)$ and

$$\lim_{n\to\infty} \frac{g(n,\varepsilon)}{n} = 0.$$

With a mistake function g we associate an auxiliary function $k_g\colon (0,\infty) \to \mathbb{N}$ by declaring that $k_g(\varepsilon)$ is the smallest $n \in \mathbb{N}$ such that $g(m,\varepsilon) < m\varepsilon$ for all $m \ge n$.

Given a mistake function g, $0 < \varepsilon < \varepsilon_0$ and $n \ge k_g(\varepsilon)$ we define the set

$$I(g;n,\varepsilon) := \{\Lambda \subset \{0,1,\dots,n-1\} : \#\Lambda \ge n - g(n,\varepsilon)\}.$$

We say that a point $y \in X$ $(g;\varepsilon,n)$-traces an orbit segment $T^{[a,b]}(x)$ over $[c,d]$ if $n = b-a+1 = d-c+1$, $k_g(\varepsilon) \ge n$ and for some $\Lambda \in I(g;n,\varepsilon)$ we have $\rho^T_\Lambda(T^a(x),T^c(y)) \le \varepsilon$. By $B_n(g;x,\varepsilon)$ we denote the set of all points which $(g;\varepsilon,n)$-trace an orbit segment $T^{[0,n)}(x)$ over $[0,n)$. Note that $B_n(g;x,\varepsilon)$ is always closed and nonempty.

A dynamical system (X,T) has the *almost specification property* if there exists a mistake function g such that for any $m \ge 1$, any $\varepsilon_1,\dots,\varepsilon_m > 0$, and any specification $\{T^{[a_j,b_j]}(x_j)\}_{j=1}^m$ with $b_j - a_j + 1 \ge k_g(\varepsilon_j)$ for every $j = 1,\dots,m$ we can find a point $z \in X$ which $(g;b_j-a_j+1,\varepsilon_j)$-traces the orbit segment $T^{[a_j,b_j]}(x_j)$ for every $j = 1,\dots,m$.

In other words, the appropriate part of the orbit of z ε_j-traces with at most $g(b_j-a_j+1,\varepsilon_j)$ mistakes the orbit of x_j over $[a_j,b_j]$.

Intuitively, it should come as no surprise that almost specification does not imply the weak specification. But we did not expect at first that the converse implication is also false (see the last section).

3. Symbolic dynamics

We assume that the reader is familiar with the basic notions of symbolic dynamics. An excellent introduction to this theory is the book of Lind and Marcus [16]. We follow the notation and terminology presented there as close as possible. We restrict our exposition to one-sided shifts, but all our results remain valid in the two-sided setting.

Let Λ be a finite set (an *alphabet*) of *symbols*. The *full shift* over Λ is the set $\Lambda^{\mathbb{N}}$ of all infinite sequences of symbols. We equip Λ with the discrete topology and $\Lambda^{\mathbb{N}}$ with the product (Tikhonov) topology, given by the metric

$$d(x,x') = \begin{cases} 0, & \text{if } x = x', \\ 2^{-\min\{j\in\mathbb{N}: x_j \neq x'_j\}}, & \text{otherwise.} \end{cases}$$

By σ we denote the shift operator given by $\sigma(x)_i = x_{i+1}$. A set $A \subset \Lambda^{\mathbb{N}}$ is *σ-invariant* if $\sigma(A) \subset A$. A *shift space* over Λ is a closed and σ-invariant subset of $\Lambda^{\mathbb{N}}$. A *block* (a *word*) over Λ is any finite sequence of symbols. The *length of a block* u, denoted $|u|$, is the number of symbols it contains. An *n-block* stands for a block of length n. An *empty block* is the unique block with no symbols and length zero. The set of all blocks over Λ (including empty block) is denoted by Λ^*. We write Λ^+ for the set of all nonempty blocks over Λ.

We say that a block $w = w_1 \dots w_n \in \Lambda^*$ *occurs in* $x = (x_i)_{i=1}^\infty \in \Lambda^{\mathbb{N}}$ and x *contains* w if $w_j = x_{i+j-1}$ for some $i \in \mathbb{N}$ and all $1 \le j \le n$. The empty block occurs in every point of $\Lambda^{\mathbb{N}}$. Similarly, given an n-block $w = w_1 \dots w_n \in \Lambda^*$, a *subblock* of w is any block of the form $v = w_i w_{i+1} \dots w_j \in \Lambda^*$ for each $1 \le i \le j \le n$. A *language* of a shift space $X \subset \Lambda^{\mathbb{N}}$ is the set $\mathcal{B}(X)$ of blocks over Λ which occur in some $x \in X$. The language of the shift space determines it: two shift spaces are equal if and only if they have the same language (see [16, Proposition 1.3.4] or [14, Proposition 3.17]). We say that a set $\mathcal{L} \subset \Lambda^*$ is *factorial*, if for any $u \in \mathcal{L}$ any subblock of u also belongs to $\mathcal{L}$. A set $\mathcal{L}$ is *right prolongable* if for every block u in $\mathcal{L}$ there is a symbol $a \in \Lambda$ such that the concatenation ua also belongs to $\mathcal{L}$. Every $\mathcal{L} \subset \Lambda^*$ which is right prolongable and factorial is a language of a unique nonempty shift space.

It is convenient to adapt definitions of the weak specification and almost specification property to symbolic dynamics.

We say a non-decreasing function $\theta\colon \mathbb{N} \to \mathbb{Z}_+$ is a *discrete mistake function* if $\theta(n) \leq n$ for all n and $\theta(n)/n \to 0$. A shift space has the *symbolic almost specification property* if there exists a discrete mistake function θ such that for every $n \in \mathbb{N}$ and $w_1, \dots, w_n \in \mathcal{B}(X)$, there exist words $v_1, \dots, v_n \in \mathcal{B}(X)$ with $|v_i| = |w_i|$ such that $v_1 v_2 \dots v_n \in \mathcal{B}(X)$ and each v_i differs from w_i in at most $\theta(|v_i|)$ places.

To see that for shift spaces the symbolic almost specification property is equivalent to the almost specification presented in the previous section for general spaces it is enough to note that:

(1) If g is a mistake function according to the general definition, then $\theta(n) = g(n, \delta)$, where $\delta = \min\{\varepsilon_0/2, 1/2\}$ is a discrete mistake function for the symbolic definition.
(2) If θ is a discrete mistake function, then a mistake function fulfilling the general definition can be defined for any n and $2^{-k} \leq \varepsilon < 2^{-k+1}$ by $k_g(\varepsilon) = k$ and $g(n, \varepsilon) = k_g(\varepsilon) \cdot (\theta(n) + 1)$ and setting $\varepsilon_0 = 1$.

We say that a shift space X has the *symbolic weak specification property* if there exists $t\colon \mathbb{N} \to \mathbb{Z}_+$ such that $t(n)/n \to 0$ as $n \to \infty$ and for any words $u, w \in \mathcal{B}(X)$ there exists a word $v \in \mathcal{B}(X)$ such that $x = uvw \in \mathcal{B}(X)$ and $|v| = t(|w|)$. We say that t is a *transition function* for X.

The equivalence of the above symbolic definition with the general one can be proved the same way as for the almost specification property.

Given an infinite collection of words $\mathcal{L}$ over an alphabet Λ, the entropy of $\mathcal{L}$ is $h(\mathcal{L}) = \limsup_{n \to \infty} \frac{1}{n} \log \#(\mathcal{L} \cap \Lambda^n)$. Note that the entropy of the language of a shift space is just the topological entropy of this shift space.

4. Almost specification and measures of maximal entropy

In this section we construct a family of shift spaces which contains examples claimed by our main result.

