ON SHRINKING TARGETS FOR PIECEWISE EXPANDING INTERVAL MAPS

TOMAS PERSSON AND MICHAŁ RAMS

Abstract. For a map $T\colon [0,1] \to [0,1]$ with an invariant measure μ, we study, for a μ-typical x, the set of points y such that the inequality $|T^n x - y| < r_n$ is satisfied for infinitely many n. We give a formula for the Hausdorff dimension of this set, under the assumption that T is piecewise expanding and μ_ϕ is a Gibbs measure. In some cases we also show that the set has a large intersection property.

1. Introduction

We consider a map $T\colon [0,1] \to [0,1]$. Let $r = (r_n)_{n=1}^\infty$ be a sequence of decreasing positive numbers. In this paper we shall investigate the size of the set

$$\begin{aligned} E(x,r) &= \{\, y \in [0,1] : d(T^n x, y) < r_n \text{ for infinitely many } n \,\} \\ &= \limsup_{n\to\infty} B(T^n x, r_n). \end{aligned}$$

Sets of this form with $T\colon x \mapsto 2x \mod 1$ were studied by Fan, Schmeling and Troubetzkoy in [4]. Li, Wang, Wu and Xu studied in [6] a related but different set in the case when T is the Gauß map.

In the paper [7], Liao and Seuret studied the case when T is an expanding Markov map with a Gibbs measure μ. They proved that if $r_n = n^{-\alpha}$, then for μ-almost all x, the set $E(x,r)$ has Hausdorff dimension $1/\alpha$ provided that $1/\alpha$ is not larger than the dimension of the measure μ.

In this paper we will consider more general maps than those studied by Liao and Seuret and prove results similar to those of the three papers mentioned above. We will use a method of statistical nature very similar to the one used in [11]. The maps we will work with are mostly piecewise expanding interval maps, but some of our results are valid for more abstract maps with certain statistical properties.

We will not assume that the maps have a Markov partition. In the case that μ is a measure that is absolutely continuous with respect to Lebesgue measure, we can consider the sets $E(x,r)$ with $r_n = n^{-\alpha}$ for any $\alpha > 1$. However, for other measures μ we have to impose extra restrictions on α and our results are only valid for sufficiently large α. This extra restriction is not present in the works of Fan, Schmeling and Troubetzkoy; Li, Wang, Wu and Xu; and Liao and Seuret.

The results of this paper are presented in two main theorems, found in Sections 2 and 3. The first theorem treats the case when μ is absolutely

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continuous with respect to the Lebesgue measure and no extra restriction is imposed on α. In this case our result is a generalisations of the corresponding result by Liao and Seuret and we also prove that for almost all x the set $E(x,r)$ has large intersections. This means that the set $E(x,r)$ belongs for some $0 < s < 1$ to the class $\mathscr{G}^s$ of G_δ-sets, with the property that any countable intersection of bi-Lipschitz images of sets in $\mathscr{G}^s$ has Hausdorff dimension at least s. See Falconer's paper [3] for more details about those classes of sets. The large intersection property was not proved in any of the papers [4], [6] and [7].

The second theorem treats more general measures and is only valid for sufficiently large α. Restriction of this type are not present in the papers [4], [6] and [7]. We have not been able to prove the large intersection property in this case in the general setting. However, we prove that if the map is a Markov map, then the large intersection property holds.

In Section 4 we provide explicit examples of maps that satisfy the assumptions of the two main theorems. Most examples are for uniformly expanding maps, but we also give some examples with non-uniformly expanding maps.

One can also study the Hausdorff dimension of the complement of $E(x,r)$. That was done both in the paper by Liao and Seuret as well as that by Fan, Schmeling and Troubetzkoy, but we shall not do so in this paper.

2. Maps with Absolutely Continuous Invariant Measures

We will first work with maps $T\colon [0,1] \to [0,1]$ satisfying the following assumptions.

Assumption 1. There exists an invariant measure μ that is absolutely continuous with respect to Lebesgue measure, with density h such that $c_h^{-1} < h < c_h$ holds Lebesgue almost everywhere for some constant $c_h > 0$.

Assumption 2. Correlations decay with summable speed for functions of bounded variation: There is a function $p\colon \mathbb{N} \to (0,\infty)$ such that if $f \in L^1$ and g is of bounded variation, then

$$\left| \int f \circ T^n \cdot g \,\mathrm{d}\mu - \int f \,\mathrm{d}\mu \int g \,\mathrm{d}\mu \right| \leq \|f\|_1 \|g\| p(n),$$

where $\|\psi\| = \|\psi\|_1 + \operatorname{var} \psi$, and we assume that the correlations are summable in the sense that

$$C := \sum_{n=0}^{\infty} p(n) < \infty.$$

We prove the following theorem. The proof is in Section 5.

Theorem 1. *Under the Assumptions 1 and 2 above,* $\dim_{\mathrm{H}} E(x,r) \geq s$ *for Lebesgue almost all* $x \in [0,1]$, *where*

$$s = \sup\{\, t : \exists c, \forall n : n^{-2} \sum_{j=1}^{n} r_j^{-t} < c \,\}.$$

Moreover, the set $E(x,r)$ *belongs to the class* $\mathscr{G}^s$ *of* G_δ*-sets with large intersections for Lebesgue almost all* x.

In particular, if $r_n = n^{-\alpha}$ *then* $\dim_{\mathrm{H}} E(x,r) = 1/\alpha$ *for Lebesgue almost all* $x \in [0,1]$.

In Section 4 we provide some examples of maps satisfying Theorem 1.

3. Maps with Gibbs Measures

We will now consider a map $T\colon [0,1] \to [0,1]$ with a Gibbs measure μ_ϕ. Our assumptions are as follows.

Assumption 3. T is piecewise monotone and expanding with respect to a finite partition, and there is bounded distortion for the derivative T'.

Assumption 4. The potential $\phi\colon [0,1] \to \mathbb{R}$ is of bounded distortion, and there is a Gibbs measure μ_ϕ to the potential ϕ, with $\mu_\phi = h_\phi \nu_\phi$ where h_ϕ is a bounded function that is bounded away from zero, and ν_ϕ is a conformal measure, that is, for any subset A of a partition element holds

$$\nu_\phi(T(A)) = \int_A e^{P(\phi)-\phi} \,\mathrm{d}\nu_\phi,$$

where $P(\phi)$ denotes the topological pressure of ϕ.

