# SUBEXPONENTIALLY INCREASING SUM OF PARTIAL QUOTIENTS IN CONTINUED FRACTION EXPANSIONS

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ABSTRACT. We investigate from multifractal analysis point of view the increasing rate of the sum of partial quotients $S_n(x) = \sum_{j=1}^n a_j(x)$, where $x = [a_1(x), a_2(x), \cdots]$ is the continued fraction expansion of an irrational $x \in (0,1)$. Precisely, for an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$, one is interested in the Hausdorff dimension of the sets

$$E_\varphi = \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{S_n(x)}{\varphi(n)} = 1 \right\}.$$

Several cases are solved by Iommi and Jordan, Wu and Xu, and Xu. We attack the remaining subexponential case $\exp(n^\beta)$, $\beta \in [1/2, 1)$. We show that when $\beta \in [1/2, 1)$, $E_\varphi$ has Hausdorff dimension $1/2$. Thus surprisingly the dimension has a jump from 1 to $1/2$ at the increasing rate $\exp(n^{1/2})$. In a similar way, the distribution of the largest partial quotients is also studied.

## 1. INTRODUCTION

Each irrational number $x \in [0,1)$ admits a unique infinite continued fraction expansion of the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}}, \tag{1.1}$$

where the integers $a_n(x)$ are called the partial quotients of $x$. Usually, (1.1) is written as $x = [a_1, a_2, \cdots]$ for simplicity. The $n$-th finite truncation of (1.1): $p_n(x)/q_n(x) = [a_1, \cdots, a_n]$ is called the $n$-th convergent of $x$. The continued fraction expansions can be induced by the Gauss transformation $T : [0,1) \to [0,1)$ defined by

$$T(0) := 0, \; T(x) = \frac{1}{x} \pmod 1, \; \text{for } x \in (0,1).$$

It is well known that $a_1(x) = \lfloor x^{-1} \rfloor$ ($\lfloor \cdot \rfloor$ stands for the integer part) and $a_n(x) = a_1(T^{n-1}(x))$ for $n \geq 2$.

For any $n \geq 1$, write $S_n(x) = \sum_{j=1}^n a_j(x)$ the sum of the $n$ first partial quotients. It is proved by Khintchine [5] in 1935 that $S_n(x)/(n \log n)$ converges in measure (Lebesgue measure) to the constant $1/\log 2$. In 1988, Philipp [7] showed that there is no reasonable normalizing sequence $\varphi(n)$

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such that a strong law of large numbers is satisfied, i.e., $S_n(x)/\varphi(n)$ will never converge to a positive constant almost surely.

From the point of view of multifractal analysis, one considers the Hausdorff dimension of the sets

$$E_\varphi = \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{S_n(x)}{\varphi(n)} = 1 \right\}.$$

where $\varphi : \mathbb{N} \to \mathbb{N}$ is an increasing function.

The case $\varphi(n) = \theta n$ with $\theta \in [1, \infty)$ was studied by Iommi and Jordan [3]. It is proved that with respect to $\theta$, the Hausdorff dimension of $E_\varphi$ is analytic, increasing from 0 to 1, and tends to 1 when $\theta$ goes to infinity. In [9], Wu and Xu proved that if $\varphi(n) = n^\alpha$ with $\alpha \in (0, \infty)$ or $\varphi(n) = \exp\{n^\beta\}$ with $\beta \in (0, 1/2)$, the Hausdorff dimension of $E_\varphi$ is always 1. It was shown by Xu [10], that if $\varphi(n) = \exp\{n\}$ then the Hausdorff dimension of $E_\varphi$ is 1/2 and if $\varphi(n) = \exp\{\gamma^n\}$ with $\gamma > 1$ then the Hausdorff dimension is $1/(\gamma+1)$. The same proofs of [10] also imply that for $\varphi(n) = \exp\{n^\beta\}$ with $\beta \in (1, \infty)$ the Hausdorff dimension of $E_\varphi$ stays at 1/2. So, only the subexponentially increasing case: $\varphi(n) = \exp\{n^\beta\}, \beta \in [1/2, 1)$ was left unknown. In this paper, we fill this gap.