Theorem 4.1. *There exist one-sided shift spaces X_1, X_2, X_3, X_4 such that*

(i) X_1 has the almost specification property and multiple measures of maximal entropy;
(ii) X_2 has the weak specification property and multiple measures of maximal entropy;
(iii) X_3 has the almost specification property but not the weak specification property;
(iv) X_4 has a shift factor Y such that
  (a) the languages of X_4 and Y have Climenhaga-Thompson decompositions $\mathcal{B}(X_4) = \mathcal{C}^{\mathrm{p}}_{\mathrm{X}} \cdot \mathcal{G}_X \cdot \mathcal{C}^{\mathrm{s}}_{\mathrm{X}}$ and $\mathcal{B}(Y) = \mathcal{C}^{\mathrm{p}}_{\mathrm{Y}} \cdot \mathcal{G}_Y \cdot \mathcal{C}^{\mathrm{s}}_{\mathrm{Y}}$,
  (b) $h(\mathcal{G}_X) > h(\mathcal{C}^{\mathrm{p}}_{\mathrm{X}} \cup \mathcal{C}^{\mathrm{s}}_{\mathrm{X}})$ and $h(\mathcal{G}_Y) \leq h(\mathcal{C}^{\mathrm{p}}_{\mathrm{Y}} \cup \mathcal{C}^{\mathrm{s}}_{\mathrm{Y}})$,
  (c) X_4 is intrinsically ergodic,
  (d) Y has multiple measures of maximal entropy.

We postpone the proof to Section 4.6 below. As we want to kill two (actually, more than two) birds with one stone, our construction is a little bit more involved than needed for each of our goals separately. We use the flexibility to shorten the total length of the paper.

4.1. **Construction of $X_{\mathbf{R}}$.** Our aim is to construct a shift space, denoted by $X_{\mathbf{R}}$, for a given integers $p,q\in\mathbb{N}$ with $q\geq 2$ and a family of sets $\mathbf{R}=\{R_n\}_{n=1}^{\infty}$. Needless to say, $X_{\mathbf{R}}$ and its properties rely on these parameters. Our notation will not reflect the dependence on p and q.

4.1.1. *Parameters.* Fix integers $p,q\in\mathbb{N}$, $p,q\geq 2$. Let $\mathbf{R}=\{R_n\}_{n=1}^{\infty}$ be a nondecreasing sequence of nonempty finite subsets of $\mathbb{N}$ such that $\max R_n\leq n$ for each $n\in\mathbb{N}$. That is,

$$\{1\}=R_1\subset R_2\subset R_3\subset\dots \text{ and } R_n\subset\{1,\dots,n\}.$$

One may think that the elements of R_n are the special positions in a word of length n.

We define a nondecreasing function $r\colon\mathbb{Z}_+\to\mathbb{Z}_+$ by $r(0)=0$ and

$$r(n)=|R_n|\quad\text{for } n\in\mathbb{N}.$$

The value $r(n)$ is the number of special positions in a word of length n for every $n\in\mathbb{N}$. We have $r(n-1)\leq r(n)$ for each $n\in\mathbb{N}$.

We say that the set R_n has *a gap of length* k if $\{1,\dots,n\}\setminus R_n$ contains k consecutive integers. By N_k we denote the smallest n such that R_n has a gap of length k (if such an n exists, otherwise we set $N_k=\infty$). We say that the set $\mathbf{R}$ has *large gaps* if $N_k<\infty$ for all $k\in\mathbb{N}$. It is easy to see that the monotonicity condition ($R_n\subset R_{n+1}$ for $n\in\mathbb{N}$) implies that some set R_m in $\mathbf{R}$ has a gap of length k if and only if $k\geq n-\max R_n$ for some $n\in\mathbb{N}$. Note that $r(n)/n\to 0$ as $n\to\infty$ implies that $\{R_n\}_{n=1}^{\infty}$ has large gaps.

4.1.2. *Definition of* $X_{\mathbf{R}}$. Let $\mathscr{A}=\{1,\dots,p\}\times\{0,1,\dots,q-1\}\cup\{(0,0)\}$. We will depict $(a,b)\in\mathscr{A}$ as $\left\lfloor\begin{smallmatrix}a\\ b\end{smallmatrix}\right\rceil$ and regard $a\in\{0,1,\dots,p\}$ as the *color* of the whole symbol. We call the symbol $\left\lfloor\begin{smallmatrix}0\\ 0\end{smallmatrix}\right\rceil$ the *marker* symbol and denote the block of length one consisting of the marker by $\mathbf{0}$. We say that a word $\left\lfloor\begin{smallmatrix}a_1\dots a_n\\ b_1\dots b_n\end{smallmatrix}\right\rceil\in\mathscr{A}^+$ is *monochromatic* if $a=a_1=\dots=a_n$, and *polychromatic* otherwise. In the former case we call a the *color* of a monochromatic word. By the definition of $\mathscr{A}$, words of the form $\mathbf{0}^k$, $k\in\mathbb{N}$ are the only monochromatic words of color 0. We use capital letters to denote blocks (words) over $\mathscr{A}$ to remind that they can be identified with matrices. We say that a subblock

$$V=\left\lfloor\begin{smallmatrix}a_i\dots a_j\\ b_i\dots b_j\end{smallmatrix}\right\rceil\in\mathscr{A}^+$$

is a *maximal monochromatic subword* of

$$W=\left\lfloor\begin{smallmatrix}a_1\dots a_n\\ b_1\dots b_n\end{smallmatrix}\right\rceil\in\mathscr{A}^+$$

if $a_i=a_{i+1}=\dots=a_j$ and $1\leq i\leq j\leq n$ are such that $i=1$ or $a_{i-1}\neq a_i$, and $j=n$ or $a_j\neq a_{j+1}$. Furthermore, if $i=1$ ($j=n$), then we say that V is a *maximal monochromatic prefix* (*suffix*, respectively) of W.

We define the shift space $X_{\mathbf{R}}$ by specifying its language, the following factorial and right prolongable set $\Lambda_{\mathbf{R}}$. We call a block u *allowed* if and only if $u\in\Lambda_{\mathbf{R}}$. We define $\Lambda_{\mathbf{R}}$ in two steps:

(1) First, we declare that the empty word and all monochromatic words are allowed. We denote the set consisting of the empty block and all monochromatic words by $\mathcal{M}$.

(2) Next, we add to Λ_R some polychromatic words. We start by introducing auxiliary notions of restricted, free and good blocks. We say that a monochromatic block
$$W = \left\lfloor\begin{smallmatrix} a_1 \dots a_n \\ b_1 \dots b_n \end{smallmatrix}\right\rceil \in \mathscr{A}^+$$
in color $a \in \{1, \dots, p\}$ is *restricted* if $b_j = 0$ for each $j \in R_n$. In other words, symbols at positions from R_n in the second row of a restricted block are set to 0. We write $\mathcal{R}$ for the set of all restricted words. We say that a block W is *free* if $W = \mathbf{0}V$, where V is a restricted block or the empty block. Let $\mathcal{F}$ be the set of all free blocks. We call any member of a set $\mathcal{G} = \mathcal{F}^*$ of all finite concatenations of free blocks a *good word*. A polychromatic word is allowed if it can be written as a concatenation of the empty or an allowed monochromatic word and a good word, that is, if W is polychromatic, then $W \in \Lambda_{\mathbf{R}}$ if and only if $W = UV_1 \dots V_k$, where $V_1, \dots, V_k \in \mathcal{F}$, $k \geq 1$, and $U \in \mathcal{M}$.

It is easy to see that the set of allowed words defined above is factorial and right prolongable, hence it is a language of a shift space, which we denote by $X_{\mathbf{R}}$. Note that every allowed block can be written as a concatenation UV, where $U \in \mathcal{M}$ and $V \in \mathcal{G}$ (V may be empty). Furthermore, a prefix of a restricted word is again restricted and a suffix of a restricted word is monochromatic.

4.2. **Dynamics of $X_{\mathbf{R}}$.** We recall that a shift space X is *synchronized* if there exists a *synchronizing word* for X, that is, there is a word $v \in \mathcal{B}(X)$ such that $uv, vw \in \mathcal{B}(X)$, implies $uvw \in \mathcal{B}(X)$.