Assumption 5. We have summable decay of correlations for functions of bounded variation. That is we assume that there is a function $p\colon \mathbb{N} \to (0,\infty)$ such that if $f \in L^1(\mu_\phi)$ and g is of bounded variation, then

$$\left| \int f \circ T^n \cdot g \,\mathrm{d}\mu_\phi - \int f \mathrm{d}\mu_\phi \int g \,\mathrm{d}\mu_\phi \right| \leq \|f\|_1 \|g\| p(n),$$

holds for all n, and we assume that

$$C := \sum_{n=0}^{\infty} p(n) < \infty.$$

Assumption 6. There is a number $s_0 > 0$ such that for any $s < s_0$ there is a constant c_s such that $\mu_\phi(I) \leq c_s |I|^s$ holds for any interval $I \subset [0,1]$.

Remark 1. We note that Assumption 6 implies that

$$\iint |x-y|^{-t} \,\mathrm{d}\mu_\phi(x)\mathrm{d}\mu_\phi(y) \leq \frac{tc_s}{s-t}. \tag{1}$$

for any $t < s < s_0$. This follows since, for any x, we have

$$\begin{aligned} \int |x-y|^{-t} \,\mathrm{d}\mu_\phi(y) &= \int_1^\infty \mu_\phi(B(x, u^{-1/t})) \,\mathrm{d}u \\ &\leq \int_1^\infty c_s u^{-s/t} \,\mathrm{d}u = \frac{tc_s}{s-t}, \end{aligned} \tag{2}$$

which implies (1).

Note also that (1) implies that the lower pointwise dimension of μ_ϕ is at least $s_0/2$ at any point in $[0,1]$. Indeed, since $|I|^{-s} \leq |x-y|^{-s}$ holds whenever $x, y \in I$, we have together with (1) that

$$\begin{aligned} |I|^{-s} \mu(I)^2 &\leq \iint_{I \times I} |x-y|^{-s} \,\mathrm{d}\mu_\phi(x)\mathrm{d}\mu_\phi(y) \\ &\leq \iint |x-y|^{-s} \,\mathrm{d}\mu_\phi(x)\mathrm{d}\mu_\phi(y) = c \end{aligned}$$

holds whenever $s < s_0$. Hence $\mu(I) \leq \sqrt{c} |I|^{s/2}$ and the claim follows.

In this setting we can prove a similar result to Theorem 1. The proof of the following theorem is in Section 6.

Theorem 2. *Assume that* $T\colon [0,1] \to [0,1]$ *satisfies the Assumptions 3, 4, 5 and 6. Then, we have that* $\dim_{\mathrm{H}} E(x,r) \geq s$ *for* μ_ϕ*-almost all* x*, where*

$$s = \sup\{\, t < s_0 : \exists c, \forall n : n^{-2} \sum_{j=1}^{n} r_j^{-t} < c \,\}.$$

In particular, if $r_n = n^{-\alpha}$ *and* $\alpha > 1/s_0$*, then* $\dim_{\mathrm{H}} E(x,r) = 1/\alpha$ *for* μ_ϕ*-almost every* x*.*

Remark 2. Note that if $\alpha \leq 1/s_0$ then Theorem 2 gives us the result that $\dim_{\mathrm{H}} E(x,r) \geq s_0$. However, one would expect that $\dim_{\mathrm{H}} E(x,r) = 1/\alpha$ as long as $1/\alpha$ is not larger than the dimension of μ_ϕ, which is the result proved by Liao and Seuret in their setting.

As is clear from Remark 1, our method cannot work for the full range of α, since we rely on Assumption 6, so that we cannot consider α such that $(2\alpha)^{-1}$ is larger than the lower pointwise dimension of μ_ϕ at any point.

If we also assume that the map is Markov, then we can prove the large intersection property of the set $E(x,r)$.

Theorem 3. *Assume that* $T\colon [0,1] \to [0,1]$ *is a Markov map that satisfies the Assumptions 3, 4, 5 and 6. Then, we have that* $E(x,r) \in \mathscr{G}^s$ *for* μ_ϕ*-almost all* x*, where*

$$s = \sup\{\, t < s_0 : \exists c, \forall n : n^{-2} \sum_{j=1}^{n} r_j^{-t} < c \,\}.$$

In the next section we give examples of maps satisfying the assumptions of Theorem 2.

4. Examples

4.1. **Examples to Theorem 1.** There exist some dynamical systems that obviously satisfy the assumptions of Theorem 1, for example $n-1$ expanding diffeomorphisms of the circle. We are going to present less obvious examples of application of our results.

For instance, the maps studied by Liverani in [8] satisfy the assumptions of Theorem 1. These maps are defined as follows. Assume that there is a finite partition $\mathscr{P}$ of $[0,1]$ into intervals, such that on every interval $I \in \mathscr{P}$, the map T can be extended to a C^2 map on a neighbourhood of the closure of I, and assume that there is a $\lambda > 1$ such that $|T'| \geq \lambda$ holds everywhere. To put it shortly, T is piecewise C^2 with respect to a finite partition, and uniformly expanding. We assume also that T is weakly covering, as defined by Liverani: The map T is said to be weakly covering if there exists an $N_0 \in \mathbb{N}$ such that if $I \in \mathscr{P}$, then

$$\bigcup_{k=0}^{N_0} T^k(I) \supset [0,1] \setminus W,$$

where W is the set of points that never hit the discontinuities of T. Under the assumptions mentioned above, it is shown in [8] that T has an invariant measure μ satisfying the assumption 1 above, and the correlations decay exponentially. Hence they are summable and Assumption 2 holds. We therefore have the following corollary.

Corollary 1. *If $T\colon [0,1] \to [0,1]$ is piecewise C^2 with respect to a finite partition, uniformly expanding, and weakly covering, then with $r_n = n^{-\alpha}$, $\alpha \geq 1$ we have*

$$\dim_{\mathrm{H}} E(x,r) = \frac{1}{\alpha}$$

and $E(x,r) \in \mathscr{G}^{1/\alpha}$ for Lebesgue almost every $x \in [0,1]$.

In fact, it is not necessary to assume that the map is piecewise C^2. It is sufficient that the derivative is of bounded variation, since then one can combine the estimates by Rychlik [12] with the method of Liverani [8] to get the same result.

If the map is piecewise expanding with an indifferent fixed point, then Assumption 2 does not hold. However, as we will see below, we can still use Theorem 1 to get the following result.

