Subexponentially increasing sums of partial quotients in continued fraction expansions

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Abstract

We investigate from a multifractal analysis point of view the increasing rate of the sums of partial quotients $S_n(x) = \sum_{j=1}^{n} a_j(x)$, where $x = [a_1(x), a_2(x), \cdots]$ is the continued fraction expansion of an irrational $x \in (0,1)$. Precisely, for an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$, one is interested in the Hausdorff dimension of the sets

$$E_\varphi = \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{S_n(x)}{\varphi(n)} = 1 \right\}.$$

Several cases are solved by Iommi and Jordan, Wu and Xu, and Xu. We attack the remaining subexponential case $\exp(n^\gamma)$, $\gamma \in [1/2, 1)$. We show that when $\gamma \in [1/2, 1)$, E_φ has Hausdorff dimension 1/2. Thus, surprisingly, the dimension has a jump from 1 to 1/2 at $\varphi(n) = \exp(n^{1/2})$. In a similar way, the distribution of the largest partial quotient is also studied.

1. *Introduction*

Each irrational number $x \in [0,1)$ admits a unique infinite continued fraction expansion of the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}}, \tag{1.1}$$

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where the positive integers $a_n(x)$ are called the partial quotients of x. Usually, (1.1) is written as $x = [a_1, a_2, \cdots]$ for simplicity. The n-th finite truncation of (1.1): $p_n(x)/q_n(x) = [a_1, \cdots, a_n]$ is called the n-th convergent of x. The continued fraction expansions can be induced by the Gauss transformation $T : [0, 1) \to [0, 1)$ defined by

$$T(0) := 0, \text{ and } T(x) := \frac{1}{x} \ (\text{mod } 1), \text{ for } x \in (0, 1).$$

It is well known that $a_1(x) = \lfloor x^{-1} \rfloor$ ($\lfloor \cdot \rfloor$ stands for the integer part) and $a_n(x) = a_1(T^{n-1}(x))$ for $n \geq 2$.

For any $n \geq 1$, we denote by $S_n(x) = \sum_{j=1}^{n} a_j(x)$ the sum of the n first partial quotients. It was proved by Khintchine [**5**] in 1935 that $S_n(x)/(n \log n)$ converges in measure (Lebesgue measure) to the constant $1/\log 2$. In 1988, Philipp [**7**] showed that there is no reasonable normalizing sequence $\varphi(n)$ such that a strong law of large numbers is satisfied, i.e., $S_n(x)/\varphi(n)$ will never converge to a positive constant almost surely.

From the point of view of multifractal analysis, one considers the Hausdorff dimension of the sets

$$E_\varphi = \left\{ x \in (0, 1) : \lim_{n\to\infty} \frac{S_n(x)}{\varphi(n)} = 1 \right\}.$$

where $\varphi : \mathbb{N} \to \mathbb{N}$ is an increasing function.

The case $\varphi(n) = \gamma n$ with $\gamma \in [1, \infty)$ was studied by Iommi and Jordan [**3**]. It is proved that with respect to γ, the Hausdorff dimension (denoted by $\dim_H$) of E_φ is analytic, increasing from 0 to 1, and tends to 1 when γ goes to infinity. In [**9**], Wu and Xu proved that if $\varphi(n) = n^\gamma$ with $\gamma \in (1, \infty)$ or $\varphi(n) = \exp(n^\gamma)$ with $\gamma \in (0, 1/2)$, then $\dim_H E_\varphi = 1$. Later, it was shown by Xu [**10**], that if $\varphi(n) = \exp(n)$ then $\dim_H E_\varphi = 1/2$ and if $\varphi(n) = \exp(\gamma^n)$ with $\gamma > 1$ then $\dim_H E_\varphi = 1/(\gamma + 1)$. The same proofs of [**10**] also imply that for $\varphi(n) = \exp(n^\gamma)$ with $\gamma \in (1, \infty)$ the Hausdorff dimension $\dim_H E_\varphi$ stays at 1/2. So, only the subexponentially increasing case: $\varphi(n) = \exp(n^\gamma), \gamma \in [1/2, 1)$ was left unknown. In this paper, we fill this gap.

THEOREM 1·1. *Let $\varphi(n) = \exp(n^\gamma)$ with $\gamma \in [1/2, 1)$. Then*

$$\dim_H E_\varphi = \frac{1}{2}.$$

We also show that there exists a jump of the Hausdorff dimension of E_φ between $\varphi(n) = \exp(n^{1/2})$ and slightly slower growing functions, for example $\varphi(n) = \exp(\sqrt{n}(\log n)^{-1})$.