Lemma 4.2. *The shift space $X_{\mathbf{R}}$ is synchronized.*

Proof. It is easy to see that $\mathbf{0}$ is a synchronizing word for $X_{\mathbf{R}}$. □

Proposition 4.3. *If $\mathbf{R}$ has large gaps, then the shift space $X_{\mathbf{R}}$ is topologically mixing with dense periodic points.*

Proof. We prove first that $X_{\mathbf{R}}$ is weakly mixing. To this end take any nonempty allowed blocks $W_1, W_2 \in \Lambda_{\mathbf{R}}$. By Furstenberg Theorem (see [10, Thm. 1.11]) it is enough to show that the set
$$N(W_1, W_2) = \{t \in \mathbb{N} : W_1 W W_2 \in \Lambda_{\mathbf{R}} \text{ for some } W \text{ with } |W| = t\}$$
is thick, that is it contains arbitrarily long intervals of consecutive integers. Write $W_2 = U_2 V_2$ where $U_2 \in \mathcal{M}$ and $V_2 \in \mathcal{G}$. If U_2 is the empty block or $U_2 = \mathbf{0}^\ell$ for some $\ell \in \mathbb{N}$ then $W_1 \mathbf{0}^k W_2 \in \Lambda_{\mathbf{R}}$ for all $k \geq 0$. Otherwise we fix any $K \in \mathbb{N}$ and we use existence of large gaps to pick $N_K \in \mathbb{N}$ such that U_2 is a suffix of a restricted word W_t of length t for each $t \in \{N_k, N_K + 1, \dots, N_K + K - 1\}$. It follows that $W_1 \mathbf{0}^{|U_2|} W_t V_2$ ends with W_2, hence $\{N_K, \dots, N_K + K - 1\} \subset N(W_1, W_2)$. We conclude that $X_{\mathbf{R}}$ is weakly mixing. This completes the proof since it is well known that every weakly mixing synchronized shift is mixing and has dense set of periodic points. (e.g. see [19, Prop. 4.8]). □

Lemma 4.4. *If $r(n)/n \to 0$ as $n \to \infty$, then the shift space $X_{\mathbf{R}}$ has the almost specification property.*

Proof. We claim that the function $\theta \colon \mathbb{Z}_+ \to \mathbb{Z}_+$ given by $\theta(n) = r(n-1) + 1$ is a mistake function for $X_{\mathbf{R}}$. It is enough to show that given
$$U = \left\lfloor\begin{smallmatrix} a_1 \dots a_k \\ b_1 \dots b_k \end{smallmatrix}\right\rceil,\ V = \left\lfloor\begin{smallmatrix} c_1 \dots c_l \\ d_1 \dots d_l \end{smallmatrix}\right\rceil \in \Lambda_{\mathbf{R}}$$
we can change V in at most $\theta(l) = r(l-1) + 1$ positions to obtain a word V' which is free. Then the concatenation $UV' \in \Lambda_{\mathbf{R}}$. To define V' we change $\left\lfloor\begin{smallmatrix} c_1 \\ d_1 \end{smallmatrix}\right\rceil$ to $\mathbf{0}$ and

modify the maximal monochromatic prefix of $\left\lfloor\begin{smallmatrix} c_2 \dots c_l \\ d_2 \dots d_l \end{smallmatrix}\right\rceil$ by putting at most $r(l-1)$ zeros on the restricted positions in the second row. Such V' is clearly free and hence $UV' \in \Lambda_{\mathbf{R}}$. $\square$

Lemma 4.5. *The shift space $X_{\mathbf{R}}$ has the weak specification property if and only if $\{R_n\}_{n=1}^{\infty}$ has large gaps and $k/N_k \to 1$ as $k \to \infty$.*

Proof. Assume that $\{R_n\}_{n=1}^{\infty}$ has large gaps and $k/N_k \to 1$ as $k \to \infty$. We claim that $t \colon \mathbb{N} \to \mathbb{Z}_+$ given by $t(k) = N_k - k + 1$ is a transition function for $X_{\mathbf{R}}$. Take nonempty blocks

$$U = \left\lfloor\begin{smallmatrix} a_1 \dots a_j \\ b_1 \dots b_j \end{smallmatrix}\right\rceil,\ W = \left\lfloor\begin{smallmatrix} c_1 \dots c_k \\ d_1 \dots d_k \end{smallmatrix}\right\rceil \in \Lambda_{\mathbf{R}}.$$

Write $W = U_W V_W$, where $U_W \in \mathcal{M}$ and $V_W \in \mathcal{G}$ (note that U_W or V_W, but not both, may be empty). Assume first that U_W is nonempty, monochromatic and in color $a \in \{1, \dots, p\}$. Let $\ell = |U_W| \leq k$. We can find an allowed word V of length $N_\ell - \ell$ such that the concatenation VU_W is a restricted word. Therefore

$$U\mathbf{0}^i V U_W V_W \in \Lambda_{\mathbf{R}}$$

for every $i \geq 1$, in particular if $i = t(k) - N_\ell + \ell > 0$. If $U_W = \mathbf{0}^\ell$ for some $\ell \geq 0$ (here, $\ell = 0$ means that U_W is the empty word), then

$$U\mathbf{0}^i U_W V_W \in \Lambda_{\mathbf{R}}$$

for every $i \geq 1$, in particular for $i = t(k)$.

Now $t(k)/k = (N_k - k + 1)/k \to 0$ as $k \to \infty$ implies that $X_{\mathbf{R}}$ has the weak specification property.

Next assume that $X_{\mathbf{R}}$ has the weak specification property. Pick a color $a \neq 0$ and let $U = \mathbf{0}$ and $W_k = \left\lfloor\begin{smallmatrix} a \\ 1 \end{smallmatrix}\right\rceil^k$ for $k \in \mathbb{N}$. Then W_k is a monochromatic, but not restricted word of length k. Weak specification implies that for each $k \in \mathbb{N}$ there is a word V_k of length $t(k)$ such that UVW_k is an allowed word. Note that $|V_k| \geq 1$, because W_k is not restricted, UV_kW_k is polychromatic, and hence the maximal monochromatic suffix of V_kW_k has to be a restricted word. Let $j(k)$ be the length of this maximal monochromatic suffix. By definition of N_k we have $N_k \leq j(k)$. Furthermore, $k < j(k) \leq |V_k| + k$. But we know that $t(k)/k = |V_k|/k \to 0$ as $k \to \infty$. It is now clear that $j(k) - \max R_{j(k)} \geq k$ and $k/j(k) \to 1$ as $k \to \infty$, which implies $k/N_k \to 1$ as $k \to \infty$, and completes the proof. $\square$

4.3. **Entropy of $X_{\mathbf{R}}$.** In this section we collect some auxiliary estimates for entropy of $X_{\mathbf{R}}$.

Lemma 4.6. *Let $\mathcal{F}_n$ and $\mathcal{G}_n$ denote the number of n-blocks in $\mathcal{F}$ and $\mathcal{G}$, respectively. Then*

(1) $\mathcal{F}_0 = \mathcal{F}_1 = 1$ and $\mathcal{F}_n = p \cdot q^{(n-1)-r(n-1)}$ for all $n > 1$;

(2) $\mathcal{G}_0 = \mathcal{G}_1 = 1$ and

$$\mathcal{G}_n = \sum_{i=1}^{n} \mathcal{F}_i \, \mathcal{G}_{n-i} = \mathcal{G}_{n-1} + \sum_{j=1}^{n-1} pq^{j-r(j)} \cdot \mathcal{G}_{n-1-j} \quad \text{for } n > 1. \tag{2}$$

Proof. The first point is obvious. The equalities $\mathcal{G}_0 = \mathcal{G}_1 = 1$ follow from the definition of $\mathcal{G}$. Let $n \geq 2$ and let $W \in \mathcal{G}_n$ be a concatenation of free words. Then W must end with a free word V of length $j \in \{1, \dots, n\}$ which has the form $V = \mathbf{0}U$ for some restricted word U. The word V can be chosen in $\mathcal{F}_j$ different ways, hence the formula. $\square$

Let us note (for later reference) the following inequality

$$1 + p\sum_{j=1}^{\infty} q^{-r(j)} \leq q. \tag{3}$$

We will show (see Proposition 4.14) that $X_{\mathbf{R}}$ is intrinsically ergodic if and only if the inequality (3) fails. We first note some conditions which should be imposed on $r(n)$ to guarantee that (3) holds for some p and q.

Lemma 4.7. *If $r(n) > 0$ for every n and*

$$\liminf_{n\to\infty} \frac{r(n)}{\ln n} > 0,$$

then there is $Q \geq 2$ such that the series

$$\sum_{n=1}^{\infty} q^{-r(n)}$$

converges for all integers $q \geq Q$ and its sum tends to 0 as $q \to \infty$.