Corollary 2. *Let $T_\beta : [0,1) \to [0,1)$ with $\beta > 1$ be the Manneville–Pomeau map*

$$x \mapsto \begin{cases} x + 2^{\beta-1}x^\beta & x < 1/2 \\ 2x - 1 & x \geq 1/2 \end{cases}$$

and $r_j = j^{-\alpha}, \alpha \geq 1$. If $1 < \beta < 2$, then for Lebesgue almost every x we have that $\dim_H E(x,r) = 1/\alpha$ and $E(x,r) \in \mathscr{G}^{1/\alpha}$. If $\beta \geq 2$, then for Lebesgue almost every x we have that $\dim_{\mathrm{H}} E(x,r) = \frac{1}{\alpha(\beta-1)}$ and $E(x,r) \in \mathscr{G}^{1/\alpha/(\beta-1)}$.

Proof. Let S_β be the first return map on the interval $[1/2, 1)$. Then there exists an S_β-invariant measure ν that is absolutely continuous with respect to Lebesgue measure, and ν is ergodic.

Let $R(x)$ be the return time of x to $[1/2, 1)$, that is, we have $T_\beta^{R(x)} = S_\beta(x)$.

It the case $1 < \beta < 2$ we will do as follows. In this case R is integrable and so, for almost all x there is a constant $c > 0$ such that

$$n \leq \sum_{k=1}^{n} R_k(x) \leq cn \tag{3}$$

for all sufficiently large n. (The lower bound always holds, since $R \geq 1$.) We put

$$r'_j = (cj)^{-\alpha} \qquad \text{and} \qquad r''_j = j^{-\alpha}.$$

Then for almost all x we will have that

$$B(S_\beta^j(x), r'_j) \subset B(T_\beta^{\sum_{k=1}^{j} R_k(x)}(x), r_{\sum_{k=1}^{j} R_k(x)}) \subset B(S_\beta^j(x), r''_j)$$

for sufficiently large j. Hence, with

$$E'(x,r') := \limsup_{j\to\infty} B(S_\beta^j(x), r'_j),$$

$$E''(x,r'') := \limsup_{j\to\infty} B(S_\beta^j(x), r''_j),$$

we have

$$E'(x,r') \subset E(x,r) \cap [1/2,1] \subset E''(x,r'')$$

for almost all x.

Now, Theorem 1 implies that $E'(x,r') \cap [1/2,1) \in \mathscr{G}^{1/\alpha}$ for almost all x and $\dim_{\mathrm{H}} E''(x,r'') = 1/\alpha$ for almost all x. This implies the desired result for $E(x,r) \cap [1/2,1)$.

In the same way we can get the result for $E(x,r) \cap I_n$ where $I_n = [x_n, 1)$, where x_n is the n-th pre-image of $1/2$ with respect to the left branch of T_β. This concludes the proof for the case $1 \leq \beta < 2$.

The method above does not quite work when $\beta \geq 2$, since then $\int R \,\mathrm{d}\nu = \infty$, and the upper bound of (3) fails. However, whenever $\varepsilon > 0$, we have for almost all x that

$$n^{\beta-1-\varepsilon} \leq \sum_{k=1}^{n} R_k(x) \leq n^{\beta-1+\varepsilon}.$$

holds for large n. The upper bound above follows from Theorem 2.3.1 of [1]. The lower bound follows using Theorem 1 in [2].

We now proceed as in the case $1 < \beta < 2$. Put

$$r_j' = (c_2 j)^{-\alpha(\beta-1+\varepsilon)} \qquad \text{and} \qquad r_j'' = (c_1 j)^{-\alpha(\beta-1-\varepsilon)}.$$

With the same notation as previously we then have that

$$E'(x,r') \subset E(x,r) \cap [1/2,1] \subset E''(x,r'')$$

for almost all x.

Theorem 1 implies that for almost all x $E'(x,r') \cap [1/2,1) \in \mathscr{G}^{1/\alpha/(\beta-1+\varepsilon)}$ and $\dim_{\mathrm{H}} E''(x,r'') \cap [1/2,1) = (\alpha(\beta-1-\varepsilon))^{-1}$. Since $\varepsilon > 0$ can be chosen arbitrarily small, this implies the result for $E(x,r) \cap [1/2,1)$. As before, we get the result stated in the corollary by considering $E(x, r \cap I_n$ in the same way. □

4.2. **Examples to Theorem 2.** Here we will show that the Assumptions 5 and 6 are satisfied for a natural class of systems. Consider a map T which is piecewise C^2 with respect to a finite partition, and uniformly expanding, as defined in Section 4.1. Then Assumption 3 is satisfied.

Suppose that ϕ satisfies the assumptions of Liverani, Saussol and Vaienti in [9], that is, e^ϕ is of bounded variation and that there exists an n_0 such that

$$\sup e^{S_{n_0}\phi} < \inf L_\phi^{n_0} 1, \tag{4}$$

where $S_{n_0}\phi = \phi + \phi \circ T + \cdots + \phi \circ T^{n_0-1}$ and

$$L_\phi f(x) = \sum_{T(y)=x} e^{\phi(y)} f(y)$$

is the transfer operator with respect to the potential ϕ. We assume moreover that ϕ is piecewise C^2 with respect to the partition of the map, so that the bounded distortion part of Assumption 4 is satisfied.

Finally, we assume that T is covering, in the sense that for any non trivial interval I there is an n such that $T^n(I) \supset [0,1] \setminus W$, where W is the set of points that never hit the discontinuities of T. Under these assumptions, there exists a unique Gibbs measure μ_ϕ and the Assumptions 4 and 5 hold, see Theorem 3.1 in [9]. In this setting, Assumption 6 will also be satisfied.

Corollary 3. *Assume that $T\colon [0,1] \to [0,1]$ is piecewise C^2 with respect to a finite partition, uniformly expanding and covering. If ϕ satisfies the assumptions above, then the Assumption 6 is satisfied with*

$$s_0 = \limsup_{m\to\infty} \inf \frac{S_m\phi - mP(\phi)}{-\log|(T^m)'|}.$$

Hence, if $r_n = n^{-\alpha}$, $\alpha > 1/s_0$, then

$$\dim_{\mathrm{H}} E(x,r) = \frac{1}{\alpha}$$

for μ_ϕ-almost every $x \in [0,1]$.

Proof. We will rely on the part of Assumption 4 that says that if A is a subset of one of the partition elements, then

$$\nu_\phi(T(A)) = \int_A e^{P(\phi)-\phi}\,\mathrm{d}\nu_\phi, \tag{5}$$

where $P(\phi) = \lim_{n\to\infty} n^{-1}\log\inf L_\phi^n 1$ denotes the topological pressure of ϕ. Since there are constants c_1 and c_2 such that $0 < c_1 < h < c_2$, it suffices to prove Assumption 6 for the measure ν_ϕ.