**Theorem 1.1.** *Let $\varphi(n) = \exp\{n^\beta\}$ with $\beta \in [1/2, 1)$. Then the Hausdorff dimension of $E_\varphi$ is one-half.*

We also show that for increasing rates slightly slower than $e^{\sqrt{n}}$, for example $\varphi(n) = e^{\sqrt{n}(\log n)^{-1}}$, the Hausdorff dimension will jump.

**Theorem 1.2.** *Let $\varphi(n) = e^{\sqrt{n}\cdot\psi(n)}$ be an increasing function with $\psi$ being a $\mathcal{C}^1$ positive function on $\mathbb{R}_+$ satisfying*

$$\lim_{n\to\infty} \psi(n) = 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{n\psi'(n)}{\psi(n)} = 0.$$

*Then the Hausdorff dimension of $E_\varphi$ is equal to one.*

Theorems 1.1 and 1.2 show that, surprisingly, there is a jump of the Hausdorff dimensions from 1 to 1/2.

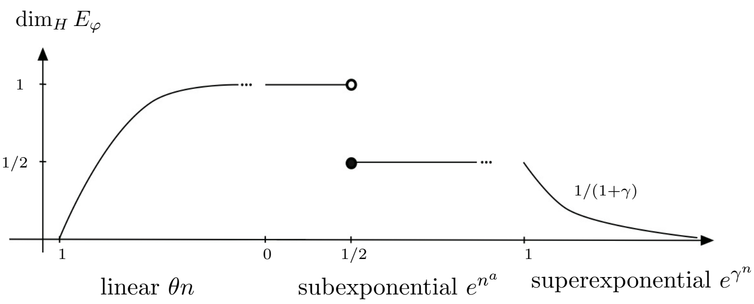


FIGURE 1. $\dim_H E_\varphi$ for $\varphi$ with different increasing rate.

By the same method, we also prove some similar results on the distribution of the largest partial quotients in continued fraction expansions. For $x \in$

$[0, 1) \cap \mathbb{Q}^c$, define

$$T_n(x) := \max\{a_k(x) : \ 1 \leq k \leq n\}.$$

One is interested in the following lower limit:

$$T(x) := \liminf_{n\to\infty} \frac{T_n(x)\log\log n}{n}.$$

It was conjectured by Erdös that almost surely $T(x) = 1$. However, it is proved by Philipp [6] that for almost all $x$, one has $T(x) = 1/\log 2$. Recently, Wu and Xu [8] showed that for all $\alpha \geq 0$ the level set

$$\left\{x \in [0, 1) \cap \mathbb{Q}^c : \lim_{n\to\infty} \frac{T_n(x)\log\log n}{n} = \alpha\right\}$$

has Hausdorff dimension 1. They also proved that if the denominator $n$ is replaced by a polynomial the same result holds. In this paper, we show the following theorem.

**Theorem 1.3.** *For all $\alpha > 0$,*

$$\left\{x \in [0, 1) \cap \mathbb{Q}^c : \lim_{n\to\infty} T_n(x)/e^{n^a} = \alpha\right\}$$

*is of Hausdorff dimension 1 if $a \in (0, 1/2)$, and is of Hausdorff dimension 1/2 if $a \in (1/2, \infty)$.*

We do not know what happens in the case $a = 1/2$.

## 2. Preliminary

For any $a_1, a_2, \cdots, a_n \in \mathbb{N}$, call

$$I_n(a_1, \cdots, a_n) := \{x \in [0, 1) : a_1(x) = a_1, \cdots, a_n(x) = a_n\}$$

a *rank-n basic interval.* Denote by $I_n(x)$ the rank-$n$ basic interval containing $x$. Write $|I|$ the length of an interval $I$. The length of the basic interval $I_n(a_1, a_2, \cdots, a_n)$ satisfies

$$\prod_{k=1}^{n}(a_k + 1)^{-2} \leq \Big|I_n(a_1, \cdots, a_n)\Big| \leq \prod_{k=1}^{n} a_k^{-2}. \tag{2.1}$$

Let $A(m, n) := \big\{(i_1, \dots, i_n) \in \{1, \dots, m\}^n : \ \sum_{k=1}^{n} i_k = m\big\}$. Let $\zeta(\cdot)$ be the Riemann zeta function.