Proof. Observe that there is an integer $N > 0$ and $c > 0$ such that $r(n) > c \ln n = \frac{c}{\log_q e} \log_q n$ for all $n > N$. Then

$$\sum_{n=1}^{\infty} q^{-r(n)} \leq \sum_{n=1}^{N} q^{-r(n)} + \sum_{n=N+1}^{\infty} \frac{1}{n^{\frac{c}{\log_q e}}},$$

hence it is enough to take $Q > 2$ so large that $c/\log_Q e > 1$. □

By Lemma 4.7 given any function $r\colon \mathbb{N} \to \mathbb{N}$ such that $r(n) > 0$ for every n and

$$\liminf_{n\to\infty} \frac{r(n)}{\ln n} > 0,$$

for any integer $p \geq 2$ we can find $q > p$ such that the inequality (3) holds. Furthermore, if

$$\lim_{n\to\infty} \frac{r(n)}{\ln n} = \infty,$$

then the series from the left hand side of (3) converges for all $q \geq 2$.

Lemma 4.8. *If (3) holds, then $\mathcal{G}_n \leq q^n$ for every $n \geq 0$.*

Proof. We use the induction on n. We have $\mathcal{G}_0 = \mathcal{G}_1 = 1$. Assume the assertion is true for $j = 0, 1, \ldots, n-1$ where $n \in \mathbb{N}$. Using recurrence relation (2) we have

$$\begin{aligned}
\mathcal{G}_n &= \mathcal{G}_{n-1} + \sum_{i=1}^{n-1} pq^{i-r(i)} \cdot \mathcal{G}_{n-1-i} \\
&\leq q^{n-1} + \sum_{i=1}^{n-1} pq^{i-r(i)} \cdot q^{n-1-i} = q^{n-1} \cdot \big(1 + p \cdot \sum_{j=1}^{n} q^{-r(j)}\big) \\
&\leq q^{n-1} \cdot \big(1 + p \cdot \sum_{j=1}^{\infty} q^{-r(j)}\big) \leq q^n.
\end{aligned}$$ □

Lemma 4.9. *If (3) does not hold, then*

$$\liminf_{n\to\infty} \frac{\log \mathcal{G}_n}{n} > \log q.$$

Proof. Assume that (3) does not hold, that is,

$$1 + p \sum_{j=1}^{\infty} q^{-r(j)} > q.$$

Then we can find $N \in \mathbb{N}$ and $z > 1$ such that

$$1 + \sum_{j=1}^{N-1} pq^{-r(j)} > qz^{N+1}. \tag{4}$$

We claim that for all k it holds

$$\mathcal{G}_k \geq (qz)^{k-N}. \tag{5}$$

It is clear that (5) is true for all $k \leq N$. For the induction step we assume that for some $n \geq N$ the inequality (5) holds for all $0 \leq k \leq n$.

By (2) we have that

$$\begin{aligned} \mathcal{G}_{n+1} &\geq \mathcal{G}_n + \sum_{j=1}^{N-1} \mathcal{G}_{n-j}\, pq^{j-r(j)} \\ &\geq (qz)^{n-N} + \sum_{j=1}^{N-1} (qz)^{n-j-N} pq^{j-r(j)} \\ &= (qz)^{n-N}\left(1 + \sum_{j=1}^{N-1} z^{-j} pq^{-r(j)}\right). \end{aligned} \tag{6}$$

We have $z^{-j} \geq z^{-N}$ hence

$$\sum_{j=1}^{N-1} z^{-j} pq^{-r(j)} \geq z^{-N} \sum_{j=1}^{N-1} pq^{-r(j)}. \tag{7}$$

Furthermore,

$$\begin{aligned} 1 + z^{-N} \sum_{j=1}^{N-1} pq^{-r(j)} &= z^{-N}\left(z^N + \sum_{j=1}^{N-1} pq^{-r(j)}\right) \\ &\geq z^{-N}\left(1 + \sum_{j=1}^{N-1} pq^{-r(j)}\right) \\ &\geq qz. \end{aligned}$$

This, combined with (6) and (7) ends the proof of (5) for $k = n+1$ completing the induction. This lemma follows by (5). □

Lemma 4.10. *If* (3) *holds, then* $h_{top}(X) = \log q$.

Proof. Every allowed word of length $n \geq 2$ is either one of $1 + pq^n$ monochromatic words or starts with a monochromatic, not necessarily restricted word of color $a \in \{1, \dots, p\}$ and length $0 \leq i \leq n-1$ followed by a concatenation of free words. Therefore

$$|\,\mathcal{B}_n(X)| = 1 + pq^n + \mathcal{G}_n + \sum_{i=1}^{n-1} (1 + pq^i)\, \mathcal{G}_{n-i}\,.$$

In particular, $|\,\mathcal{B}_n(X)| \geq q^n$. It follows from Lemma 4.8 that $\mathcal{G}_n \leq q^n$ for all n, hence

$$|\,\mathcal{B}_n(X)| \leq 1 + pq^n + q^n + \sum_{i=1}^{n-1} (1 + pq^i) q^{n-i} \leq 1 + n(p+1)q^n.$$

It is now clear that

$$h_{\mathrm{top}}(X) = \lim_{n\to\infty} \frac{1}{n} \log |\,\mathcal{B}_n(X)| = \log q.$$ □

Corollary 4.11. *If* (3) *holds, then* $X_{\mathbf{R}}$ *has at least* p *ergodic measures of maximal entropy.*

Proof. The set X_a of all sequences with all symbols in the upper row in color $a \in \{1, \dots, p\}$ is clearly an invariant subsystem with entropy $\log q$. The result follows by Lemma 4.10. □

Note that the supports of the p measures of maximal entropy found by Corollary 4.11 are nowhere dense and pairwise disjoint.

4.4. **Climenhaga-Thompson decompositions.** We recall the notion introduced in [5]. We say that the language $\mathcal{B}(X)$ of a shift space X *admits Climenhaga-Thompson decomposition* if there are subsets $\mathcal{C}^{\mathrm{p}}$, $\mathcal{G}$, $\mathcal{C}^{\mathrm{s}}$ satisfying following conditions:

(I) for every $w \in \mathcal{B}(X)$ there are $u_p \in \mathcal{C}^{\mathrm{p}}$, $v \in \mathcal{G}$, $u_s \in \mathcal{C}^{\mathrm{s}}$ such that $w = u_p v u_s$.
(II) there exists $t \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ and $w_1, \dots, w_n \in \mathcal{G}$, there exist $v_1, \dots, v_{n-1} \in \mathcal{B}(X)$ such that $x = w_1 v_1 w_2 v_2 \dots v_{n-1} w_n \in \mathcal{B}(X)$ and $|v_i| = t$ for $i = 1, \dots, n-1$.
(III) For every $M \in \mathbb{N}$, there exists τ such that given $w \in \mathcal{B}(X)$ satisfying $w = u_p v u_s$ for some $u_p \in \mathcal{C}^{\mathrm{p}}$, $v \in \mathcal{G}$, $u_s \in \mathcal{C}^{\mathrm{s}}$, with $|u_p| \le M$ and $|u_s| \le M$, there exist words u', u'' with $|u'| \le \tau$, $|u''| \le \tau$ for which $u' w u'' \in \mathcal{G}$.

Proposition 4.12. *Let $\mathcal{C}^{\mathrm{p}} = \mathcal{M}$, $\mathcal{C}^{\mathrm{s}} = \emptyset$ and $\mathcal{G}$ be the collection of all good words. If the sequence $\{R_n\}_{n=1}^{\infty}$ has large gaps, then $\mathcal{C}^{\mathrm{s}}, \mathcal{G}, \mathcal{C}^{\mathrm{p}}$ is a Climenhaga-Thompson decomposition for $X_{\mathbf{R}}$.*

Proof. The condition (I) is a direct consequence of the definition of $X_{\mathbf{R}}$.

To prove that (II) holds with $t = 0$ just note that the concatenation of any two good words is again a good word.

For a proof of (III) we fix $M \in \mathbb{N}$ and take any $W \in \mathcal{B}(X_{\mathbf{R}})$ such that $W = U_p V$ for some $U_p \in \mathcal{C}^{\mathrm{p}} = \mathcal{M}$ with $|U_p| = M$ and $V \in \mathcal{G}$.

