UPPER AND LOWER FAST KHINTCHINE SPECTRA IN CONTINUED FRACTIONS

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Abstract. Every $x \in [0,1)$ can be expanded as a continued fraction: $x = [a_1(x), a_2(x), \cdots]$. Let $\psi : \mathbb{N} \to \mathbb{N}$ be a function with $\psi(n)/n \to \infty$ as $n \to \infty$. The (upper, lower) fast Khintchine spectrum for ψ is defined as the Hausdorff dimension of the set of numbers $x \in (0,1)$ for which the (upper, lower) limit of $\frac{1}{\psi(n)} \sum_{j=1}^{n} \log a_j(x)$ is equal to 1. The fast Khintchine spectrum was determined by Fan, Liao, Wang, and Wu. We calculate the upper and lower fast Khintchine spectra. These three spectra can be different.

1. Introduction

Each irrational number $x \in [0,1)$ admits a unique infinite continued fraction expansion of the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}}, \tag{1.1}$$

where the integers $a_n(x)$, called the partial quotients of x, can be generated by using the Gauss transformation $T : [0,1) \to [0,1)$ defined by

$$T(0) := 0, \ T(x) = \frac{1}{x} \pmod 1, \ \text{for } x \in (0,1).$$

In fact, let $a_1(x) = \lfloor x^{-1} \rfloor$ ($\lfloor \cdot \rfloor$ stands for the integral part), then $a_n(x) = a_1(T^{n-1}(x))$ for $n \geq 2$. For simplicity, (1.1) is often written as $x = [a_1, a_2, \cdots]$.

For any $x \in (0,1)$, the Khintchine exponent of x is defined by the limit (if it exists)

$$\xi(x) := \lim_{n\to\infty} \frac{\log a_1(x) + \cdots + \log a_n(x)}{n}.$$

Khintchine [5] proved that for Lebesgue almost all points x, we have

$$\xi(x) = \int_0^1 \frac{\log a_1(x)}{(1+x)\log 2} dx = 2.6854....$$

Let $\psi : \mathbb{N} \to \mathbb{N}$ and let $\alpha > 0$. Define

$$E(\psi, \alpha) = \left\{ x \in [0,1) : \lim_{n\to\infty} \frac{\log a_1(x) + \cdots + \log a_n(x)}{\psi(n)} = \alpha \right\}.$$

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When $\psi(n) = n$, the set $E(\psi, \alpha)$ is a level set of the Khintchine exponent, whose Hausdorff dimension is determined in [2]. The function of the Hausdorff dimension associated to each α is called the *Khintchine spectrum.*

Later, in [3], the authors studied the fast Khintchine spectrum, i.e. the Hausdorff dimension of $E(\psi, \alpha)$ where ψ satisfies that $\psi(n)/n \to \infty$ as $n \to \infty$. In this case, it turns out that the Hausdorff dimension does not depend on the level α, but only on the increasing rate of ψ. More precisely, let ψ and $\tilde{\psi}$ be two functions defined on $\mathbb{N}$. We say ψ and $\tilde{\psi}$ are *equivalent* if $\frac{\psi(n)}{\tilde{\psi}(n)} \to 1$ as $n \to \infty$. We denote the Hausdorff dimension by $\dim_H$. The authors of [3] proved the following theorem.

Theorem 1.1 ([3]). *Let $\psi : \mathbb{N} \to \mathbb{N}$ with $\psi(n)/n \to \infty$ as $n \to \infty$. If ψ is equivalent to an increasing function, then for all $\alpha > 0$, $E(\psi) \neq \emptyset$ and*

$$\dim_H E(\psi, \alpha) = \frac{1}{1 + \beta}, \quad \text{with } \beta = \limsup_{n\to\infty} \frac{\psi(n+1)}{\psi(n)}.$$

Otherwise, $E(\psi, \alpha) = \emptyset$ for all $\alpha > 0$.

When the sets $E(\psi, \alpha)$ are not empty, the dimensional function associated to ψ (and α) is called the *fast Khintchine spectrum* in [3].