**Lemma 2.1.** *For any $s > 1/2$, for all $n \geq 1$ and for all $m \geq n$, we have*

$$\sum_{(i_1,\dots,i_n)\in A(m,n)} \prod_{k=1}^{n} i_k^{-2s} \leq \left(\frac{9}{2}\big(1 + \zeta(2s)\big)\right)^n m^{-2s}.$$

*Proof.* The proof goes by induction. First consider the case $n = 2$. We will estimate the sum $\sum_{i=1}^{m-1} i^{-2s}(m-i)^{-2s}$. We have

$$\begin{aligned}
&\sum_{i=1}^{m-1} i^{-2s}(m-i)^{-2s} \\
=&2\sum_{i=1}^{u-1} i^{-2s}(m-i)^{-2s} + \sum_{i=u}^{m-u} i^{-2s}(m-i)^{-2s} \\
\le&2\Big(\sum_{i=1}^{u-1} i^{-2s}\Big)(m-u)^{-2s} + (m-2u+1)u^{-2s}(m-u)^{-2s} \\
\le&2\zeta(2s)(m-u)^{-2s} + (m-2u+1)u^{-2s}(m-u)^{-2s}.
\end{aligned}$$

Take $u = \lfloor m/3 \rfloor$. Then for $m$ large enough, one can have

$$(m-2u+1)u^{-2s} \le \Big((m-2(\frac{m}{3}-1)+1\Big)\,(\frac{m}{3})^{-2s} \le 2,$$

Hence, the above sum is bounded from above by

$$(2+2\zeta(2s))\cdot(\frac{2m}{3})^{-2s} \le \frac{9}{2}(1+\zeta(2s))\cdot m^{-2s}.$$

Suppose that we have the estimation for $n$. Then for $n+1$, we have

$$\begin{aligned}
&\sum_{(i_1,\dots,i_{n+1})\in\{1,\dots,m\}^{n+1},\ \sum i_k = m}\ \prod_{k=1}^{n+1} i_k^{-2s} \\
=&\sum_{i=1}^{m-1} i^{-2s} \sum_{(i_1,\dots,i_n)\in\{1,\dots,m\}^n,\ \sum i_k = m-i}\ \prod_{k=1}^{n} i_k^{-2s} \\
\le&\sum_{i=1}^{m-1} i^{-2s}\left(\frac{9}{2}\big(1+\zeta(2s)\big)\right)^n (m-i)^{-2s} \\
=&\left(\frac{9}{2}\big(1+\zeta(2s)\big)\right)^n \cdot \sum_{i=1}^{m-1} i^{-2s}(m-i)^{-2s} \\
\le&\left(\frac{9}{2}\big(1+\zeta(2s)\big)\right)^n \cdot \left(\frac{9}{2}\big(1+\zeta(2s)\big)\right) m^{-2s} \\
=&\left(\frac{9}{2}\big(1+\zeta(2s)\big)\right)^{n+1} m^{-2s}.
\end{aligned}$$

□

Let

$$A(a,c_1,c_2,N) := \left\{x \in (0,1) : c_1 < \frac{a_n(x)}{e^{n^a}} < c_2,\ \forall n \ge N\right\}.$$

**Lemma 2.2.** *For any $a > 0$, any $N \ge 1$ and any $0 < c_1 < c_2$*

$$\dim_H A(a,c_1,c_2,N) = \frac{1}{2}.$$

*Proof.* This lemma is only a simplest special case of [2, Lemma 3.2], but we will sketch the proof (based on [4]), needed for the next lemma. For simplicity, let $N = 1$ (the proof for other $N$ is almost identical).

Let $a_1, a_2, \ldots, a_n$ satisfy $c_1 < a_j e^{-j^a} < c_2$ for all $j$. Those are exactly the possible sequences for which the basic interval $I_n(a_1, \ldots, a_n)$ has nonempty intersection with $A(a, c_1, c_2, 1)$.