If $U_p = \mathbf{0}^{\ell}$ for $\ell \ge 0$, then W is already a good word and there is nothing to prove.

If U_p is of color $a \in \{1, \dots, p\}$, then the length of the maximal monochromatic prefix of W is equal to $|U_p| = M$. Hence we can extend U_p to a restricted word by adding a prefix in the same color of length $N_M - M$. Then adding $\mathbf{0}$ as a prefix we obtain a free word which we can freely concatenate with V to obtain a good word. Therefore (III) holds with $\tau = N_M$. □

Proposition 4.13. *Let $\mathcal{G}$, $\mathcal{C}^{\mathrm{p}}$ and $\mathcal{C}^{\mathrm{s}}$ be as above. The condition* (3) *does not hold if and only if*

$$h(\mathcal{G}) > h(\mathcal{C}^{\mathrm{s}} \cup \mathcal{C}^{\mathrm{p}}) = h(\mathcal{C}^{\mathrm{p}}).$$

Proof. Note that

$$h_{\mathrm{top}}(\mathcal{C}^{\mathrm{s}} \cup \mathcal{C}^{\mathrm{p}}) = h_{\mathrm{top}}(\mathcal{M}) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{M}_n = \log q.$$

If $h_{\mathrm{top}}(\mathcal{G}) > h_{\mathrm{top}}(\mathcal{C}^{\mathrm{s}} \cup \mathcal{C}^{\mathrm{p}}) = \log q$ then $h_{\mathrm{top}}(X) > \log q$ and hence by Lemma 4.10 condition (3) does not hold. The converse implication follows from Lemma 4.9. □

Proposition 4.14. *The shift space $X_{\mathbf{R}}$ is intrinsically ergodic if and only if the condition* (3) *does not hold.*

Proof. If the condition (3) does not hold, then Propositions 4.13 and 4.12 allow us to apply the Climenhaga-Thompson result ([5, Theorem C]) and deduce that $X_{\mathbf{R}}$ is intrinsically ergodic. If condition (3) holds, then Corollary 4.11 implies that $X_{\mathbf{R}}$ is not intrinsically ergodic. □

4.5. **Some concrete examples of R.** So far, we have not proved that suitable sequences $\{R_n\}_{n=1}^{\infty}$ exist. We fill this gap and provide concrete examples of $X_{\mathbf{R}}$.

Remark 4.15. *Let $q \ge 2$. If $r(n) = \lfloor \sqrt{n} \rfloor$ for each $n \in \mathbb{N}$, then*

$$\sum_{n=1}^{\infty} q^{-r(n)} = \sum_{k=1}^{\infty} (2k+1) q^{-k} = \frac{3q-1}{(q-1)^2}. \tag{8}$$

Proposition 4.16. *There exists a sequence $\{R_n\}_{n=1}^{\infty}$ and $q > p \geq 2$ such that $r(n)/n \to 0$ as $n \to \infty$, condition* (3) *is satisfied, and the shift space $X_{\mathbf{R}}$ has the almost specification property, but it does not have the weak specification property.*

Proof. If we denote
$$R_n = [1, n] \cap \{k^2 : k \in \mathbb{N}\},$$
then $r(n) = |R_n| = \lfloor\sqrt{n}\rfloor$, hence $X_{\mathbf{R}}$ has the almost specification property by Lemma 4.4. By (8) taking $p = 2$ and $q = 4$ we assure that (3) holds. Next, observe that in R_{k^2} the largest gap has length $k^2 - (k-1)^2 = 2k - 1$ hence $N_{2k} > k^2$, in particular $\liminf_{k\to\infty} k/N_k = 0$ and so the proof is finished by Lemma 4.5. □

Proposition 4.17. *There exists a sequence $\{R_n\}_{n=1}^{\infty}$ and $q > p \geq 2$ such that $r(n)/n \to 0$ as $n \to \infty$, condition* (3) *is satisfied and the shift space $X_{\mathbf{R}}$ has the weak specification property.*

Proof. If we denote
$$R_n = \{1, 2, \ldots, \lfloor\sqrt{n}\rfloor\}$$
then $r(n) = |R_n| = \lfloor\sqrt{n}\rfloor$. By (8) taking $p = 2$ and $q = 4$ we assure that (3) holds. Note that for every $\varepsilon > 0$ there is $K \in \mathbb{N}$ such that $\varepsilon K > 1$ and $\sqrt{k} < \varepsilon k$ for every integer $k > K$. Fix any $k > K$. It is clear that N_k is the minimal m such that $k + \sqrt{m} \leq m$. By the choice of ε, $k + \sqrt{m} \leq m$ holds when $k \leq (1-\varepsilon)m$. In particular, $N_k \leq 1 + k/(1-\varepsilon)$ which implies that $\lim_{k\to\infty} k/N_k \geq (1-\varepsilon)$. But ε can be arbitrarily small, hence $\lim_{k\to\infty} k/N_k = 1$ and so the proof is finished by Lemma 4.5. □

4.6. **Proof of the main theorem.** We are now in position to prove Theorem 4.1.

Proof of Theorem 4.1. Condition (i) follows by Proposition 4.16 and Corollary 4.11, (ii) follows by Proposition 4.17 and Corollary 4.11, while (iii) follows by Proposition 4.16.

Note that $R_n = \{1, 2, \ldots, \lfloor\sqrt{n}\rfloor\}$ has large gaps. Let $\mathbf{R} = \{R_n\}_{n=1}^{\infty}$. Put $q = 4$ and let X be given by our construction for $p = 3$ and Y be given by our construction for $p = 2$. There is a factor map $\pi\colon X \to Y$ given by the 1-block map given by $\left\lfloor{a \atop b}\right\rceil \mapsto \left\lfloor{a \atop b}\right\rceil$, if $a \in \{0, 1, 2\}$ and $b \in \{0, 1, 2, 3\}$; and $\left\lfloor{3 \atop b}\right\rceil \mapsto \left\lfloor{2 \atop b}\right\rceil$. By Remark 4.15 condition (3) is not satisfied when $p = 3$ and is satisfied when $p = 2$. By Proposition 4.14 the shift space X is intrinsically ergodic while Y has at least two measures of maximal entropy by Corollary 4.11. Proposition 4.13 implies (iva) and (ivb). □

Remark 4.18. *Climenhaga and Thompson proved that if a shift space X has the decomposition named after them given by the sets $\mathcal{C}^{\mathrm{p}}$, $\mathcal{G}$, $\mathcal{C}^{\mathrm{s}}$, and $h(\mathcal{G}) > h(\mathcal{C}^{\mathrm{p}} \cup \mathcal{C}^{\mathrm{s}})$, then X is intrinsically ergodic (*[5, Theorem C]*). Furthermore, if $h(\mathcal{C}^{\mathrm{p}} \cup \mathcal{C}^{\mathrm{s}}) = 0$, then every shift factor of X is intrinsically ergodic as well (*[5, Theorem D]*). Shift space constructed in Theorem 4.1*(iv) *shows that the assumptions of Theorem D in* [5] *cannot be altogether removed — Theorem D may not hold when $h(\mathcal{C}^{\mathrm{p}}_{\mathrm{X}} \cup \mathcal{C}^{\mathrm{s}}_{\mathrm{X}}) > 0$.*

Remark 4.19. *Invariant measures of $X_{\mathbf{R}}$ and their entropy can be analyzed by methods introduced by Thomsen* [24]*. Thomsen's theory applies because $X_{\mathbf{R}}$ is synchronized. The following claims can be proved in a straightforward way:*

1. *the Markov boundary of $X_{\mathbf{R}}$ is a disjoint union of subsystems X_a of monochromatic sequences in a given color $a \in \{1, \ldots, p\}$,*
2. *the entropy of the Fisher cover of $X_{\mathbf{R}}$ is equal to $h(\mathcal{G}) \geq \log q$.*

It follows from [24, Thm. 7.4] *that $X_{\mathbf{R}}$ has either $p+1$ ergodic measures of maximal entropy if $h(\mathcal{G}) = \log q$, or a unique, fully supported measure of maximal entropy if $h(\mathcal{G}) > \log q$. We refer the reader to* [24] *for the definitions of Markov boundary and Fisher cover.*