In this note, we consider the following sets

$$\overline{E}(\psi) = \left\{ x \in [0, 1] : \limsup_{n\to\infty} \frac{\log a_1(x) + \cdots + \log a_n(x)}{\psi(n)} = 1 \right\},$$

and

$$\underline{E}(\psi) = \left\{ x \in [0, 1] : \liminf_{n\to\infty} \frac{\log a_1(x) + \cdots + \log a_n(x)}{\psi(n)} = 1 \right\}.$$

Their Hausdorff dimensions are called *upper* and *lower fast Khintchine spectra.*

Remark that we only consider the level $\alpha = 1$ here, since for other levels the Hausdorff dimension will not change, as in Theorem 1.1.

Theorem 1.2. *Assume that $\psi : \mathbb{N} \to \mathbb{N}$ satisfy $\psi(n)/n \to \infty$ as $n \to \infty$. Write*

$$\liminf_{n\to\infty} \frac{\log \psi(n)}{n} = \log b \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log \psi(n)}{n} = \log B.$$

Assume $b, B \in (1, \infty]$. Then

$$\dim_H \overline{E}(\psi) = \frac{1}{1 + b} \quad \text{and} \quad \dim_H \underline{E}(\psi) = \frac{1}{1 + B}.$$

We remark that the three values β, b and B are in general different even though we always have the relation $b \leq B \leq \beta$. We also remark that the set $\overline{E}(\psi)$ and $\underline{E}(\psi)$ are always nonempty.

2. Preliminary

For any $n \geq 1$ and $(a_1, a_2, \cdots, a_n) \in \mathbb{N}^n$, define

$$I_n(a_1, a_2, \cdots, a_n) = \big\{x \in [0,1) : \ a_1(x) = a_1, \cdots, a_n(x) = a_n\big\},$$

which is the set of numbers starting with $(a_1, \cdots, a_n)$ in their continued fraction expansions, and is called a *basic interval* of order n. The length of a basic interval will be denoted by $|I_n|$.

Proposition 2.1 ([5])**.** *For any $n \geq 1$ and $(a_1, \cdots, a_n) \in \mathbb{N}^n$,*

$$\left(2^n \prod_{k=1}^{n} a_k\right)^{-2} \leq |I_n(a_1, \cdots, a_n)| \leq \left(\prod_{k=1}^{n} a_k\right)^{-2}. \tag{2.1}$$

The following lemma is used to calculate the lower bound of the Hausdorff dimension of $\overline{E}(\psi)$.

Let $\{s_n\}_{n\geq 1}$ be a sequence of integers and $\ell \geq 2$ be some fixed integer. Set

$$F(\{s_n\}_{n=1}^{\infty}; \ell) := \big\{x \in [0,1) : s_n \leq a_n(x) < \ell s_n, \text{ for all } n \geq 1\big\}.$$

Lemma 2.2 ([2])**.** *Under the assumption that $s_n \to \infty$ as $n \to \infty$, one has*

$$\dim_H F(\{s_n\}_{n=1}^{\infty}; \ell) = \left(2 + \limsup_{n\to\infty} \frac{\log s_{n+1}}{\log s_1 s_2 \cdots s_n}\right)^{-1}.$$

In fact, Lemma 2.2 has a more general form. Let $s := \{s_n\}_{n\geq 1}$ and $t := \{t_n\}_{n\geq 1}$ be two sequences of real numbers such that $s_n > 1, t_n > 1$ for all $n \geq 1$. Consider the following set

$$F(s,t) := \big\{x \in [0,1) : s_n \leq a_n(x) < s_n t_n, \text{ for all } n \geq 1\big\}.$$

Lemma 2.3. *Assume that $s_n \to \infty$ as $n \to \infty$, and*

$$\lim_{n\to\infty} \frac{\log(t_n - 1)}{\log s_n} = 0.$$

Then

$$\dim_H F(s,t) = \left(2 + \limsup_{n\to\infty} \frac{\log s_{n+1}}{\log s_1 s_2 \cdots s_n}\right)^{-1}.$$

The proof of Lemma 2.3 is essentially contained in the proof of the lower bound of the dimension of $\underline{E}(\psi)$ in Subsection 3.2. So the details are left for the reader. A special case of Lemma 2.3 can be found in [6].