Let $r_0 > 0$ be such that any interval of length r_0 intersects at most two partition elements. If $r < r_0$ and I is an interval of length r, then I intersects at most two different partition elements and therefore $T(I)$ consists of at most two intervals of length at most $r\sup|T'|$. By (5), it follows that

$$\nu_\phi(I)\inf_I e^{P(\phi)-\phi} \leq 2 \sup_{|I_1| = r\sup_I|T'|} \nu(I_1).$$

Hence

$$\nu_\phi(I) \leq 2\sup_I e^{\phi-P(\phi)} \sup_{|I_1| = r\sup_I|T'|} \nu(I_1).$$

By induction, we conclude that

$$\nu_\phi(I) \leq \bigl(2\sup_I e^{\phi-P(\phi)}\bigr)^n,$$

where n is the largest integer such that $r(\sup_I|T'|)^n \leq r_0$. Hence we have that there is a constant C_1, that does not depend on I, such that

$$\nu_\phi(I) \leq C_1 r^{\theta_1} = C_1|I|^{\theta_1}, \quad \theta_1 = \frac{\log 2 + \log\sup_I e^{\phi-P(\phi)}}{-\log\sup_I|T'|}$$

By making the constant C_1 sufficiently large, we can ensure that the estimate above holds for all intervals I, not only those that are sufficiently small.

By considering T^m instead of T, where m is a positive integer, the same argument gives us the existence of a constant C_m such that

$$\nu_\phi(I) \leq C_m|I|^{\theta_m}, \quad \theta_m = \frac{\log 2 + \log\sup_I e^{S_m\phi-mP(\phi)}}{-\log\sup_I|(T^m)'|}$$

holds for any interval I.

This shows that we may take $s_0 = \limsup_{m\to\infty}\inf\frac{S_m\phi-mP(\phi)}{-\log|(T^m)'|}$. The assumption (4) guaranties that $s_0 > 0$. □

5. Proof of Theorem 1

The proof of Theorem 1 will be based on the following lemma. It is a special case of Theorem 1 in [10]. We refer to [10] for a proof.

Lemma 1. *Let E_n be open subsets of $[0,1]$, and μ_n Borel probability measures with support in E_n, that converge weakly to a measure μ that is absolutely continuous with respect to Lebesgue measure and with density that is bounded and bounded away for zero. Suppose there exists a constant C such that*

$$\iint |x-y|^{-s}\,\mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y) < C$$

holds for all n. Then the set $\limsup_{n\to\infty} E_n$ belongs to the class $\mathscr{G}^s$ and has Hausdorff dimension at least s.

We will also make use of the following two lemmata.

Lemma 2. *Let $0 < s < 1$. There is a constant $c_s > 0$ such that if $B_1 = B(x_1, r_1)$ and $B_2 = B(x_2, r_2)$ are two balls, then*

$$\frac{1}{r_1 r_2}\int_{B_1}\int_{B_2} |x-y|^{-s}\,\mathrm{d}x\mathrm{d}y \leq c_s \min\{|x_1-x_2|^{-s}, r_1^{-s}, r_2^{-s}\},$$

and for any fixed x_2, the variation of the function

$$x_1 \mapsto \frac{1}{r_1 r_2}\int_{B_1}\int_{B_2} |x-y|^{-s}\,\mathrm{d}x\mathrm{d}y,$$

is less than $2c_s \min\{r_1^{-s}, r_2^{-s}\}$.

Proof. This is intuitively clear, but we provide a proof.

We suppose that $r_1 \geq r_2$. Let

$$I(x_1,x_2) = \frac{1}{r_1 r_2}\int_{B_1}\int_{B_2} |x-y|^{-s}\,\mathrm{d}x\mathrm{d}y.$$

It is clear that I achieves it's maximal value when $x_1 = x_2$, for instance when $x_1 = x_2 = 1/2$. Then a direct calculation shows that there is a constant c_1 such that

$$I(1/2,1/2) \leq c_1 r_1^{-s}.$$

Hence $I(x_1,x_2) \leq c_1 r_1^{-s} = c_1 \min\{r_1^{-s}, r_2^{-s}\}$.

Suppose that $|x_1 - x_2| > r_1$. It suffices to show that $I(x_1,x_2) \leq c_2|x_1 - x_2|^{-s}$ holds for some constant c_2. By a change of variables, we have that

$$\begin{aligned} I(x_1,x_2) &= |x_1-x_2|^{-s}\int_{-1}^{1}\int_{-1}^{1}\Big|1 - \frac{r_1}{|x_1-x_2|}u - \frac{r_2}{|x_1-x_2|}v\Big|^{-s}\mathrm{d}u\mathrm{d}v \\ &\leq 4|x_1-x_2|^{-s}\int_{0}^{1}\int_{0}^{1}\Big|1 - \frac{r_1}{|x_1-x_2|}u - \frac{r_2}{|x_1-x_2|}v\Big|^{-s}\mathrm{d}u\mathrm{d}v. \end{aligned}$$

Since $r_1/|x_1-x_2|$ and $r_2/|x_1-x_2|$ are not larger than 1, we have that

$$I(x_1,x_2) \leq 4|x_1-x_2|^{-s}\int_0^1\int_0^1 |1-u-v|^{-s}\,\mathrm{d}u\mathrm{d}v = c_2|x_1-x_2|^{-s}.$$

We can now conclude that $I(x_1,x_2) \leq c_s \min\{|x_1-x_2|^{-s}, r_1^{-s}, r_2^{-s}\}$, with $c_s = \max\{c_1, c_2\}$.