5. Almost specification, u.p.e. and horseshoes

The notion of *uniform positive entropy (u.p.e.)* dynamical systems was introduced in [2], as an analogue in topological dynamics of the notion of a K-system in ergodic theory (see [10, Def. 3.49] for the definition of the later). In particular, every non-trivial factor of a u.p.e. system has positive topological entropy. A pair $(x, x') \in X \times X$ is an *entropy pair* if for every standard cover $\mathscr{U} = \{U, V\}$ with $x \in \operatorname{int}(X \setminus U)$ and $x' \in \operatorname{int}(X \setminus V)$ we have $h(T, \mathscr{U}) > 0$. A dynamical system (X, T) has *uniform positive entropy (u.p.e.)* if every non-diagonal pair in $X \times X$ is an entropy pair. Generalizing the notion of an entropy pair, Glasner and Weiss [11] call an n-tuple $(x_1, x_2, \dots, x_n) \in X \times \dots \times X$ an *entropy n-tuple* if at least two of the points $\{x_j\}_{j=1}^n$ are different and whenever U_j are closed pairwise disjoint neighborhoods of the distinct points x_j, the open cover $\mathscr{U} = \{X \setminus U_j : 0 < j \leq n\}$ satisfies $h(T, \mathscr{U}) > 0$. We say that (X, T) is a *topological K system* if every non-diagonal tuple is an entropy tuple. We say that a Borel set $U \subset X$ is *universally null* for T if $\mu(U) = 0$ for every T-invariant measure μ. The *measure center* of a dynamical system (X, T) is the complement of the union of all universally null open sets. We note the following simple lemma for further reference.

Lemma 5.1. *For every dynamical system (X, T) there exists an invariant measure μ such that* $\operatorname{supp}\mu$ *is the measure center of* (X, T).

Proof. Denote the measure center of (X, T) by Y. Let $\mathcal{U}$ be a countable base of the topology on X. Enumerate elements of the family $\{U \in \mathcal{U} : U \cap Y \neq \emptyset\}$ as $\{V_i : i \in I\}$, where I is an at most countable set. For every $i \in I$ there is a T-invariant measure μ_i such that $\mu_i(V_i) > 0$. Let $\{\alpha_i\}_{i \in I}$ be any set of positive reals such that
$$\sum_{i \in I} \alpha_i = 1.$$
Set $\mu = \sum \alpha_i \mu_i$. It is easy to see that $\operatorname{supp}\mu = Y$. □

Standing assumption. In the remainder of this section we assume that (X, T) is a dynamical system with the almost specification property, g is a mistake function and k_g corresponds to g.

The following result is proved implicitly in the proof of [26, Theorem 6.8]. For the reader's convenience we provide it with a proof. Recall that $x \in X$ is a *minimal point* for $T \colon X \to X$ if its orbit closure is a minimal set.

Theorem 5.2. *For every $m \geq 1$, $\varepsilon_1, \dots, \varepsilon_m > 0$, $x_1, \dots, x_m \in X$, and integers $l_0 = 0 < l_1 < \dots < l_{m-1} < l_m = L$ with $l_j - l_{j-1} \geq k_g(\varepsilon_j)$ for $j = 1, \dots, m$ there is a minimal point $q \in X$ which $(g; l_j - l_{j-1}, \varepsilon_j)$ traces $T^{[l_{j-1}, l_j)}(x_j)$ over $[l_{j-1} + sL, l_j + sL)$ for every $j = 1, \dots, m$ and $s \in \mathbb{Z}_+$.*

Proof. Fix $m \geq 1$, $\varepsilon_1, \dots, \varepsilon_m > 0$, $x_1, \dots, x_m \in X$, and integers $l_0 = 0 < l_1 < \dots < l_{m-1} < l_m = L$ with $n_j = l_j - l_{j-1} \geq k_g(\varepsilon_j)$ for $j = 1, \dots, m$. By the almost specification property closed sets
$$C = \bigcap_{j=1}^{m} T^{-l_{j-1}}(B_{n_j}(g; x_j, \varepsilon_j), \text{ and } C_s = \bigcap_{j=0}^{s} T^{-jL}(C)$$
are nonempty. The set
$$Z = \bigcap_{s=0}^{\infty} C_s$$
is closed and nonempty because it is an intersection of a non-increasing family of closed nonempty subsets. Moreover, Z is T^L invariant, hence it contains a T^L-minimal point q. But q must be then also minimal for T. □

It follows easily from Theorem 5.2 that the minimal points are dense in the measure center of a dynamical system with the almost specification property. For a proof see [26, Theorem 6.8]. The above result suggests that a system with the almost specification should have a lot of minimal subsystems. However, there are examples of proximal dynamical systems with the almost specification property and a unique minimal point, which is then necessarily fixed (see Lemma 8.2 and Example 8.3 in [13]). The measure center of theses examples is trivial (it is the singleton of that fixed point) and hence they all have topological entropy zero.

We say that a dynamical system (X, T) *contains a horseshoe* if there are an integer $K > 0$ and a closed, T^K-invariant set Z such that the full shift over a finite alphabet with at least two elements is a factor of $(Z, T^k|_Z)$. Our definition generalizes the notion of horseshoe introduced by Misiurewicz and Szlenk [18] in the setting of interval maps (see also [3, Prop. II.15]).

Observe that a minimal system contains no horseshoes. Furthermore, there are non-trivial minimal systems which are topological K systems. This can be seen as follows: Recall that a dynamical system (X, T) is said to be *strictly ergodic* if there is a unique T-invariant Borel probability measure on X and it has the full support. Every strictly ergodic system is minimal because existence of a proper closed invariant set would lead to the existence of another invariant measure concentrated on it. A remarkable result due to R. Jewett and W. Krieger (see [10, Thm. 15.28]) says that any ergodic action is isomorphic to a strictly ergodic system. In particular, there is a strictly ergodic system whose unique ergodic measure is a K-measure. By [12, Thm. 3.4] such system is minimal and topological K.

Theorem 5.3. *Let (X, T) be a dynamical system with the almost specification property. Then the restriction of (X, T) to the measure center is a topological K system. If the measure center is non-trivial, then (X, T) contains a horseshoe.*

Proof. Let C denotes the measure center of (X, T). If $C \neq X$, then replace (X, T) by $(C, T|_C)$. The system $(C, T|_C)$ also has the almost specification property since the almost specification is preserved by passing to the measure center by [26, Thm. 6.7]. We shall, by convenient abuse of notation, still denote the resulting system by (X, T). By Lemma 5.1 we may assume that (X, T) admits a fully supported invariant measure denoted by μ. Fix $m \geq 1$ and let $U_1, \ldots, U_m$ be nonempty open sets and let $\delta > 0$ and $W_j \subset U_j$ for $j = 1, \ldots, m$ be nonempty open sets such that $V_j^\delta = \{y \in X : \rho(x, y) \leq \delta \text{ for some } x \in \overline{W}_j\} \subset U_j$ for $j = 1, \ldots, m$. By [13, Thm. 4.3] we know that (X, T) is topologically weakly mixing, hence X is either a singleton or a perfect space. In the former case, (X, T) is trivially a topological K system. In the later case we may assume that $V_1^\delta, \ldots, V_m^\delta$ are pairwise disjoint.

Let $\mu^\star = \mu \times \ldots \times \mu$ be the m-fold product measure on X^m. By ergodic decomposition theorem there are an ergodic measure ν on X^m and $\varepsilon > 0$ such that $\nu(W_1 \times \ldots \times W_m) > \varepsilon$. Let $(x_1, \ldots, x_m) \in X^m$ be a generic point for ν. For $j = 1, \ldots, m$ and $k \in \mathbb{N}$ define $N_k(x_j, W_j) = \{0 \leq l < k : T^l(x_j) \in W_j\}$. Furthermore, let

$$J_k = \{0 \leq l < k : T^l(x_j) \in W_j \text{ for any } j = 1, \ldots, m\} = \bigcap_{j=1}^m N_k(x_j, W_j).$$

By ergodic theorem there exists $K \in \mathbb{N}$ such that for $k \geq K$ we have $|J_k| \geq k\varepsilon$. Take $k > K$ such that $k \geq k_g(\delta)$ and $g(k, \delta) < k\varepsilon/m$. Note that for every $A_1, \ldots, A_m \in I(g; k, \delta)$ we have

$$|\bigcap_{i=1}^m A_i| \geq k - mg(k, \delta) > k(1 - \varepsilon).$$

Therefore $J_k \cap A_1 \cap \ldots \cap A_m \neq \emptyset$.