The next lemma is useful for the upper bound of the Hausdorff dimensions of $\overline{E}(\psi)$ and $\underline{E}(\psi)$.

Lemma 2.4 ([7])**.** *For any $a > 1, b > 1$,*

$$\dim_H\{x : a_n(x) \geq a^{b^n}, \forall n \geq 1\} = \dim_H\{x : a_n(x) \geq a^{b^n}, i.o.\} = \frac{1}{b+1}.$$

3. Proofs

3.1. **Dimension of $\overline{E}(\psi)$.** We first calculate the Hausdorff dimension of $\overline{E}(\psi)$. Recall that

$$\overline{E}(\psi) = \Big\{x \in [0,1) : \limsup_{n\to\infty} \frac{\log a_1(x) + \cdots + \log a_n(x)}{\psi(n)} = 1\Big\}.$$

We will only give the proof for $1 < b < \infty$. The case $b = \infty$ can be obtained by a standard limit procedure.

Upper bound: For $x \in \overline{E}(\psi)$, let $S_n(x) := \log a_1(x) + \cdots + \log a_n(x)$. Then for any $\delta > 0$, there are infinitely many n's such that $S_n(x) \geq \psi(n)(1-\delta)$. This implies that there exist infinitely many $i \leq n$ such that

$$\log a_i(x) \geq \frac{\psi(n)}{n}(1-\delta).$$

By the definition of b, for any $\varepsilon > 0$, $\psi(n) > (b-\varepsilon)^n$ for all $n \geq 1$. Thus, we have infinitely many i's, such that

$$\log a_i(x) > \frac{(b-\varepsilon)^n}{n}(1-\delta) > (b-2\varepsilon)^i.$$

By Lemma 2.4, the Hausdorff dimension of $\underline{E}(\psi)$ is bounded by $1/(1+(b-2\varepsilon))$ from above. Letting $\varepsilon \to 0$, we obtain the upper bound.

Lower bound: We define a real sequence $\{\tilde{c}_n\}_{n=1}^\infty$ as follows. Let $\tilde{c}_1 = e^{\psi(1)}$ and

$$\tilde{c}_2 = \min\left\{\frac{e^{\psi(2)}}{\tilde{c}_1},\ \tilde{c}_1^{b-1+\varepsilon}\right\}.$$

Assume that $\tilde{c}_n$ has been already well defined, then set

$$\tilde{c}_{n+1} = \min\left\{\frac{e^{\psi(n+1)}}{\prod_{k=1}^n \tilde{c}_k},\ \prod_{k=1}^n \tilde{c}_k^{b-1+\varepsilon}\right\}.$$

Now for all $n \geq 1$, take $c_n = \lfloor \tilde{c}_n \rfloor + 2$, where $\lfloor\cdot\rfloor$ stands for the integer part. Then we can check that

$$\limsup_{n\to\infty} \frac{\log c_{n+1}}{\log c_1 + \cdots + \log c_n} \leq b - 1 + \varepsilon. \tag{3.1}$$

By the definition of b, we can further check that there exist infinitely many n, such that $\tilde{c}_{n+1} = \frac{e^{\psi(n+1)}}{\prod_{k=1}^n \tilde{c}_k}$. Thus we have

$$\limsup_{n\to\infty} \frac{\log c_1 + \cdots + \log c_n}{\psi(n)} = 1. \tag{3.2}$$

Define

$$E(\{c_n\}) := \{x \in [0,1) : c_n \leq a_n(x) < 2c_n,\ \text{for all } n \geq 1\}.$$

By (3.2), $E(\{c_n\}) \subset \overline{E}(\psi)$.