There are approximately

$$\prod_{j=1}^{n} (c_2 - c_1) e^{j^a} \approx e^{\sum_1^n j^a} \tag{2.2}$$

of such basic intervals, each of diameter

$$|I_n(a_1, \ldots, a_n)| \approx e^{-2\sum_1^n j^a}, \tag{2.3}$$

(both estimations are up to a factor exponential in $n$). Hence, using the intervals $\{I_n(a_1, \ldots, a_n)\}$ as a cover, we get

$$\dim_H A(a, c_1, c_2, 1) \leq \frac{1}{2}.$$

To get the lower bound, we consider a probabilistic measure $\mu$ uniformly distributed on $A(a, c_1, c_2, 1)$, in the following sense: given $a_1, \ldots, a_{n-1}$, probability of $a_n$ taking any particular value between $c_1 e^{n^a}$ and $c_2 e^{n^a}$ is the same.

The basic intervals $I_n(a_1, \ldots, a_n)$ have length $e^{-2\sum_1^n j^a}$ and measure $e^{-\sum_1^n j^a}$, each (up to a factor $c^n$). They are distributed in clusters: all $I_n(a_1, \ldots, a_n)$ contained in single $I_n(a_1, \ldots, a_{n-1})$ form an interval of length $e^{-2\sum_1^{n-1} j^a - n^a}$ (up to a factor $c^n$, with $c$ being a constant), then there is a gap, then there is another cluster. Hence, for any $r \in (e^{-2\sum_1^n j^a}, e^{-2\sum_1^{n-1} j^a})$ and any $x \in A(a, c_1, c_2, 1)$ it is easy to estimate the measure of $B(x, r)$:

$$\mu(B(x, r)) \approx \begin{cases} r \cdot e^{-\sum_1^n j^a} & \text{if } r < e^{-2\sum_1^{n-1} j^a - n^a} \\ e^{-\sum_1^{n-1} j^a} & \text{if } r > e^{-2\sum_1^{n-1} j^a - n^a} \end{cases}$$

(up to a factor $c^n$). The minimum of $\log \mu(B(x, r))/\log r$ is thus achieved for $r = e^{-2\sum_1^{n-1} j^a - n^a}$, and this minimum equals

$$\frac{-\sum_1^{n-1} j^a}{-2\sum_1^{n-1} j^a - n^a} \approx \frac{-n^{a+1}/(a+1)}{-2n^{a+1}/(a+1) - n^a} = \frac{1}{2} - O(1/n).$$

Hence, the lower local dimension of $\mu$ equals $1/2$ at each point of $A(a, c_1, c_2, 1)$, which implies

$$\dim_H A(a, c_1, c_2, 1) \geq \frac{1}{2}$$

by the Frostman Lemma (see [1]). □

Let now $c_1$ and $c_2$ not be constant but depend on $n$:

$$B(a, c_1, c_2, N) = \left\{ x \in (0, 1) : c_1(n) < \frac{a_n(x)}{e^{n^a}} < c_2(n) \ \forall n \geq N \right\}.$$

A slight modification of the proof of Lemma 2.2 gives the following.

**Lemma 2.3.** *Fix $a$ and $N$. Assume $0 < c_1(n) < c_2(n)$ for all $n$. Assume also that*

$$\lim_{n\to\infty} \frac{\log(c_2(n) - c_1(n))}{n^a} = 0$$

*and*

$$\liminf_{n\to\infty} \frac{\log c_1(n)}{\log n} > -\infty \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\log c_2(n)}{\log n} < +\infty.$$

*Then*

$$\dim_H B(a, c_1, c_2, N) = 1/2.$$

*Proof.* We need only to replace the constants $c_1$ and $c_2$ by $c_1(n)$ and $c_2(n)$ in the proof of Lemma 2.2. Notice that by the assumptions of Lemma 2.3, the formulas (2.2) and (2.3) still hold, up to a factor $e^{cn\log n}$ for some bounded $c$, much smaller than the main term $e^{\sum_1^n j^a}$ which is of order $e^{n^{1+a}}$. The rest of the proofs are the same. □

## 3. Proofs

*Proof of Theorem 1.1.* Let $\varphi : \mathbb{N} \to \mathbb{N}$ be defined by $\varphi(n) = \exp\{n^a\}$ with $a > 0$. For this case, we will denote $E_\varphi$ by $E_a$.