We claim that for any $s, t \in \{1, \dots, m\}$ with $s \neq t$ the sets $B_k(g; x_s, \delta)$ and $B_k(g; x_t, \delta)$ are disjoint. To reach a contradiction we assume that there is $x \in B_k(g; x_s, \delta) \cap B_k(g; x_t, \delta)$ with $s \neq t$. Let $u \in \{s, t\}$ and $A_u \in I(g; k, \delta)$ be such that

$$A_u = \{0 \le l < k : \rho(T^l(x_u), T^l(x)) \le \delta\}.$$

Then for $l \in J_k \cap A_s \cap A_t \neq \emptyset$ we have $\rho(T^l(x_u), T^l(x)) \le \delta$ and $T^l(x_u) \in W_u$, hence $T^l(x) \in V_u^\delta$ for $u \in \{s, t\}$. But V_s^δ and V_t^δ are disjoint for $s \neq t$, which is a contradiction.

Set $C_j = B_k(g; x_j, \delta)$ for $j = 1, \dots, m$. Define

$$Z = \bigcap_{s \in \mathbb{Z}_+} T^{-sk}(C_1 \cup \ldots \cup C_m).$$

We have $T^k(Z) \subset Z$. Given $\xi \colon I \to \{1, \dots, m\}$, where $I = \{0, 1, \dots, n-1\}$ or $I = \mathbb{Z}_+$ let

$$Z_\xi = \bigcap_{s \in I} T^{-sk}(C_{\xi(s)}). \tag{9}$$

Clearly,

$$Z = \bigcup_{\xi \in \{1, \dots, m\}^{\mathbb{Z}_+}} Z_\xi.$$

It is easy to see that Z and Z_ξ are always closed and nonempty by the almost specification property. Observe also that $z \in Z$ if and only if there is some $\xi \in \{1, \dots, m\}^{\mathbb{Z}_+}$ such that $z \in Z_\xi$. Moreover, for any I as above we have $Z_{\xi'} \neq Z_{\xi''}$ provided $\xi', \xi'' \colon I \to \{1, \dots, m\}$ and $\xi' \neq \xi''$. Therefore $\pi \colon Z \to \{1, \dots, m\}^{\mathbb{Z}_+}$ given by $\pi(z) = \xi$, where $\xi \in \{1, \dots, m\}^{\mathbb{Z}_+}$ is such that $z \in Z_\xi$ is a well-defined surjection. To see that π is continuous note that for every word $w \in \{1, \dots, m\}^{\{0,1,\dots,n-1\}}$ we have $\pi^{-1}(C[w]) = Z_w$ is closed, where $C[w]$ denotes the cylinder of w. Moreover, Z_w is open as

$$Z_w = Z \setminus \bigcup \left\{ Z_{w'} : w' \in \{1, \dots, m\}^{\{0,1,\dots,n-1\}}, \, w' \neq w \right\}$$

has closed complement. It is now easy to see that π is a factor map form a T^k invariant set Z onto the full shift on m symbols. In other words, (X, T) contains a horseshoe.

It remains to prove that (X, T) is a topological K system. To this end, assume that the open sets $U_1, \dots, U_m$ have pairwise disjoint closures. We need to show that the cover $\mathscr{U} = \left\{ X \setminus \overline{U}_1, \dots, X \setminus \overline{U}_m \right\}$ has positive entropy. Assume on the contrary that $h(T, \mathscr{U}) = 0$. Therefore there is n such that

$$N\Big(\bigvee_{i=0}^{kn-1} T^{-i}\mathscr{U}\Big) < \left(\frac{m}{m-1}\right)^n$$

as otherwise $h(T, \mathscr{U}) \ge \log(\frac{m}{m-1})/k > 0$. Let $\mathscr{V}$ be a subcover of $\bigvee_{i=0}^{kn-1} T^{-i}\mathscr{U}$ with less than $m^n/(m-1)^n$ elements. For each $\xi \in \{1, \dots, m\}^{\{0,1,\dots,n-1\}}$ fix a point $z_\xi \in Z_\xi$ (see (9)) and recall that the set $\{z_\xi\}$ has exactly m^n elements. Since $\mathscr{V}$ is a cover of X there is $V \in \mathscr{V}$ such that the set $A = \{\xi : z_\xi \in V\} \subset \{1, \dots, m\}^{\{0,1,\dots,n-1\}}$ satisfies $|A| > (m-1)^n$. It follows that there are $\xi^{(1)}, \dots, \xi^{(m)} \in A$ and $0 \le j < n$ such that $\left|\left\{\xi_j^{(s)} : 1 \le s \le m\right\}\right| = m$. Then there is $i \in J_k$ such that $T^{jk+i}(z_{\xi^{(s)}}) \in U_{\xi_j^{(s)}}$. By the definition of V there exists $U \in \mathscr{U}$ such that $V \subset T^{-jk-i}(U)$, which means that $T^{jk+i}(z_{\xi^{(s)}}) \in U$ for each $s = 1, \dots, m$. But $U = X \setminus \overline{U}_s$ for some $1 \le s \le m$ and there is also r such that $\xi_j^{(r)} = s$ which is a contradiction. Hence $h(T, \mathscr{U}) > 0$ and the proof is completed. □

6. Weak specification does not imply almost specification

In this section we are going to present a construction which proves the following:

Theorem 6.1. *There exists a dynamical system* $(\mathbf{X}, S)$ *with the periodic weak specification property, for which the almost specification property fails.*

For the rest of this section $(\mathbf{X}, S)$ denotes the dynamical system defined below. Theorem 6.1 is a consequence of Lemmas 6.2 and 6.3.

6.1. Definition of $(\mathbf{X}, S)$.

For every integer $m \geq 0$ let X_m be the shift space over the alphabet $\{a, b, c\}$ given by the following set of forbidden words:

$$\mathcal{F} = \{bc, cb\} \cup \left\{ x a^k y^l : x, y \in \{b, c\}\,,\ l \geq 1,\ 1 \leq k \leq 2^m \lceil \log_2(l+1) \rceil \right\}.$$

Roughly speaking, the words allowed in X_m consist of runs of a's, b's or c's subject to the condition on the length of the run of a's separating runs of b's or c's. Note that if u, w are words allowed in X_m, then $ua^l w$ is also allowed provided that $l \geq 2^m(\lceil \log_2 |w| \rceil + 1)$. This shows that X_m has the weak specification property.

Let $\mathbb{X} = \prod_{m=0}^{\infty} \{a, b, c\}^{\mathbb{N}}$. It is customary to think of elements of $\mathbb{X}$ as of infinite matrices from $\{a, b, c\}^{\mathbb{N} \times \mathbb{N}}$. Hence we will use the matrix notation to denote points $\mathbf{x}$ in $\prod_{m=0}^{\infty} X_m$. We write $\mathbf{x}_{i\star}$ for the i-th row of $\mathbf{x}$, $\mathbf{x}_{\star j}$ for the j-th column and $\mathbf{x}_{ij}$ for the symbol in the row i and column j. We endow $\mathbb{X}$ with the metric $\rho(\mathbf{x}, \mathbf{y}) = \sup_{i=0,1,\ldots} 2^{-i} d(\mathbf{x}_{i\star}, \mathbf{y}_{i\star})$. It follows that for points $\mathbf{x}, \mathbf{y} \in \mathbb{X}$ we have

$$\rho(\mathbf{x}, \mathbf{y}) < 2^{-n} \text{ if and only if } \mathbf{x}_{ij} = \mathbf{y}_{ij} \text{ for } i + j < n.$$

In other words, in order to $\mathbf{x}$ be close to $\mathbf{y}$ with respect to ρ the elements in the big upper left corners of matrices $\mathbf{x}, \mathbf{y}$ must agree. We define $S\colon \mathbb{X} \to \mathbb{X}$ to be the left shift, which acts on $\mathbf{x} \in \mathbb{X}$ by removing the first column and shifting the remaining columns one position to the left. It is easy to see that S is continuous on $\mathbb{X}$