The statement about the variation is now a direct consequence since the function

$$x_1 \mapsto \frac{1}{r_1 r_2} \int_{B_1} \int_{B_2} |x - y|^{-s} \, \mathrm{d}x \mathrm{d}y,$$

is positive, unimodal and with maximal value at most $c_s \min\{r_1^{-s}, r_2^{-s}\}$. $\square$

Lemma 3. *Suppose that* $F \colon [0,1]^2 \to \mathbb{R}$ *is a continuous and non-negative function, and that* D *and* E *are constants such that for each fixed* x *the function* $f \colon y \mapsto F(x,y)$ *satisfies* $\operatorname{var} f \leq D$ *and* $\int f \, \mathrm{d}\mu \leq E$. *Then*

$$\int F(T^n x, x) \, \mathrm{d}\mu(x) \leq E + (D + E) p(n).$$

Proof. Let $\varepsilon > 0$. Let $I_k = [k/m, (k+1)/m)$. There is an m such that if

$$G(x,y) = \sum_{k=0}^{m-1} F(k/m, y) 1_{I_k}(x),$$

where 1_{I_k} denotes the indicator function on I_k, then

$$|F(x,y) - G(x,y)| < \varepsilon.$$

Hence we have

$$\left| \int F(T^n x, x) \, \mathrm{d}\mu(x) - \int G(T^n x, x) \, \mathrm{d}\mu(x) \right| < \varepsilon.$$

For each term $F(k/m, y) 1_{I_k}(x)$ in the sum defining G, we have

$$\left| \int F(k/m, x) 1_{I_k}(T^n x) \, \mathrm{d}\mu(x) - \int F(k/m, x) \, \mathrm{d}\mu(x) \int 1_{I_k} \, \mathrm{d}\mu \right| \\ \leq \mu(I_k)(D + E) p(n).$$

by the decay of correlations. As a consequence, we have

$$\int F(k/m, x) 1_{I_k}(T^n x) \, \mathrm{d}\mu(x) \leq E \mu(I_k) + \mu(I_k)(D + E) p(n).$$

and so

$$\int F(T^n x, x) \, \mathrm{d}\mu(x) \leq \varepsilon + \int G(T^n x, x) \, \mathrm{d}\mu(x) \leq \varepsilon + E + (D + E) p(n).$$

Let $\varepsilon \to 0$. $\square$

Proof of Theorem 1. Let $B_n(x) = B(T^n x, r_n)$. We consider the sets

$$V_n(x) = \bigcup_{k=m(n)}^{n} B_k(x)$$

where $m(n)$ is a slowly increasing sequence such that $m(n) < n$ and $m(n) \to \infty$ as $n \to \infty$. It then holds that $\limsup V_n(x) = \limsup B_n(x)$.

We define probability measures $\mu_{n,x}$ with support in $V_n(x)$ by

$$\mu_{n,x} = \frac{1}{n - m(n) + 1} \sum_{k=m(n)}^{n} \lambda_{B_k(x)},$$

where λ_A denotes the Lebesgue measure restricted to the set A and normalised so that $\lambda_A(A) = 1$. It is clear that $\mu_{n,x}$ converges weakly to μ as $n \to \infty$ for almost every x.

We shall consider the quantities

$$I_s(\mu_{n,x}) = \iint |y - z|^{-s} \, \mathrm{d}\mu_{n,x}(y) \mathrm{d}\mu_{n,x}(z).$$

From the definition of the measure $\mu_{n,x}$ it follows that

$$I_s(\mu_{n,x}) = \frac{1}{(n - m(n) + 1)^2} \sum_{i=m(n)}^{n} \sum_{j=m(n)}^{n} \frac{1}{4 r_i r_j} \int_{B_i} \int_{B_j} |y - z|^{-s} \, \mathrm{d}y \mathrm{d}z,$$

We now assume that $m(n) < n/2$. Together with Lemma 2 we then get that

$$I_s(\mu_{n,x}) \leq \frac{4c_s}{n^2} \sum_{m(n) \leq i \leq j \leq n} \min\{|T^i x - T^j x|^{-s}, r_i^{-s}\}.$$

Using that μ is T-invariant, we can write

$$\int I_s(\mu_{n,x}) \, \mathrm{d}\mu(x) \leq \frac{4c_s}{n^2} \sum_{m(n) \leq i \leq j \leq n} \int \min\{|T^{j-i} x - x|^{-s}, r_i^{-s} \wedge r_j^{-s}\} \, \mathrm{d}\mu(x),$$

where $a \wedge b$ denotes the minimum of a and b.

An application of Lemma 3 gives that

$$\begin{aligned} \int I_s(\mu_{n,x}) \, \mathrm{d}\mu(x) &\leq \frac{1}{n^2} \sum_{m(n) \leq i \leq j \leq n} \big(C_1 + (C_1 + 2(r_i^{-s} \wedge r_j^{-s})) p(j - i)\big) \\ &\leq \frac{1}{n^2} \sum_{m(n) \leq i \leq j \leq n} C_2 (1 + (r_i^{-s} \wedge r_j^{-s}) p(j - i)) \\ &\leq C_2 + \frac{C_2}{n^2} \sum_{j=1}^{n} \sum_{i=1}^{j} (r_i^{-s} \wedge r_j^{-s}) p(j - i). \end{aligned}$$

Since p is summable, we can estimate that

$$\sum_{j=1}^{n} \sum_{i=1}^{j} (r_i^{-s} \wedge r_j^{-s}) p(j - i) \leq \sum_{j=1}^{n} \sum_{i=1}^{j} r_j^{-s} p(j - i) = \sum_{j=1}^{n} \sum_{i=0}^{j-1} r_j^{-s} p(i) \leq C \sum_{j=1}^{n} r_j^{-s}.$$

(This estimate is actually not too rough, since

$$\sum_{j=1}^{n} \sum_{i=1}^{j} (r_i^{-s} \wedge r_j^{-s}) p(j - i) = \sum_{j=1}^{n} \sum_{i=0}^{j-1} (r_{j-i}^{-s} \wedge r_j^{-s}) p(i) \geq \sum_{j=1}^{n} r_j^{-s} p(0),$$

which is of the same order of magnitude if $r_i \to 0$ as $i \to \infty$.)

We conclude that

$$\int I_s(\mu_{n,x}) \, \mathrm{d}\mu(x) \leq C_2 + \frac{C C_2}{n^2} \sum_{j=1}^{n} r_j^{-s},$$

and this is uniformly bounded for all n if

$$s < \sup\{\, t : \exists c, \forall n : n^{-2} \sum_{j=1}^{n} r_j^{-t} < c \,\}.$$

Suppose s satisfies the inequality above. Then, by Birkhoff's ergodic theorem, for μ-almost all x the measures $\mu_{n,x}$ converges weakly to the measure μ, and, as follows from the considerations above, for μ-almost all x, there is a sequence n_k, with $n_k \to \infty$, such that the sequence $(I_s(\mu_{n_k,x}))_{k=1}^{\infty}$ is bounded. We can now apply Lemma 1 and conclude that for μ-almost all x the set $E(x,r)$ belongs to the class $\mathscr{G}^s$. This proves the first part of Theorem 1.