Let us start from some easy observations, giving (among other things) a simple proof of $\dim_H E_a = 1/2$ for $a \geq 1$.

Consider first $a \geq 1/2$. If $x \in E_a$ then for any $\varepsilon > 0$ for $n$ large enough,

$$(1-\varepsilon)e^{n^a} \leq S_n(x) \leq (1+\varepsilon)e^{n^a}$$

and

$$(1-\varepsilon)e^{(n+1)^a} \leq S_{n+1}(x) \leq (1+\varepsilon)e^{(n+1)^a}$$

Hence,

$$(1-\varepsilon)e^{(n+1)^a} - (1+\varepsilon)e^{n^a} \leq a_{n+1}(x) \leq (1+\varepsilon)e^{(n+1)^a} - (1-\varepsilon)e^{n^a}.$$

For $a \geq 1$ this implies

$$E_a \subset \bigcup_N A(a, c_1, c_2, N)$$

for some constants $c_1, c_2$. By Lemma 2.2,

$$\dim_H E_a \leq \frac{1}{2}, \quad \forall a \geq 1.$$

Consider now any $a > 0$. Set

$$c_1(n) = (e^{n^a} - e^{(n-1)^a})e^{-n^a} \quad \text{and} \quad c_2(n) = \frac{n+1}{n} c_1(n).$$

As

$$(e^{n^a} - e^{(n-1)^a})e^{-n^a} \approx a n^{a-1},$$

the assumptions of Lemma 2.3 are satisfied. As $B(a, c_1, c_2, N) \subset E_a$, by Lemma 2.3,

$$\dim_H E_a \geq \frac{1}{2}, \quad \forall a > 0.$$

Thus we have obtained $\dim_H E_a = 1/2$ for $a \geq 1$ and $\dim_H E_a \geq 1/2$ for $a > 0$. What is left to prove is that for $a \in [1/2, 1)$ $\dim_H E_a \leq 1/2$.

Let us first assume that $a > 1/2$. We will once again use the fact that for any $\varepsilon > 0$, if $x \in E_a$, then for $n$ large enough,

$$(1-\varepsilon)e^{n^a} \le S_n(x) \le (1+\varepsilon)e^{n^a}.$$

Take a subsequence $n_0 = 1$, and $n_k = k^{1/a}$ $(k \ge 1)$. Then there exists an integer $N \ge 1$ such that for all $k \ge N$,

$$(1-\varepsilon)e^k \le S_{n_k}(x) \le (1+\varepsilon)e^k,$$

and

$$(1-\varepsilon)e^k - (1+\varepsilon)e^{k-1} \le S_{n_k}(x) - S_{n_{k-1}}(x) \le (1+\varepsilon)e^k - (1-\varepsilon)e^{k-1}.$$

Thus

$$E_a \subset \bigcup_N A(a,N),$$

with $A(a,N)$ being the union of the intervals $\{I_{n_k}(a_1, a_2, \cdots, a_{n_k})\}_{k \ge N}$ such that

$$\sum_{j=n_{\ell-1}+1}^{n_\ell} a_j = m \quad \text{with} \quad m \in D_\ell, \quad N \le \ell \le k,$$

where $D_\ell := [(1-\varepsilon)e^\ell - (1+\varepsilon)e^{\ell-1}, (1+\varepsilon)e^\ell - (1-\varepsilon)e^{\ell-1}]$.

Now, we are going to estimate the upper bound of the Hausdorff dimension of $A(a,1)$. For $A(a,N)$ with $N \ge 2$, we will have the same bound and the proofs are almost the same.