Denote by $\mathbf{X}$ the subset of $\prod_{m=0}^{\infty} X_m \subset \mathbb{X}$ consisting of points $\mathbf{x}$ constructed by the following inductive procedure:

(1) Pick $\mathbf{x}_{0\star} \in X_0$.
(2) Assume that $\mathbf{x}_{i\star} \in X_0$ are given for $i = 0, 1, \ldots, m-1$ for some $m > 0$. Pick any $\mathbf{x}_{m\star} \in X_m$ fulfilling

$$\mathbf{x}_{mj} \in \left\{ b, \mathbf{x}_{(m-1)j} \right\} \text{ for every } j.$$

Roughly speaking, when rows $0, 1, \ldots, m-1$ are defined, we pick row m so that $\mathbf{x}_{m\star}$ is in X_m and for each column we either rewrite a symbol from the same column in the row above, or we write b. Note that it means that b's are persistent. In other words if $\mathbf{x}_{(m-1)j} = b$ for some m and j then we have to fill the rest of the column j with b's, that is $\mathbf{x}_{ij} = b$ for all $i \geq m$. Clearly $\mathbf{X}$ is nonempty, closed and S-invariant ($S(\mathbf{X}) = \mathbf{X}$).

6.2. Properties of $(\mathbf{X}, S)$.

Lemma 6.2. *The dynamical system* $(\mathbf{X}, S)$ *has the weak specification property.*

Proof. Given any $\varepsilon > 0$ pick $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$ and let $m = 2^n$. We claim that $M_\varepsilon\colon \mathbb{N} \to \mathbb{N}$ given by

$$M_\varepsilon(l) = 2^m \left(\lceil \log_2(l + m) \rceil + 1 \right) + m$$

is an ε-gap function for S on $\mathbf{X}$. Clearly, $M_\varepsilon(l)/l \to 0$ as $l \to \infty$. Fix $K \in \mathbb{N}$, $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(K)} \in \mathbf{X}$ and integers $0 = \beta_0 < \alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2 < \ldots \alpha_K \leq \beta_K$.

$$
\begin{array}{|ccc|ccc|ccc|}
\hline
\mathbf{x}^{(k)}_{0\alpha_k} & \cdots & \mathbf{x}^{(k)}_{0(\beta_k+m)} & a & \cdots & a & \mathbf{x}^{(k+1)}_{0\alpha_{k+1}} & \cdots & \mathbf{x}^{(k+1)}_{0(\beta_{k+1}+m)} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{x}^{(k)}_{(m-1)\alpha_k} & \cdots & \mathbf{x}^{(k)}_{(m-1)(\beta_k+m)} & a & \cdots & a & \mathbf{x}^{(k+1)}_{(m-1)0\alpha_{k+1}} & \cdots & \mathbf{x}^{(k+1)}_{(m-1)(\beta_{k+1}+m)} \\
\hline
b & \cdots & b & b & \cdots & b & b & \cdots & b \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots
\end{array}
$$

FIGURE 1. An illustration of the point $\mathbf{x}$ defined in the proof of Lemma 6.2

Assume that $\alpha_k - \beta_{k-1} \geq M_\varepsilon(\beta_k - \alpha_k)$ for $k = 1, 2, \dots, K$. Let $r = \alpha_1 + \beta_K + m$. Define $\mathbf{x}$ (see Figure 1) by

$$
\mathbf{x}_{ij} = \begin{cases} \mathbf{x}^{(k)}_{ij}, & \text{if } i < m \text{ and } (j \bmod r) \in [\alpha_k, \beta_k + m] \text{ for some } k \in \{1, \dots, K\}, \\ a, & \text{if } i < m \text{ and } (j \bmod r) \notin [\alpha_k, \beta_k + m] \text{ for all } k \in \{1, \dots, K\}, \\ b, & \text{if } i \geq m. \end{cases}
$$

It is easy to see that $\mathbf{x} \in \mathbf{X}$ and a simple computation shows that

$$\rho(S^j(\mathbf{x}), S^j(\mathbf{x}^{(k)})) < \varepsilon$$

for each $1 \leq k \leq K$ and $\alpha_k \leq j \leq \beta_k$. □

Lemma 6.3. *The dynamical system $(\mathbf{X}, S)$ does not have the almost specification property.*

Proof. Assume on the contrary that S has the almost specification property with the mistake function g. In particular, this means that there is $\varepsilon_0 > 0$ such that $g\colon \mathbb{N} \times [0, \varepsilon_0) \to \mathbb{N}$ and for every $0 < \varepsilon < \varepsilon_0$ it holds $\lim_{l\to\infty} g(\varepsilon, l)/l = 0$. We are going to show that there is a specification and parameters which cannot be traced with g as a mistake function for S.

First we set up some constants. Let $n \in \mathbb{N}$ be such that $0 < 2^{-n} < \varepsilon_0$ and fix $\varepsilon = 2^{-n}$. Pick $N > k_g(\varepsilon)$ hence $g(N, \varepsilon) < N$. Note that by the definition of ρ, our choice of ε and N implies that for any point $\mathbf{y}$ which (g, N, ε)-traces the orbit segment $S^{[a,b]}(\mathbf{x})$ of length N there is $a \leq j < b$ such that $\mathbf{x}_{0j} = \mathbf{y}_{0j}$. Take any $M \in \mathbb{N}$ such that $2^M > 2N$ and let $m \geq k_g(2^{-M})$. Note that every point $\mathbf{y}$, which is $(g; m, 2^{-M})$-tracing an orbit segment $S^{[0,m)}(\mathbf{x})$ we can find some $0 \leq j < m$ such that $\mathbf{x}_{Mj} = \mathbf{y}_{Mj}$. Let $s > 2$ be an integer such that

$$2^M \left(\lceil \log_2((s-2) \cdot N + 1) \rceil \right) > N + m.$$

Let $\mathbf{x}^{(0)} = c^{\mathbb{N}\times\mathbb{N}}$ and $\mathbf{x}^{(1)} = \ldots = \mathbf{x}^{(s)} = b^{\mathbb{N}\times\mathbb{N}}$. Let $n_0 = m$ and $n_j = m + j \cdot N$ for $j = 1, \dots, s$. Define a specification

$$\xi = \{S^{[0,n_0)}(\mathbf{x}^{(0)})\} \cup \{S^{[n_{j-1},n_j)}(\mathbf{x}^{(j)}) : \text{ for } j = 1, 2, \dots, s\}.$$

By our choice of parameters there is a point $\mathbf{y}$ which $(g; 2^{-M}, m)$-traces ξ over $[0, n_0)$ and $(g; N, \varepsilon)$-traces ξ over $[n_{j-1}, n_j)$ for every $j = 1, \dots, s$. It follows that for each $j = 1, \dots, s$ there is $p_j \in [n_{j-1}, n_j)$ such that $\mathbf{y}_{0p_j} = b$. Therefore in the M-th row of $\mathbf{y}$ there are symbols b in columns $p_1, \dots, p_s$. Note that $p_{j+1} - p_j < 2N$ for $j = 1, 2, \dots, s-1$. Hence by $2^M > 2N$ together with the definition of $\mathbf{X}$ imply that $\mathbf{y}_{Mt} = b$ for every $p_1 \leq t \leq p_s$, where $n_0 \leq p_1 < n_1$ and $n_{s-1} \leq p_s < n_s$. In particular, in the M-th row of $\mathbf{y}$ there are at least $(s-2) \cdot N$ consecutive b's (all symbols between columns n_2 and n_{s-1} inclusively). It follows that all symbols in M-th row of $\mathbf{y}$ in columns from 0 to $m - 1 = n_1 - 1$ are either a's or b's, i.e. symbol c cannot appear there. We have reached a contradiction since $\mathbf{y}$ is a point $(g; 2^{-M}, m)$-tracing $\mathbf{x}^{(0)}$ over $[0, m)$ which implies $\mathbf{x}_{Mj} = \mathbf{y}_{Mj} = c$ for some $0 \leq j < m$. □

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