To apply Lemma 2.2, we need the condition $c_n \to \infty$ as $n \to \infty$ which is not necessarily satisfied. So, some modifications on the subset $E(\{c_n\})$ are needed. By the condition that $\psi(n)/n \to \infty$ as $n \to \infty$, we can choose a sequence $\{n_k\}_{k=1}^\infty$ such that for each $k \geq 1$,

$$\frac{\psi(n)}{n} \geq k^2, \text{ when } n \geq n_k.$$

Take $\alpha_n = 2$ if $1 \le n < n_1$ and

$$\alpha_n = k+1, \text{ when } n_k \le n < n_{k+1}.$$

Then it is easy to see

$$\lim_{n\to\infty} \frac{\log \alpha_1 + \cdots + \log \alpha_n}{\psi(n)} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\log \alpha_{n+1}}{\log \alpha_1 + \cdots + \log \alpha_n} = 0.$$

Since $c_n \ge 2$ and $\alpha_n \ge 2$ for all $n \ge 1$, we have

$$\log c_n \le \log(c_n + \alpha_n) \le \log c_n + \log \alpha_n \quad \forall n \ge 1.$$

So, by taking $s_n = c_n + \alpha_n$ for each $n \ge 1$, we get

$$\limsup_{n\to\infty} \frac{\log s_1 + \cdots + \log s_n}{\psi(n)} = 1.$$

Define

$$E(\{s_n\}) := \{x \in [0,1) : s_n \le a_n(x) < 2s_n, \text{ for all } n \ge 1\}.$$

Then $E(\{s_n\}) \subset \overline{E}(\psi)$. As $s_n \to \infty$ as $n \to \infty$, by Lemma 2.2, we have

$$\dim_H E(\{s_n\}) = \left(2 + \limsup_{n\to\infty} \frac{\log s_{n+1}}{\log s_1 + \cdots + \log s_n}\right)^{-1}.$$

Note that

$$\begin{aligned}
& \limsup_{n\to\infty} \frac{\log s_{n+1}}{\log s_1 + \cdots + \log s_n} \\
= \ & \limsup_{n\to\infty} \frac{\log(c_{n+1} + \alpha_{n+1})}{\log(c_1+\alpha_1) + \cdots + \log(c_n + \alpha_n)} \\
\le \ & \limsup_{n\to\infty} \frac{\log c_{n+1} + \log \alpha_{n+1}}{\log(c_1+\alpha_1) + \cdots + \log(c_n + \alpha_n)} \\
\le \ & \limsup_{n\to\infty} \frac{\log c_{n+1}}{\log c_1 + \cdots + \log c_n} + \limsup_{n\to\infty} \frac{\log \alpha_{n+1}}{\log \alpha_1 + \cdots + \log \alpha_n} \\
\le \ & b - 1 + \varepsilon.
\end{aligned}$$

Hence,

$$\dim_H \overline{E}(\psi) \ge \dim_H E(\{s_n\}) \ge \frac{1}{b+1+\varepsilon}.$$

3.2. **Dimension of $\underline{E}(\psi)$.** Recall that

$$\underline{E}(\psi) = \left\{x \in [0,1) : \liminf_{n\to\infty} \frac{\log a_1(x) + \cdots + \log a_n(x)}{\psi(n)} = 1\right\}.$$

As in the calculation of the Hausdorff dimension of $\overline{E}(\psi)$, we will only give the proof for $1 < B < \infty$ and the easy case $B = \infty$ is left for the reader.

Upper bound: By the definition of B, for any $\varepsilon > 0$, there is a sequence $\{n_i\}$ such that

$$\psi(n_i) > (B - \varepsilon)^{n_i}.$$

Denoting $S_n(x) = \log a_1(x) + \cdots + \log a_n(x)$, for all $x \in \underline{E}(\psi)$, for any $\delta > 0$, we have

$$S_n(x) \ge \psi(n)(1-\delta), \ \forall n \ge 1.$$

Thus

$$S_{n_i}(x) \ge (B-\varepsilon)^{n_i}(1-\delta).$$

Then there exists $j \le n_i$ such that

$$\log a_j(x) \ge (B-\varepsilon)^{n_i}(1-\delta)/n_i > (B-2\varepsilon)^j.$$

As n_i goes to infinity, we will have infinitely many such j's. Thus by Lemma 2.4, the Hausdorff dimension of $\underline{E}(\psi)$ is bounded by $1/(1+(B-2\varepsilon))$ from above. The upper bound then follows.

Lower bound: We will construct a nonempty subset of $\underline{E}(\psi)$. Thus the following proof also shows that the set $\underline{E}(\psi)$ is always nonempty.