THEOREM 1·2. *Let $\varphi(n) = \exp(\sqrt{n} \cdot \psi(n))$ be an increasing function with ψ being a $\mathcal{C}^1$ positive function on $\mathbb{R}_+$ satisfying*

$$\lim_{x\to\infty} \frac{\sup_{y\geq x} \psi(y)^2}{\psi(x)} = 0 \quad \textit{and} \quad \lim_{x\to\infty} \frac{x\psi'(x)}{\psi(x)} = 0. \tag{1.2}$$

Then

$$\dim_H E_\varphi = 1.$$

We remark that the assumption (1.2) on the function ψ says that ψ decreases to 0 slower than any polynomial. We also remark that when ψ is decreasing, then the first condition of (1.2) is automatically satisfied.

Theorems 1·1 and 1·2 show that, surprisingly, there is a jump of the Hausdorff dimensions from 1 to 1/2 in the class $\varphi(n) = \exp(n^\gamma)$ at $\gamma = 1/2$ and that this jump cannot be easily removed by considering another class of functions. See Figure 1 for an illustration of the jump of the Hausdorff dimension.

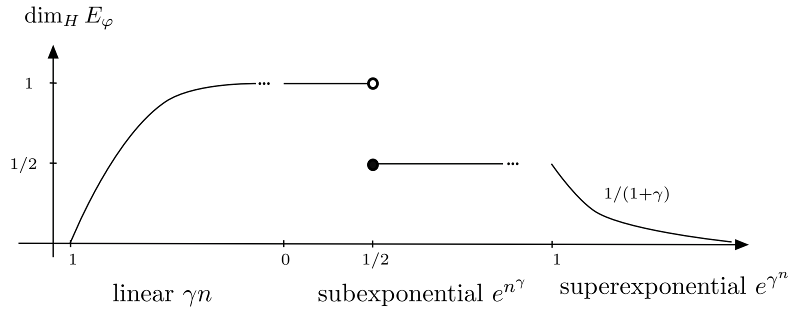

Fig. 1. $\dim_H E_\varphi$ for different φ.

By the same method, we also prove some similar results on the distribution of the largest partial quotient in continued fraction expansions. For $x \in [0,1) \setminus \mathbb{Q}$, define

$$T_n(x) := \max\{a_k(x) : \ 1 \le k \le n\}.$$

One is interested in the following lower limit:

$$T(x) := \liminf_{n\to\infty} \frac{T_n(x)\log\log n}{n}.$$

It was conjectured by Erdös that almost surely $T(x) = 1$. However, it was proved by Philipp [**6**] that for almost all x, one has $T(x) = 1/\log 2$. Recently, Wu and Xu [**8**] showed that

$$\forall \alpha \ge 0, \quad \dim_H \left\{ x \in [0,1) \setminus \mathbb{Q} : \lim_{n\to\infty} \frac{T_n(x)\log\log n}{n} = \alpha \right\} = 1.$$

They also proved that if the denominator n is replaced by any polynomial the same result holds. In this paper, we show the following theorem.

THEOREM 1·3. *For all $\alpha > 0$,*

$$F(\gamma,\alpha) = \left\{ x \in [0,1) \setminus \mathbb{Q} : \lim_{n\to\infty} T_n(x)/\exp(n^\gamma) = \alpha \right\}$$

satisfies

$$\dim_H F(\gamma,\alpha) = \begin{cases} 1 & \textit{if } \gamma \in (0,1/2) \\ \frac{1}{2} & \textit{if } \gamma \in (1/2,\infty). \end{cases}$$

We do not know what happens in the case $\gamma = 1/2$.

2. *Preliminaries*

For any $a_1, a_2, \cdots, a_n \in \mathbb{N}$, call

$$I_n(a_1,\cdots,a_n) := \{x \in [0,1) : a_1(x) = a_1, \cdots, a_n(x) = a_n\}$$

a *rank-n basic interval.* Denote by $I_n(x)$ the rank-n basic interval containing x. Write $|I|$ for the length of an interval I. The length of the basic interval $I_n(a_1,a_2,\cdots,a_n)$ satisfies

$$\prod_{k=1}^{n}(a_k+1)^{-2} \le \Big|I_n(a_1,\cdots,a_n)\Big| \le \prod_{k=1}^{n} a_k^{-2}. \tag{2.1}$$

Let $A(m,n) := \big\{(i_1,\dots,i_n)\in\{1,\dots,m\}^n :\ \sum_{k=1}^n i_k = m\big\}$. Let $\zeta(\cdot)$ be the Riemann zeta function.