If $r_n = n^{-\alpha}$, then it is easy to check that the result above gives us that the set $E(x,r)$ belongs to $\mathscr{G}^{1/\alpha}$ for almost all x. A simple covering argument shows that in fact the dimension is not larger than $1/\alpha$. □

6. Proof of Theorems 2 and 3

Assume that we have a sequence of open sets E_n, such that each E_n is a finite union of disjoint intervals, and that the diameters of these intervals go to zero as n grows. We are first going to study the Hausdorff dimension of the set $\limsup E_n$ in the following lemmata. The proof of Theorem 2 will then be similar to that of Theorem 1, but will instead be based on the lemmata below.

Lemma 4. *Let E_n be open subsets of $[0,1]$. Suppose there are Borel probability measures μ_n with support in E_n, that converge weakly to a measure μ that satisfies assumption (1). If for some $t < s < s_0$ there is a constant C such that*

$$\iint |x-y|^{-s}\,\mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y) < C$$

for all n, then, whenever I is an interval with

$$\iint_{I\times I} |x-y|^{-t}\,\mathrm{d}\mu(x)\mathrm{d}\mu(y) < c|I|^{-t}\mu(I)^2,$$

there is an n_I such that

$$\sum |U_k|^t \geq \frac{1}{2c}|I|^t$$

holds for any cover $\{U_k\}$ of $E_n \cap I$, $n > n_I$.

Proof. The assumptions implies that for any $t < s$

$$\iint_{I\times I} |x-y|^{-t}\,\mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y) \to \iint_{I\times I} |x-y|^{-t}\,\mathrm{d}\mu(x)\mathrm{d}\mu(y),$$

as $n \to \infty$. (See [10].)

For a measure ν on I we write $R_t\nu(x) = \int |x-y|^{-t}\,\mathrm{d}\nu(y)$.

Take an interval $I \subset [0,1]$ and define the measure ν_n on I by

$$\nu_n(A) = \frac{\int_A (R_t\mu_n|_I)^{-1}\,\mathrm{d}\mu_n}{\int_I (R_t\mu_n|_I)^{-1}\,\mathrm{d}\mu_n}, \tag{6}$$

where $\mu_n|_I$ denotes the restriction of μ_n to I.

There are constants c and n_I such that if $n > n_I$ then

$$\nu_n(U) \leq 2c\frac{|U|^t}{|I|^t} \tag{7}$$

holds for all intervals $U \subset I$. This is proved as follows. By the definition of ν_n the estimate (7) is equivalent to

$$\frac{1}{|U|^t} \int_U (R_t \mu_n|_I)^{-1} \, \mathrm{d}\mu_n \leq \frac{2c}{|I|^t} \int_I (R_t \mu_n|_I)^{-1} \, \mathrm{d}\mu_n.$$

We prove the stronger statement that

$$\frac{1}{|U|^t} \int_U (R_t \mu_n|_I)^{-1} \, \mathrm{d}\mu_n \leq 1 \leq \frac{2c}{|I|^t} \int_I (R_t \mu_n|_I)^{-1} \, \mathrm{d}\mu_n. \tag{8}$$

The first inequality in (8) is proved in [10]. To prove the second inequality we use Jensen's inequality and Assumption 1 to conclude that

$$\begin{aligned}
\int_I (R_t \mu_n|_I)^{-1} \, \frac{\mathrm{d}\mu_n}{\mu_n(I)} &\geq \biggl(\int_I (R_t \mu_n|_I) \, \frac{\mathrm{d}\mu_n}{\mu_n(I)} \biggr)^{-1} \\
&= \biggl(\frac{1}{\mu_n(I)} \iint_{I \times I} |x-y|^{-s} \mathrm{d}\mu_n(x) \mathrm{d}\mu_n(y) \biggr)^{-1} \\
&\geq \frac{1}{\sqrt{2}} \biggl(\frac{1}{\mu(I)} \iint_{I \times I} |x-y|^{-s} \mathrm{d}\mu(x) \mathrm{d}\mu(y) \biggr)^{-1} \\
&\geq \frac{1}{2c} |I|^t \mu_n(I)^{-1},
\end{aligned}$$

provided $n > n_I$ for some n_I. Hence

$$\frac{1}{|I|^t} \int_I (R_t \mu_n|_I)^{-1} \, \mathrm{d}\mu_n \geq \frac{1}{2c}$$

and (8) follows.

We have now proved (7), and will use it as follows. Suppose that $\{U_k\}$ is a cover of $E_n \cap I$, and $n > n_I$. Then

$$1 = \nu_n(\bigcup_k U_k) \leq \sum_k \nu_n(E_k) \leq \frac{2c}{|I|^t} \sum_k |U_k|^t.$$

This shows that $\sum_k |U_k|^t \geq \frac{1}{2c} |I|^t$ for any cover $\{U_k\}$ of $E_n \cap I$. □

If we would have known that for some constant c, the estimate

$$\iint_{I \times I} |x-y|^{-t} \, \mathrm{d}\mu_\phi(x) \mathrm{d}\mu_\phi(y) < c |I|^{-t} \mu_\phi(I)^2,$$

holds for any I, then we could have used this to prove that the set $E(x,r)$ has a large intersection property, see the proof of Theorem 2. However, we are unable to prove that such a constant exists, and our strategy is instead to prove that we have such an estimate for sufficiently many intervals to get the dimension result. The lemma below is what we need.

If $\mathscr{Z}$ is the partition with respect to which T is piecewise expanding, then the elements of the partition $\mathscr{Z} \vee T^{-1} \mathscr{Z} \vee \cdots \vee T^{-n+1} \mathscr{Z}$ are called cylinders of generation n.