For any $s > 1/2$ we can apply Lemma 2.1 together with the formula

$$|I_{n_k}|^s \le \prod_{\ell=1}^{k} (a_{n_{\ell-1}+1} a_{n_{\ell-1}+2} \cdots a_{n_\ell})^{-2s}$$

to obtain

$$\sum_{I_{n_k} \in \mathcal{A}} |I_{n_k}|^s \le \prod_{\ell=1}^{k} \sum_{m \in D_\ell} \left(\frac{9}{2}\big(1+\zeta(2s)\big)\right)^{n_\ell - n_{\ell-1}} m^{-2s}$$

$$\le \prod_{\ell=1}^{k} 2\varepsilon\big(1+\frac{1}{e}\big)e^\ell \cdot \left(\frac{9}{2}\big(1+\zeta(2s)\big)\right)^{\ell^{1/a}-(\ell-1)^{1/a}} \cdot \left(\frac{e}{e-1-\varepsilon e-\varepsilon}\right)^{2s} e^{-2s\ell}.$$

We have $\ell^{1/a} - (\ell-1)^{1/a} \approx \ell^{1/a-1}$. As $a > 1/2$, we have $1/a - 1 < 1$, and the main term in the above estimation is $e^{(1-2s)\ell}$. Thus for any $s > 1/2$, the product is uniformly bounded and we have the Hausdorff dimension of $A(a,1)$ is not greater than $1/2$.

If $a = 1/2$, we take $n_k = k^2/L$ with $L$ being a constant and we repeat the same argument. Then the same estimation will lead to

$$\sum_{I_{n_k} \in \mathcal{A}} |I_{n_k}|^s \le \prod_{\ell=1}^{k} C \cdot e^{\ell/\sqrt{L}} \left(\frac{9}{2}\big(1+\zeta(2s)\big)\right)^{(\ell^2-(\ell-1)^2)/L} e^{-2s\ell/\sqrt{L}}.$$

The main term of the right side of the above inequality should be

$$\left(\frac{9}{2}\big(1+\zeta(2s)\big)\right)^{2\ell/L} \cdot e^{(1-2s)\ell/\sqrt{L}}.$$

We solve the equation

$$\left(\frac{9}{2}(1+\zeta(2s))\right)^{2/L} \cdot e^{(1-2s)/\sqrt{L}} = 1,$$

which is equivalent to

$$\left(\frac{9}{2}(1+\zeta(2s))\right) = e^{\frac{2s-1}{2}\sqrt{L}}. \tag{3.1}$$

Observe that the two curves (of the variable $s$) of the two side of (3.1) always have a unique intersection for some $s_L \in [1/2, 1]$, when $L$ is large enough. These $s_L$ are all upper bounds for the Hausdorff dimension of $A(a,1)$. Notice that the intersecting point $s_L \to 1/2$ as $L \to \infty$ since the zeta function $\zeta$ has a pole at 1. Thus the dimension of $A(a,1)$ is not greater than 1/2.

So, in both cases, we have obtained $\dim_H E_a \leq 1/2$. □

*Sketch proof of Theorem 1.2.* The proof goes like Section 4 of [9] with the following changes. We choose $\varepsilon_k = \psi(k)$. Then by the hypothesis on the function $\psi$, we have

$$\sum_{k=1}^{r(n)} \varepsilon_k \approx r(n)\psi(r(n)),$$

and we obtain the key formula (10) in [9] in the form

$$r(n)\psi(r(n)) \ll \sqrt{n}\psi(n).$$

The other key point, the formula (15) in [9] follows by the estimation

$$\log(a_{n_1}a_{n_2}\cdots a_{n_{r(n)}}) \ll r(n)\sqrt{n}\psi(n) + r(n) \ll \frac{n\psi^2(n)}{\psi(r(n))} + r(n) \ll n.$$

□

*Proof of Theorem 1.3.* For the case $a < 1/2$, the set constructed in Section 4 of [9] (as a subset of the set of points for which $S_n(x) \approx e^{n^a}$) satisfies also $T_n(x) \approx e^{n^a}$ and has Hausdorff dimension one. We proceed to the case $a > 1/2$.

The lower bound is a corollary of Lemma 2.3. Take $c_1(n) = \alpha(1-\frac{1}{n})$ and $c_2(n) = \alpha$. Then the conditions of Lemma 2.3 are satisfied, and for all points $x$ such that $c_1(n)e^{n^a} < a_n(x) < c_2(n)e^{n^a}$, we have

$$T_n(x)/e^{n^a} \geq c_1(n) = \alpha\left(1-\frac{1}{n}\right),$$

and

$$T_n(x)/e^{n^a} = a_k/e^{n^a} \leq \alpha e^{k^a}/e^{n^a} \leq \alpha$$

(where $k \leq n$ is the position at which the sequence $a_1, \ldots, a_n$ achieves a maximum). Thus we have

$$\lim_{n\to\infty} T_n(x)/e^{n^a} = \alpha$$

for all $x \in B(a, c_1, c_2, 1)$. The similar argument works for $B(a, c_1, c_2, N)$ for any $N$. Hence, the lower bound follows directly from Lemma 2.3.