LEMMA 2·1. *For any $s\in(1/2,1)$, for all $n\ge 1$ and for all $m\ge n$, we have*

$$\sum_{(i_1,\dots,i_n)\in A(m,n)} \prod_{k=1}^n i_k^{-2s} \le \left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^n m^{-2s}.$$

Proof. The proof goes by induction. First consider the case $n=2$. For $m=2$ the assertion holds, assume that $m>2$. We will estimate the sum $\sum_{i=1}^{m-1} i^{-2s}(m-i)^{-2s}$. For any $u\in[1,m/2]$ we have

$$\begin{aligned}
&\sum_{i=1}^{m-1} i^{-2s}(m-i)^{-2s} = 2\sum_{i=1}^{u-1} i^{-2s}(m-i)^{-2s} + \sum_{i=u}^{m-u} i^{-2s}(m-i)^{-2s}\\
&\le 2\Big(\sum_{i=1}^{u-1} i^{-2s}\Big)(m-u)^{-2s} + (m-2u+1)u^{-2s}(m-u)^{-2s}\\
&\le 2\zeta(2s)(m-u)^{-2s} + (m-2u+1)u^{-2s}(m-u)^{-2s}.
\end{aligned}$$

Take $u=\lfloor m/3\rfloor$. Then one has

$$(m-2u+1)u^{-2s} = (m+1)u^{-2s} - 2u^{1-2s} \le (m+1)\big\lfloor\frac{m}{3}\big\rfloor^{-2s} - 2 \le 4.$$

Hence, the above sum is bounded from above by

$$(4+2\zeta(2s))\cdot\big(\frac{2m}{3}\big)^{-2s} \le \frac{9}{2}(2+\zeta(2s))\cdot m^{-2s}.$$

Suppose now that the assertion holds for $n\in\{2,n_0\}$. Then for $n=n_0+1$, we have

$$\begin{aligned}
&\sum_{(i_1,\dots,i_{n_0+1})\in\{1,\dots,m\}^{n_0+1},\ \sum i_k=m}\ \prod_{k=1}^{n_0+1} i_k^{-2s}\\
&=\sum_{i=1}^{m-1} i^{-2s} \sum_{(i_1,\dots,i_{n_0})\in\{1,\dots,m\}^{n_0},\ \sum i_k=m-i}\ \prod_{k=1}^{n_0} i_k^{-2s}\\
&\le \sum_{i=1}^{m-1} i^{-2s}\left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^{n_0}(m-i)^{-2s}\\
&=\left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^{n_0}\cdot\sum_{i=1}^{m-1} i^{-2s}(m-i)^{-2s}\\
&\le\left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^{n_0}\cdot\left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)m^{-2s}\\
&=\left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^{n_0+1} m^{-2s}.
\end{aligned}$$

□

Let

$$A(\gamma,c_1,c_2,N) := \left\{x\in(0,1) : c_1 < \frac{a_n(x)}{e^{n^\gamma}} < c_2,\ \forall n\ge N\right\}.$$

Denote by N_0 the smallest integer n such that $(c_2-c_1)\cdot e^{n^\gamma} > 1$. Then the set $A(\gamma,c_1,c_2,N)$ is non-empty when $N\ge N_0$.

LEMMA 2·2. *For any $\gamma > 0$, any $N \geq N_0$ and any $0 < c_1 < c_2$,*

$$\dim_H A(\gamma, c_1, c_2, N) = \frac{1}{2}.$$

Proof. This lemma is only a simple special case of [**2**, Lemma 3.2], but we will sketch the proof (based on [**4**]), needed for the next lemma. Without loss of generality, we suppose $N_0 = 1$ and let $N = 1$ (the proof for other N is almost identical).

Let $a_1, a_2, \ldots, a_n$ satisfy $c_1 < a_j e^{-j^\gamma} < c_2$ for all j. Those are exactly the possible sequences for which the basic interval $I_n(a_1, \ldots, a_n)$ has nonempty intersection with $A(\gamma, c_1, c_2, 1)$.

There are approximately

$$\prod_{j=1}^{n} (c_2 - c_1) e^{j^\gamma} \approx e^{\sum_1^n j^\gamma} \tag{2.2}$$

of such basic intervals, each of diameter

$$|I_n(a_1, \ldots, a_n)| \approx e^{-2\sum_1^n j^\gamma}, \tag{2.3}$$

(both estimations are up to a factor exponential in n). Hence, by using the intervals $\{I_n(a_1, \ldots, a_n)\}$ as a cover, we obtain

$$\dim_H A(\gamma, c_1, c_2, 1) \leq \frac{1}{2}.$$

To get the lower bound, we consider a probability measure μ uniformly distributed on $A(\gamma, c_1, c_2, 1)$, in the following sense: given $a_1, \ldots, a_{n-1}$, the probability of a_n taking any particular value between $c_1 e^{n^\gamma}$ and $c_2 e^{n^\gamma}$ is the same.