For any $\varepsilon > 0$, define

$$A_i = \sup_{n \ge i} \exp\{\psi(n)(B+\varepsilon)^{i-n}\}.$$

This is the smallest function satisfying

$$A_{i+1} \le A_i^{B+\varepsilon} \quad \text{and} \quad A_i \ge e^{\psi(i)}. \tag{3.3}$$

Let

$$Z := \liminf \frac{\sum_{i=1}^n \log A_i}{\psi(n)}.$$

Since for all $i \in \mathbb{N}$, $A_i \ge \exp\{\psi(i)(B+\varepsilon)^{i-i}\} = e^{\psi(i)}$, we have

$$Z \ge \frac{\log A_n}{\psi(n)} \ge 1.$$

We start by showing the following proposition.

Proposition 3.1. *We have $Z < \infty$.*

Since $\limsup_{n\to\infty} \frac{\log \psi(n)}{n} = \log B$, we have for n large enough,

$$\psi(n) \le (B+\varepsilon/2)^n.$$

Thus

$$A_i \le \exp\{(B+\varepsilon/2)^n (B+\varepsilon)^{i-n}\}.$$

Since $(B+\varepsilon/2)^n/(B+\varepsilon)^n$ goes to 0 as $n \to \infty$, we have the supremum in the definition of A_i can be obtained for the first time by some $t_i \ge i$. Remark that for many consecutive i's the t_i will be the same. More precisely, $t_i = t_{i+1} = \cdots = t_{t_i}$. Let us write $n_i = t_{t_i}$. Then $n_i < n_{i+1}$. Notice that for these n_i, we have

$$\log A_{n_i} = \psi(n_i),$$

and for $k \in (n_{i-1}, n_i]$,

$$\log A_k = \psi(n_i)(B+\varepsilon)^{k-n_i}.$$

Thus

$$\sum_{k=n_{i-1}+1}^{n_i} \log A_k = \sum_{k=n_{i-1}+1}^{n_i} \psi(n_i)(B+\varepsilon)^{k-n_i} \le C \cdot \psi(n_i).$$

Suppose $\{n_i\}$ are defined as above. Denote $S_n\psi := \sum_{k=1}^n \psi(k)$. Proposition 3.1 follows directly from the following two lemmas.

Lemma 3.2. *The following liminf is finite:*

$$\liminf_{n\to\infty} \frac{S_n\psi}{\psi(n)} < \infty.$$

Proof. For $\varepsilon > 0$, we will show that there exist infinitely many i, such that $\psi(i) > \varepsilon S_{i-1}\psi$. If not, we will have

$$S_n\psi = S_{n-1}\psi + \psi(n) \leq (1+\varepsilon)S_{n-1}\psi.$$

Thus

$$\limsup \frac{\log S_n\psi}{n} \leq \log(1+\varepsilon),$$

which is impossible since we have

$$\limsup \frac{\log \psi(n)}{n} = B > \log(1+\varepsilon).$$

Write l_i the sequence such that $\psi(l_i) > \varepsilon S_{l_i-1}\psi$. Then

$$\frac{S_{l_i}\psi}{\psi(l_i)} = \frac{S_{l_i-1}\psi + \psi(l_i)}{\psi(l_i)} \leq 1 + \frac{1}{\varepsilon} < \infty,$$

and the conclusion follows. □

Lemma 3.3. *If*

$$L := \liminf_{n\to\infty} \frac{S_n\psi}{\psi(n)} < \infty,$$

then

$$\liminf_{i\to\infty} \frac{S_{n_i}\psi}{\psi(n_i)} < \infty.$$

Proof. Let m_k be the sequence such that

$$\lim_{k\to\infty} \frac{S_{m_k}\psi}{\psi(m_k)} = L.$$

Then each m_k is in some $(n_{i-1}, n_i]$. Thus

$$\begin{aligned} S_{n_i}\psi = S_{m_k}\psi + \sum_{j=m_k+1}^{n_i} \psi(j) &\leq (L+\varepsilon)\psi(m_i) + \sum_{j=m_k+1}^{n_i} \psi(j) \\ &\leq (L+\varepsilon) \sum_{j=m_k}^{n_i} \psi(j). \end{aligned}$$