The upper bound is a modification of that of Theorem 1.1. Denote the set in question by $E(a, \alpha)$. We consider the case $\alpha = 1$ only, since for other $\alpha > 0$, the proofs are similar.

Notice that for any $\varepsilon > 0$, if $x \in E(a,1)$, then for $n$ large enough,

$$(1-\varepsilon)e^{n^a} \leq S_n(x) \leq n(1+\varepsilon)e^{n^a}.$$

Take a subsequence $n_k = k^{1/a}(\log k)^{1/a^2}$. Then

$$(1-\varepsilon)e^{k(\log k)^{1/a}} \leq S_{n_k}(x) \leq k^{1/a}(\log k)^{1/a^2}(1+\varepsilon)e^{k(\log k)^{1/a}},$$

and

$$u_k \leq S_{n_k}(x) - S_{n_{k-1}}(x) \leq v_k,$$

with

$$u_k := (1-\varepsilon)e^{k(\log k)^{1/a}} - (k-1)^{1/a}(\log(k-1))^{1/a^2}(1+\varepsilon)e^{(k-1)(\log(k-1))^{1/a}},$$

and

$$v_k := k^{1/a}(\log k)^{1/a^2}(1+\varepsilon)e^{k(\log k)^{1/a}} - (1-\varepsilon)e^{(k-1)(\log(k-1))^{1/a}}.$$

We remark that

$$u_k > \frac{1}{2}e^{k(\log k)^{1/a}}, \quad v_k < \frac{3}{2}k^{1/a}(\log k)^{1/a^2}e^{k(\log k)^{1/a}}$$

when $k$ is large enough.

Observe that

$$E(a,1) \subset \bigcup_N B(a,N),$$

with $B(a,N)$ being the union of the intervals $\{I_{n_k}(a_1,a_2,\cdots,a_{n_k})\}_{k\geq N}$ such that

$$\sum_{j=n_{\ell-1}+1}^{n_\ell} a_j = m \quad \text{with} \quad m \in D_\ell, \quad N \leq \ell \leq k,$$

where $D_\ell$ is the set of integers in the interval $[u_\ell, v_\ell]$.

As in the proof the Theorem 1.1, we need only study the set $B(a,1)$. We have for any $s > 1/2$, since

$$|I_{n_k}|^s \leq \prod_{\ell=1}^{k}(a_{n_{\ell-1}+1}a_{n_{\ell-1}+2}\cdots a_{n_\ell})^{-2s},$$

by Lemma 2.1,

$$\sum_{I_{n_k}\in\mathcal{A}} |I_{n_k}|^s \leq \prod_{\ell=1}^{k}\sum_{m\in D_\ell}\left(\frac{9}{2}\big(1+\zeta(2s)\big)\right)^{n_\ell-n_{\ell-1}} m^{-2s}$$

$$\leq \prod_{\ell=1}^{k}\frac{3}{2}\cdot \ell^{1/a}(\log \ell)^{1/a^2}e^{\ell(\log \ell)^{1/a}}\left(\frac{9}{2}\big(1+\zeta(2s)\big)\right)^{n_\ell-n_{\ell-1}} 2^{2s}e^{-2s\ell(\log \ell)^{1/a}}.$$

Since $n_\ell - n_{\ell-1} \approx \ell^{1/a-1+o(\varepsilon)}$ and $1/a-1<1$, the main term in the above estimation is $e^{(1-2s)\ell(\log \ell)^{1/a}}$. Thus for any $s > 1/2$, the product is uniformly bounded and we have the Hausdorff dimension of $B(a,1)$ is not greater than $1/2$. Then we can conclude $\dim_H E(a,1) \leq 1/2$ and the proof is completed. □

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