The basic intervals $I_n(a_1, \ldots, a_n)$ have, up to a factor c^n, the length $\exp(-2\sum_1^n j^\gamma)$ and the measure $\exp(-\sum_1^n j^\gamma)$. They are distributed in clusters: all $I_n(a_1, \ldots, a_n)$ contained in a single $I_n(a_1, \ldots, a_{n-1})$ form an interval of length $\exp(n^\gamma) \cdot \exp(-2\sum_1^n j^\gamma)$ (up to a factor c^n, with c being a constant), then there is a gap, then there is another cluster. Hence, for any $r \in (\exp(-2\sum_1^n j^\gamma),\ \exp(-2\sum_1^{n-1} j^\gamma))$ and any $x \in A(\gamma, c_1, c_2, 1)$ we can estimate the measure of $B(x, r)$:

$$\mu(B(x,r)) \approx \begin{cases} r \cdot e^{-\sum_1^n j^\gamma} & \text{if } r < e^{-2\sum_1^n j^\gamma + n^\gamma} \\ e^{-\sum_1^{n-1} j^\gamma} & \text{if } r > e^{-2\sum_1^n j^\gamma + n^\gamma} \end{cases}$$

(up to a factor c^n). The minimum of $\log \mu(B(x,r))/\log r$ is thus achieved for $r = e^{-2\sum_1^n j^\gamma + n^\gamma}$, and this minimum equals

$$\frac{-\sum_1^{n-1} j^\gamma}{-2\sum_1^n j^\gamma + n^\gamma} \approx \frac{-n^{\gamma+1}/(\gamma+1)}{-2n^{\gamma+1}/(\gamma+1) - n^\gamma} = \frac{1}{2} - O(1/n).$$

Hence, the lower local dimension of μ equals $1/2$ at each point of $A(\gamma, c_1, c_2, 1)$, which implies

$$\dim_H A(\gamma, c_1, c_2, 1) \geq \frac{1}{2}$$

by the Frostman Lemma (see [**1**, Principle 4.2]). $\square$

Let now c_1 and c_2 not be constant but depend on n:

$$B(\gamma, c_1, c_2, N) = \left\{ x \in (0,1) : c_1(n) < \frac{a_n(x)}{e^{n^\gamma}} < c_2(n)\ \forall n \geq N \right\}.$$

A slight modification of the proof of Lemma 2·2 gives the following.

LEMMA 2·3. *Fix $\gamma > 0$. Assume $0 < c_1(n) < c_2(n)$ for all n. Assume also that*

$$\lim_{n\to\infty} \frac{\log(c_2(n) - c_1(n))}{n^\gamma} = 0$$

and

$$\liminf_{n\to\infty} \frac{\log c_1(n)}{\log n} > -\infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log c_2(n)}{\log n} < +\infty.$$

Then there exists an integer N_1 such that $(c_2(n) - c_1(n)) \cdot e^{n^\gamma} > 1$ for all $n \geq N_1$, and for all $N \geq N_1$,

$$\dim_H B(\gamma, c_1, c_2, N) = 1/2.$$

Proof. We need only to replace the constants c_1 and c_2 by $c_1(n)$ and $c_2(n)$ in the proof of Lemma 2·2. Notice that by the assumptions of Lemma 2·3, the formula (2.2) holds up to a factor $\exp(\varepsilon \sum_1^n j^\gamma)$ for a sufficiently small $\varepsilon > 0$. While the formula (2.3) holds up to a factor $\exp(cn \log n)$ for some bounded c. All these factors are much smaller than the main term $\exp(\sum_1^n j^\gamma)$ which is of order $\exp(n^{1+\gamma})$. The rest of the proof is the same as that of Lemma 2·2. □

3. *Proofs*

Proof of Theorem 1·1 Let $\varphi : \mathbb{N} \to \mathbb{N}$ be defined by $\varphi(n) = \exp(n^\gamma)$ with $\gamma > 0$. For this case, we will denote E_φ by E_γ.

Let us start from some easy observations, giving (among other things) a simple proof of $\dim_H E_\gamma = 1/2$ for $\gamma \geq 1$.

Consider first $\gamma \geq 1/2$. If $x \in E_\gamma$ then for any $\varepsilon > 0$ and for n large enough

$$(1-\varepsilon)e^{n^\gamma} \leq S_n(x) \leq (1+\varepsilon)e^{n^\gamma} \tag{3.1}$$

and

$$(1-\varepsilon)e^{(n+1)^\gamma} \leq S_{n+1}(x) \leq (1+\varepsilon)e^{(n+1)^\gamma}.$$

Hence

$$(1-\varepsilon)e^{(n+1)^\gamma} - (1+\varepsilon)e^{n^\gamma} \leq a_{n+1}(x) \leq (1+\varepsilon)e^{(n+1)^\gamma} - (1-\varepsilon)e^{n^\gamma}.$$