Lemma 5. *Let $d_0 > 0$ be given and suppose that (1) holds and that $s < s_0$. Then there is a constant $K = K(d_0)$ such that if I is an interval that is a subset of a cylinder of generation n and $|T^n(I)| > d_0$, then*

$$\iint_{I \times I} |x-y|^{-s} \, \mathrm{d}\mu_\phi(x) \mathrm{d}\mu_\phi(y) < K |I|^{-s} \mu_\phi(I)^2. \tag{9}$$

Proof. Let

$$K_0 = \sup_{|I|>d_0} \frac{|I|^s}{\mu_\phi(I)^2} \iint_{I\times I} |x-y|^{-s} \, \mathrm{d}\mu_\phi(x)\mathrm{d}\mu_\phi(y) < \infty.$$

By the bounded distortion, there exists a constant K_1 such that

$$\frac{|I|^s}{\mu_\phi(I)^2} \iint_{I\times I} |x-y|^{-s} \, \mathrm{d}\nu_\phi(x)\mathrm{d}\nu_\phi(y) \\ < K_1 \frac{|T^n(I)|^s}{\nu_\phi(T^n(I))^2} \iint_{T^n(I)\times T^n(I)} |x-y|^{-s} \, \mathrm{d}\nu_\phi(x)\mathrm{d}\nu_\phi(y),$$

whenever I is an interval contained in a cylinder of generation n. Since $\mu_\phi = h_\phi \nu_\phi$, where h_ϕ is bounded and bounded away from zero, the combination of these two estimates gives us the desired result. □

By Lemma 5 we know that some particular intervals are good, in the sense that we have the estimate (9). We will now use these intervals to construct a Cantor set $N = \cap N_n \subset \limsup E_n$ with large dimension. The following lemma describes the important properties of this construction.

Lemma 6. *Suppose that the assumptions of Lemma 4 hold with $\mu = \mu_\phi$, and that* (1) *is satisfied. Then, for any $\varepsilon > 0$, there is a sequence of sets N_n with the following properties.*

(i) *All N_n are compact, each $N_n = \cup N_{n,i}$ is a finite and disjoint union of intervals $N_{n,i}$, and $N_{n+1} \subset N_n$.*
(ii) *There is an increasing sequence m_n such that $N_n \subset E_{m_n}$.*
(iii) *For any $N_{n,i}$ we have*

$$\sum |U_k|^t \geq \frac{1}{4K} |N_{n,i}|^t,$$

for any cover $\{U_k\}$ of $N_{n,i}$.
(vi) *For any $N_{n,i}$ and $N_{n+1,j}$ we have*

$$\frac{|N_{n,i}|}{|N_{n+1,j}|} > (4K)^{1/\varepsilon}.$$

Proof. By Hofbauer [5], Lemma 13, we have that if we choose d_0 sufficiently small, then the Hausdorff dimension of the set of points, for which $|T^n I_n(x)| > d_0$ does not hold for infinitely many different n, is arbitrarily close to 0. In particular, if we choose d_0 sufficiently small, then there is a set A of full measure such that for any $x \in A$ there are infinitely many n with $|T^n I_n(x)| > d_0$.

If $x \in A$ and $I_n(x)$ has the property that $|T^n I_n(x)| > r_0$, then we let $J_{x,n} = I_n(x)$. We denote by $\mathscr{J}$ the set of all $J_{x,n}$, that is

$$\mathscr{J} = \{ \, J_{x,n} : x \in A \, \}.$$

We will define the sets N_n inductively as follows. We set $N_0 = [0,1]$. Clearly $[0,1]$ satisfies the assumptions of Lemma 5. We let $m_0 = n_{[0,1]}$, where $n_{[0,1]}$ is by Lemma 4.

Suppose that N_n has been defined together with a number m_n such that for any $N_{n,i}$, Lemma 4 is satisfied with $n_{N_{n,i}} \leq m_n$.

We wish to define N_{n+1}. The set $A \cap N_n$ has full measure in N_n. Hence, for any $\varepsilon_n > 0$, we can find a finite and disjoint collection $\mathscr{J}_n \subset \mathscr{J}$ such that for all $J_{x,n} \in \mathscr{J}_n$ we have $|J_{x,n}| < \varepsilon_n$ and $J_{x,n} \subset E_{m_n}$. Moreover, we can choose the collection $\mathscr{J}_n$ such that for any $N_{n,i}$, if $\mathscr{J}_n'$ denotes the elements of $\mathscr{J}_n$ that are subsets of $N_{n,i}$, then

$$\nu_n(\cup \mathscr{J}_n') > \frac{1}{2},$$

where the measure ν_n is defined by (6). As in the proof of Lemma 4, we can then conclude that for any $N_{n,i}$ we have

$$\sum |U_k|^t \geq \frac{1}{4} K^{-1} |N_{n,i}|^t,$$

for any cover $\{U_k\}$ of $N_{n,i}$.

We put $N_{n+1} = \cup \mathscr{J}_n$ and $\{N_{n+1,i}\} = \mathscr{J}_n$. The number m_{n+1} is taken to be an upper bound of $\{\, n_I : I \in J_n \,\}$. By taking ε_n sufficiently small we can achieve that

$$\frac{|N_{n,i}|}{|N_{n+1,j}|} > (4K)^{1/\varepsilon}$$

holds for any $N_{n,i}$ and $N_{n+1,j}$.

By induction, we now get the sets N_n with the desired properties. □

Lemma 7. *With the assumptions and notation of Lemma 6 we have that* $\dim_{\mathrm{H}} N \geq t - \varepsilon$, *where* $N = \cap N_n$.

Proof. Consider any countable cover $\mathscr{U} = \{U_k\}$ of the set N. Since N is compact, we can assume that $\mathscr{U}$ is a finite cover. We will consider the sum

$$Z_{t-\varepsilon}(\mathscr{U}) = \sum_k |U_k|^{t-\varepsilon},$$

trying to prove that it is uniformly bounded away from 0.

Step 1. There exists n_0 such that there is a finite cover $\mathscr{U}' = \{U_k'\}$ of N such that each intersection $U_k' \cap N$ is a finite union of $N \cap N_{n_0,i_\ell}$ and

$$Z_{t-\varepsilon}(\mathscr{U}) \geq \frac{1}{2} Z_{t-\varepsilon}(\mathscr{U}'). \tag{10}$$

This can be done by taking n_0 so large that the intervals $N_{n_0,i}$ are much smaller than all the (finitely many) elements of the cover $\mathscr{U}$, and then perturb each U_k so that it is aligned with the intervals $N_{n_0,i}$.

Step 3. Consider a new cover $\mathscr{U}''$, obtained in the following way. For any U_k', the set $U_k' \cap N$ must be contained in some $N_{n,i}$. There are at most two sets $N_{n+1,j}$ that intersect U_k' but are not contained in U_k'. We replace U_k' by at most three open sets: $U_k' \cap N_{n,i} \cap N_{n+1,j_1}$, $U_k' \cap N_{n,i} \cap N_{n+1,j_2}$, and $U_k' \cap N_{n,i} \setminus \overline{(N_{n+1,j_1} \cup N_{n+1,j_2})}$. The latter we leave as is, with the former two we repeat the procedure. The end result of this procedure: instead of U_k' we have a finite family of open sets U_ℓ'', each of which contains a finite union of $N_{n',i}$ for some n' and does not intersect other $N_{n',j}$ (we will call this the *wholeness property*). We will call such U_ℓ'' a n'-th level element.