Since for $j \in [m_k, n_i] \subset (n_{i-1}, n_i]$,

$$\psi(j) \leq \frac{1}{(B+\varepsilon)^{n_i-j}}\psi(n_i),$$

we have

$$S_{n_i}\psi \leq C \cdot (L+\varepsilon)\psi(n_i).$$

Then the result follows. □

We continue the estimation of the lower bound. Let ε_i be a sequence decreasing to 0. (We will see $\varepsilon_i = 1/i$ are OK.) Construct x by choosing $a_i(x)$ in the interval

$$[A_i^{1/Z}(1-\varepsilon_i),\ A_i^{1/Z}(1+\varepsilon_i)].$$

Choose ε_i such that

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} \log(1 \pm \varepsilon_j)}{\psi(n)} = 0.$$

Then

$$\liminf_{n\to\infty} \frac{\sum_{j=1}^n \log a_j(x)}{\psi(n)} = \liminf_{n\to\infty} \frac{\frac{1}{Z}\sum_{j=1}^n \log A_j}{\psi(n)} = 1.$$

So such constructed x's are indeed in the set $\underline{E}(\psi)$. Denote by E the set of those x's.

To estimate the Hausdorff dimension, we define a probability measure μ on E. For each position, we distribute the probability evenly. That is for each possible a_i, we give the probability

$$\frac{1}{|[A_i^{1/Z}(1-\varepsilon_i), A_i^{1/Z}(1+\varepsilon_i)]|} = \frac{1}{2\varepsilon_i A_i^{1/Z}}.$$

Thus for each basic interval $I_n = I_n(a_1, \dots a_n)$, we have

$$\mu(I_n) = \prod (2\varepsilon_i A_i^{1/Z})^{-1}.$$

By (2.1)

$$|I_n| \approx \prod (A_i^{1/Z})^{-2}.$$

To calculate the local dimension of $x \in E$, we will use a smaller interval D_n included in I_n:

$$D_n = \cup_{a_{n+1} \geq A_{n+1}^{1/Z}(1-\varepsilon_{n+1})} I_{n+1}(a_1, \cdots, a_n a_{n+1}).$$

Since $a_i \in [A_i^{1/Z}(1-\varepsilon_i),\ A_i^{1/Z}(1+\varepsilon_i)]$, and A_i grows super-exponentially, the Hausdorff dimension will be determined by calculating the local dimension

$$\liminf \frac{\log \mu(D_n)}{\log |D_n|}.$$

(See Section 4 of Jordan and Rams [4].)

The length of this interval is

$$|D_n| \approx |I_n| \cdot A_{n+1}^{-1/Z}.$$

We have

$$-\log \mu(D_n) = -\log \mu(I_n) = \sum_{i=1}^n \log(2\varepsilon_i) + \frac{1}{Z}\sum_{i=1}^n \log A_i.$$

Let us choose ε_i such that $|\sum_{i=1}^n \log(2\varepsilon_i)| \ll \log A_{n+1}$. By the property that $A_{i+1} \leq A_i^{B+\varepsilon}$, we deduce that for big n,

$$-\log \mu(D_n) \geq \sum_{i=1}^n \log(2\varepsilon_i) + \frac{1}{Z}\sum_{i=1}^n \frac{1}{(B+\varepsilon)^i} \log A_{n+1} \approx \frac{1}{Z(B+\varepsilon-1)} \log A_{n+1}.$$

Now we calculate $-\log |D_n|$:

$$-\log |D_n| \approx -\log |I_n| + \frac{1}{Z}\log A_{n+1} \approx -2\log \mu(D_n) + \frac{1}{Z}\log A_{n+1}.$$

Thus

$$\begin{aligned} \frac{-\log \mu(D_n)}{-\log |D_n|} &\approx \frac{-\log \mu(D_n)}{-2\log \mu(D_n) + \frac{1}{Z}\log A_{n+1}} = \frac{1}{2 + \frac{\frac{1}{Z}\log A_{n+1}}{-\log \mu(D_n)}} \\ &\geq \frac{1}{2+B+\varepsilon-1} = \frac{1}{B+1+\varepsilon}. \end{aligned}$$

Then the lower bound follows from the Frostman Lemma (see [1]).

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