For $\gamma \geq 1$ this implies

$$E_\gamma \subset \bigcup_N A(\gamma, c_1, c_2, N)$$

for some constants c_1, c_2. By Lemma 2·2,

$$\dim_H E_\gamma \leq \frac{1}{2}, \quad \forall \gamma \geq 1.$$

Consider now any $\gamma > 0$. Set

$$c_1(n) = (e^{n^\gamma} - e^{(n-1)^\gamma})e^{-n^\gamma} \quad \text{and} \quad c_2(n) = \frac{n+1}{n} c_1(n).$$

For $\gamma \geq 1$, $c_1(n)$ and $c_2(n)$ are bounded from below. For $\gamma < 1$ and n large, we have

$$(e^{n^\gamma} - e^{(n-1)^\gamma})e^{-n^\gamma} \approx \gamma n^{\gamma - 1}.$$

Thus, in both cases the assumptions of Lemma 2·3 are satisfied. Checking $B(\gamma, c_1, c_2, N) \subset E_\gamma$, we deduce by Lemma 2·3 that

$$\dim_H E_\gamma \geq \frac{1}{2}, \quad \forall \gamma > 0.$$

Therefore, we have obtained $\dim_H E_\gamma = 1/2$ for $\gamma \geq 1$ and $\dim_H E_\gamma \geq 1/2$ for $\gamma > 0$. What is left to prove is that for $\gamma \in [1/2, 1)$ we have $\dim_H E_\gamma \leq 1/2$.

Let us first assume that $\gamma > 1/2$. Remember that if $x \in E_\gamma$, then for any $\varepsilon > 0$ and for n large enough we have (3.1). Take a subsequence $n_0 = 1$, and $n_k = k^{1/\gamma}$ $(k \geq 1)$. Then there exists an integer $N \geq 1$ such that for all $k \geq N$,

$$(1-\varepsilon)e^{n_k^\gamma} \leq S_{n_k}(x) \leq (1+\varepsilon)e^{n_k^\gamma},$$

and (as $\exp(n_k^\gamma) = e^k$)

$$(1-\varepsilon)e^k - (1+\varepsilon)e^{k-1} \leq S_{n_k}(x) - S_{n_{k-1}}(x) \leq (1+\varepsilon)e^k - (1-\varepsilon)e^{k-1}.$$

Thus

$$E_\gamma \subset \bigcup_N \bigcap_{k \geq N} A(\gamma, k, N),$$

with $A(\gamma, k, N)$ being the union of the intervals $\{I_{n_k}(a_1, a_2, \cdots, a_{n_k})\}$ such that

$$\sum_{j=n_{\ell-1}+1}^{n_\ell} a_j = m \quad \text{with} \quad m \in D_\ell, \quad N \leq \ell \leq k,$$

where $D_\ell := [(1-\varepsilon)e^{n_\ell^\gamma} - (1+\varepsilon)e^{n_\ell^\gamma - 1}, (1+\varepsilon)e^{n_\ell^\gamma} - (1-\varepsilon)e^{n_\ell^\gamma - 1}]$.

Now, we are going to estimate the upper bound of the Hausdorff dimension of $E_\varphi^{(1)} = \bigcap_k A(\gamma, k, 1)$. For $E_\varphi^{(N)} = \bigcap_{k \geq N} A(\gamma, k, N)$ with $N \geq 2$ we have the same bound and the proofs are almost the same.

Observe that every set $A(\gamma, k, N)$ has a product structure: the conditions on a_i for $i \in (n_{\ell_1}, n_{\ell_1+1}]$ and for $i \in (n_{\ell_2}, n_{\ell_2+1}]$ are independent from each other. Hence, for any $s \in (1/2, 1)$ we can apply Lemma 2·1 together with the formula

$$|I_{n_k}|^s \leq \prod_{\ell=1}^{k} (a_{n_{\ell-1}+1} a_{n_{\ell-1}+2} \cdots a_{n_\ell})^{-2s}$$

to obtain

$$\sum_{I_{n_k} \subset A(\gamma,k,1)} |I_{n_k}|^s \leq \prod_{\ell=1}^{k} \sum_{m \in D_\ell} \left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^{n_\ell - n_{\ell-1}} m^{-2s}.$$

Denote $r_1 := 2\varepsilon(1-e^{-1})$ and $r_2 := (e - 1 - \varepsilon e - \varepsilon)/e$. Then we have $|D_\ell| \leq r_1 e^\ell$ and any $m \in D_\ell$ is not smaller than $r_2 e^\ell$. Thus we get

$$\sum_{I_{n_k} \subset A(\gamma,k,1)} |I_{n_k}|^s \leq \prod_{\ell=1}^{k} r_1 e^\ell \cdot \left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^{\ell^{1/\gamma} - (\ell-1)^{1/\gamma}} \cdot r_2^{2s} e^{-2s\ell}. \tag{3.2}$$

We have $\ell^{1/\gamma} - (\ell-1)^{1/\gamma} \approx \ell^{1/\gamma - 1}$. As $\gamma > 1/2$, we have $1/\gamma - 1 < 1$, and the main term in the above estimate is $e^{(1-2s)\ell}$. Thus for any $s > 1/2$, the product is uniformly bounded. Thus $\dim_H E_\varphi^{(1)} \leq 1/2$.

If $\gamma = 1/2$, we take $n_k = k^2/L^2$ with L being a constant and we repeat the same argument. Observe that now $\exp(n_k^\gamma) = e^{k/L}$. Then the same estimation will lead to

$$\sum_{I_{n_k}\subset A(\gamma,k,1)} |I_{n_k}|^s \le \prod_{\ell=1}^{k} r_1 r_2^{2s} \cdot \left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^{\frac{\ell^2-(\ell-1)^2}{L^2}} e^{(1-2s)\ell/L}. \tag{3.3}$$

The main term of the right side of the above inequality should be

$$\left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^{2\ell/L^2} \cdot e^{(1-2s)\ell/L}.$$

We solve the equation

$$\left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^{2/L^2} \cdot e^{(1-2s)/L} = 1,$$

which is equivalent to

$$\left(\frac{9}{2}\big(2+\zeta(2s)\big)\right) = e^{\frac{2s-1}{2}L}. \tag{3.4}$$

Observe that the graphs of the two sides of (3.4) (as functions of the variable s) always have a unique intersection for some $s_L \in [1/2, 1]$, when L is large enough. These s_L are upper bounds for the Hausdorff dimension of $E_\varphi^{(1)}$. Notice that the intersecting point $s_L \to 1/2$ as $L \to \infty$ since the zeta function ζ has a pole at 1. Thus the dimension of $E_\varphi^{(1)}$ is not greater than 1/2.

So, in both cases, we have obtained $\dim_H E_\gamma \le 1/2$. $\square$

Sketch proof of Theorem 1·2 The proof goes like Section 4 of [**9**] with the following changes. We choose $\varepsilon_k = \psi(k)$. Let n_1 be such that $\varphi(n_1) \ge 1$ and define n_k as the smallest positive integer such that

$$\varphi(n_k) \ge (1+\varepsilon_{k-1})\varphi(n_{k-1}). \tag{3.5}$$

For a large enough integer M, set

$$\begin{aligned} E_M(\varphi) := \Big\{ x \in [0,1) : \ & a_{n_1}(x) = \lfloor (1+\varepsilon_1)\varphi(n_1)\rfloor + 1, \\ & a_{n_k}(x) = \lfloor (1+\varepsilon_k)\varphi(n_k)\rfloor - \lfloor (1+\varepsilon_{k-1})\varphi(n_{k-1})\rfloor + 1 \text{ for all } k \ge 2, \\ & \text{and } 1 \le a_i(x) \le M \text{ for } i \ne n_k \text{ for any } k \ge 1 \Big\}. \end{aligned}$$

We can check that $E_M(\varphi) \subset E_\varphi$.

To prove $\dim_H E_\varphi = 1$, for any $\varepsilon > 0$, we construct a $(1/(1+\varepsilon))$-Lipschitz map from $E_M(\varphi)$ to E_M, the set of numbers with partial quotients less than some M in its continued fraction expansion. The theorem will be proved by letting $\varepsilon \to 0$ and $M \to \infty$.

Such a Lipschitz map can be constructed by send a point x in $E_M(\varphi)$ to a point $\tilde{x}$ by deleting all the partial quotients a_{n_k} in its continued fraction expansion. Define $r(n) := \min\{k : n_k \le n\}$. The $(1/(1+\varepsilon))$-Lipschitz property will be assured if

$$\lim_{n\to\infty} \frac{r(n)}{n} = 0, \tag{3.6}$$

and

$$\lim_{n\to\infty} \frac{\log(a_{n_1}a_{n_2}\cdots a_{n_{r(n)}})}{n} = 0. \tag{3.7}$$

In fact, by (1.2), we can check for any $\delta > 0$, $\psi(n) \le n^\delta$ for n large enough. Thus by definition of n_k, we can deduce that $r(n) \le n^{1/2+\delta}$. Hence (3.6) is satisfied.

Further, we have

$$\sum_{k=1}^{r(n)} \varepsilon_k \approx r(n)\psi(r(n)). \tag{3.8}$$

By (3.5)

$$\varphi(n) \ge \varphi(n_{r(n)}) \ge \prod_{k=1}^{r(n)-1} (1+\varepsilon_k)\varphi(n_1) \ge e^{\sum_{k=1}^{r(n)} \varepsilon_k/2}\varphi(n_1).$$

Thus (3.8) implies

$$r(n)\psi(r(n)) \ll \sqrt{n}\psi(n), \tag{3.9}$$

where $a_n \ll b_n$ means that a_n/b_n is bounded by some constant when $n \to \infty$.

On the other hand, by (2.1) and (3.5), we have

$$\log(a_{n_1}a_{n_2}\cdots a_{n_{r(n)}}) \le r(n)\log(2\varphi(n)) + \sum_{k=1}^{r(n)} \varepsilon_k.$$

Hence (3.8) and (3.9) give

$$\log(a_{n_1}a_{n_2}\cdots a_{n_{r(n)}}) \ll r(n)\sqrt{n}\psi(n) + r(n)\psi(r(n)) \ll \frac{n\psi^2(n)}{\psi(r(n))} + r(n).$$

Finally, (3.7) follows from the assumption (1.2) and the already proved formula (3.6). □

Proof of Theorem 1·3 For the case $\gamma < 1/2$, the set constructed in Section 4 of [**9**] (as a subset of the set of points for which $S_n(x) \approx e^{n^\gamma}$) satisfies also $T_n(x) \approx e^{n^\gamma}$ and has Hausdorff dimension one. We proceed to the case $\gamma > 1/2$.

The lower bound is a corollary of Lemma 2·3. Take $c_1(n) = \alpha(1 - \frac{1}{n})$ and $c_2(n) = \alpha$. Let N_1 be the smallest integer n such that $\frac{\alpha}{n}e^{n^\gamma} > 1$. Then the conditions of Lemma 2·3 are satisfied, and for all points x such that $c_1(n)e^{n^\gamma} < a_n(x) < c_2(n)e^{n^\gamma}$, we have

$$T_n(x)/e^{n^\gamma} \ge c_1(n) = \alpha\left(1 - \frac{1}{n}\right),$$

and

$$T_n(x)/e^{n^\gamma} = a_k/e^{n^\gamma} \le \alpha e^{k^\gamma}/e^{n^\gamma} \le \alpha,$$

where $k \le n$ is the position at which the sequence $a_1, \ldots, a_n$ achieves a maximum. Thus for all $x \in B(\gamma, c_1, c_2, N_1)$

$$\lim_{n\to\infty} T_n(x)/e^{n^\gamma} = \alpha.$$

Hence, $B(\gamma, c_1, c_2, N_1) \subset F(\gamma, \alpha)$ and the lower bound follows directly from Lemma 2·3.

The upper bound is a modification of that of Theorem 1·1. We consider the case $\alpha = 1$ only, since for other $\alpha > 0$, the proofs are similar.

Notice that for any $\varepsilon > 0$, if $x \in F(\gamma, 1)$, then for n large enough,

$$(1-\varepsilon)e^{n^\gamma} \le S_n(x) \le n(1+\varepsilon)e^{n^\gamma}.$$

Take a subsequence $n_k = k^{1/\gamma}(\log k)^{1/\gamma^2}$. Then

$$(1-\varepsilon)e^{k(\log k)^{1/\gamma}} \le S_{n_k}(x) \le k^{1/\gamma}(\log k)^{1/\gamma^2}(1+\varepsilon)e^{k(\log k)^{1/\gamma}},$$

and

$$u_k \le S_{n_k}(x) - S_{n_{k-1}}(x) \le v_k,$$

with

$$u_k := (1-\varepsilon)e^{k(\log k)^{1/\gamma}} - (k-1)^{1/\gamma}(\log(k-1))^{1/\gamma^2}(1+\varepsilon)e^{(k-1)(\log(k-1))^{1/\gamma}},$$

and

$$v_k := k^{1/\gamma}(\log k)^{1/\gamma^2}(1+\varepsilon)e^{k(\log k)^{1/\gamma}} - (1-\varepsilon)e^{(k-1)(\log(k-1))^{1/\gamma}}.$$

We remark that

$$u_k > \frac{1}{2}e^{k(\log k)^{1/\gamma}}, \quad v_k < \frac{3}{2}k^{1/\gamma}(\log k)^{1/\gamma^2}e^{k(\log k)^{1/\gamma}} \tag{3.10}$$

when k is large enough.

Observe that

$$F(\gamma,1) \subset \bigcup_N B(\gamma,N),$$

with $B(\gamma,N)$ being the union of the intervals $\{I_{n_k}(a_1,a_2,\cdots,a_{n_k})\}_{k\ge N}$ such that

$$\sum_{j=n_{\ell-1}+1}^{n_\ell} a_j = m \quad \text{with} \quad m\in D_\ell, \quad N\le \ell\le k,$$

where D_ℓ is the set of integers in the interval $[u_\ell, v_\ell]$.

As in the proof of Theorem 1·1, we need only study the set $B(\gamma,1)$. For any $s\in(1/2,1)$, since

$$|I_{n_k}|^s \le \prod_{\ell=1}^{k}(a_{n_{\ell-1}+1}a_{n_{\ell-1}+2}\cdots a_{n_\ell})^{-2s},$$

by Lemma 2·1,

$$\sum_{I_{n_k}\subset B(\gamma,N)} |I_{n_k}|^s \le \prod_{\ell=1}^{k}\sum_{m\in D_\ell}\left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^{n_\ell-n_{\ell-1}} m^{-2s}.$$

Note that by (3.10) the number of integers in D_ℓ satisfies

$$|D_\ell| \le v_\ell - u_\ell \le v_\ell < \frac{3}{2}\cdot \ell^{1/\gamma}(\log \ell)^{1/\gamma^2}.$$

By (3.10), we also have

$$m \ge u_\ell > \frac{1}{2}e^{\ell(\log\ell)^{1/\gamma}} \quad \text{for any } m\in D_\ell.$$

Similar to (3.2) and (3.3), we deduce that $\sum_{I_{n_k}\subset B(\gamma,N)}|I_{n_k}|^s$ is less than

$$\prod_{\ell=1}^{k}\frac{3}{2}\cdot\ell^{1/\gamma}(\log\ell)^{1/\gamma^2}e^{\ell(\log\ell)^{1/\gamma}}\left(\frac{9}{2}\big(2+\zeta(2s)\big)\right)^{n_\ell-n_{\ell-1}} 2^{2s}e^{-2s\ell(\log\ell)^{1/\gamma}}.$$

Since $n_\ell - n_{\ell-1} \approx \ell^{1/\gamma-1+o(\varepsilon)}$ and $1/\gamma-1<1$, the main term in the above estimation is $e^{(1-2s)\ell(\log\ell)^{1/\gamma}}$. Thus for any $s>1/2$ the product is uniformly bounded and we have the Hausdorff dimension of $B(\gamma,1)$ is not greater than $1/2$. Then we can conclude $\dim_H F(\gamma,1)\le 1/2$ and the proof is completed. □

4. *Generalizations*

In this section we consider after [**4**] certain infinite iterated function systems that are natural generalizations of the Gauss map. For each $n \in \mathbb{N}$, let $f_n : [0,1] \to [0,1]$ be C^1 maps such that

(1) there exists $m \in \mathbb{N}$ and $0 < A < 1$ such that for all $(a_1, ..., a_m) \in \mathbb{N}^m$ and for all $x \in [0,1]$
$$0 < |(f_{a_1} \circ \cdots \circ f_{a_m})'(x)| \leq A < 1,$$
(2) for any $i, j \in \mathbb{N}$ $f_i((0,1)) \cap f_j((0,1)) = \emptyset$,
(3) there exists $d > 1$ such that for any $\varepsilon > 0$ there exist $C_1(\varepsilon), C_2(\varepsilon) > 0$ such that for $i \in \mathbb{N}$ there exist constants ξ_i, λ_i such that for all $x \in [0,1]$ $\xi_i \leq |f_i'(x)| \leq \lambda_i$ and
$$\frac{C_1}{i^{d+\varepsilon}} \leq \xi_i \leq \lambda_i \leq \frac{C_2}{i^{d-\varepsilon}}.$$

We will call such an iterated function system a *d-decaying system.* It will be further called *Gauss like* if
$$\bigcup_{i=1}^{\infty} f_i([0,1]) = [0,1)$$
and if for all $x \in [0,1]$ we have that $f_i(x) < f_j(x)$ implies $i < j$.

We have a natural projection $\Pi : \mathbb{N}^{\mathbb{N}} \to [0,1]$ defined by
$$\Pi(\underline{a}) = \lim_{n\to\infty} f_{a_1} \circ \cdots \circ f_{a_n}(1),$$
which gives for any point $x \in [0,1]$ its symbolic expansion $(a_1(x), a_2(x), \ldots)$. This expansion is not uniquely defined, but there are only countably many points with more than one symbolic expansions.

For a d-decaying Gauss like system we consider $S_n(x) = \sum_1^n a_i(x)$. Given an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ we denote
$$E_d(\varphi) = \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{S_n(x)}{\varphi(n)} = 1 \right\}.$$

THEOREM 4·1. *Let $\{f_i\}$ be a d-decaying Gauss like system. We have*

i) *if $\varphi(n) = e^{n^\gamma}$ with $\gamma < 1/d$,*
$$\dim_H E_d(\varphi) = 1,$$
ii) *if $\varphi(n) = e^{n^\gamma}$ with $\gamma > 1/d$,*
$$\dim_H E_d(\varphi) = \frac{1}{d},$$
iii) *if $\varphi(n) = e^{\gamma^n}$ with $\gamma > 1$,*
$$\dim_H E_d(\varphi) = \frac{1}{\gamma + d - 1}.$$

The proofs (both from Section 3 and from [**9**, **10**]) go through without significant changes.

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