Note that in this subfamily there will be at most one element of level n and at most two elements of each level n', $n < n' \leq n_0$. The lengths of elements of level n or $n+1$ are not greater than of the original $|U'_k|$, and by Lemma 6, for any element U''_ℓ of level $n' \geq n+2$ we have

$$|U''_\ell| \leq (4K)^{-(n'-n-1)/\varepsilon}|U'_k|.$$

Hence,

$$\sum_\ell |U''_\ell|^{t-\varepsilon} \leq K'|U'_k|^{t-\varepsilon}$$

for

$$K' = 3 + \frac{2}{(4K)^{(t-\varepsilon)/\varepsilon} - 1}.$$

Repeating this procedure for all U'_k and combining the subfamilies $\{U''_\ell\}$, we get a new cover $\mathscr{U}''$ consisting only of the elements with the wholeness property and satisfying

$$Z_{t-\varepsilon}(\mathscr{U}'') \leq K' Z_{t-\varepsilon}(\mathscr{U}'). \tag{11}$$

Step 4. Rename $\mathscr{U}''$ by $\mathscr{U}^{(n_0)}$. We remind that n_0 is the smallest n for which $N_n \subset \bigcup U''_k$ (that is, the maximal level of elements in U'').

We construct the sequence of covers $\mathscr{U}^{(n)}$ in the following way: let $\mathscr{U}^{(n+1)}$ be a cover with the wholeness property and with maximal level of elements $n+1$. Whenever for some $N_{n,i}$ there are elements $U^{(n+1)}_{k_1}, \dots, U^{(n+1)}_{k_\ell}$ that together cover all $N_{n+1,j} \subset N_{n,i}$, we replace those elements by $N_{n,i}$. The cover constructed in this way has wholeness property and does not have elements of level greater than n. Moreover, by Lemma 6,

$$|N_{n,i}|^t \leq 4K \sum |U^{(n+1)}_{k_i}|^t. \tag{12}$$

Let us divide the elements of $\mathscr{U}^{(n+1)}$ into three subcategories. An element $U^{(n+1)}_k$ is called

- *simple* if it is of the form $N_{n+1,j}$,
- *imminent* if it is not simple but of level $n+1$ (hence $U^{(n+1)}_k \cap N$ is contained in some $N_{n,j}$),
- *nonimminent* if it is of level not greater than n.

We divide the sum correspondingly:

$$Z_{t-\varepsilon}(\mathscr{U}^{(n+1)}) = Z^{(\mathrm{s})}_{t-\varepsilon}(\mathscr{U}^{(n+1)}) + Z^{(\mathrm{i})}_{t-\varepsilon}(\mathscr{U}^{(n+1)}) + Z^{(\mathrm{n})}_{t-\varepsilon}(\mathscr{U}^{(n+1)}).$$

Observe that by the construction of $\mathscr{U}^{(n)}$, the simple and imminent elements of $\mathscr{U}^{(n+1)}$ are replaced by simple elements of $\mathscr{U}^{(n)}$, while the nonimminent elements of $\mathscr{U}^{(n+1)}$ pass to $\mathscr{U}^{(n)}$ unchanged (where some of them become imminent, the other stay nonimminent). Hence,

$$Z^{(\mathrm{i})}_{t-\varepsilon}(\mathscr{U}^{(n)}) + Z^{(\mathrm{n})}_{t-\varepsilon}(\mathscr{U}^{(n)}) = Z^{(\mathrm{n})}_{t-\varepsilon}(\mathscr{U}^{(n+1)}). \tag{13}$$

As for $Z^{(\mathrm{s})}_{t-\varepsilon}(\mathscr{U}^{(n)})$, (12) implies

$$Z^{(\mathrm{s})}_t(\mathscr{U}^{(n)}) \leq 4K\big(Z^{(\mathrm{s})}_t(\mathscr{U}^{(n+1)}) + Z^{(\mathrm{i})}_t(\mathscr{U}^{(n+1)})\big).$$

By Lemma 6, if $N_{n+1,i} \subset N_{n,j}$ then

$$\frac{1}{4K}|N_{n+1,i}|^{-\varepsilon} \geq |N_{n,j}|^{-\varepsilon}.$$

hence

$$Z^{(\mathrm{s})}_{t-\varepsilon}(\mathscr{U}^{(n)}) \leq Z^{(\mathrm{s})}_{t-\varepsilon}(\mathscr{U}^{(n+1)}) + 4K Z^{(\mathrm{i})}_{t-\varepsilon}(\mathscr{U}^{(n+1)}). \tag{14}$$

Step 5. Induction procedure leads us to the cover $\mathscr{U}^{(0)} = \{[0,1]\}$. We have

$$Z_{t-\varepsilon}(\mathscr{U}^{(0)}) = Z^{(\mathrm{s})}_{t-\varepsilon}(\mathscr{U}^{(0)}) = |[0,1]|^{t-\varepsilon} = 1.$$

Combining equations (13) and (14) and repeating the inductive procedure from n_0 to 0, we observe that over the procedure, the nonimminent element of $\mathscr{U}^{(n_0)}$ first stays nonimminent for some time, then it becomes imminent, one step later it is combined into a simple element, and then it is combined with other elements into another simple element at each step. The only moment in this procedure when $Z_{t-\varepsilon}$ can increase is when the imminent element is combined into a simple element, which happens at most once for each element of $\mathscr{U}^{(n_0)}$. Moreover, at this time the corresponding term in the sum $Z_{t-\varepsilon}$ can increase at most by a factor $4K$. Hence,

$$Z_{t-\varepsilon}(\mathscr{U}^{(n_0)}) \geq \frac{1}{4K}.$$

Combining this with (10) and (11) we get

$$Z_{t-\varepsilon}(\mathscr{U}) \geq \frac{1}{8KK'}.$$

Since the cover $\mathscr{U}$ is arbitrary, it follows that $\dim_{\mathrm{H}} N \geq t - \varepsilon$. □

Proof of Theorem 2. We can now prove Theorem 2 in the same way as Theorem 1, by replacing the use of Lemma 1 with that of Lemmata 4, 5, 6 and 7.

Since the proof is very similar to that of Theorem 1, we will only sketch the proof. We define the sets $V_n(x)$ and the measures $\mu_{n,x}$ as in the proof of Theorem 1. We will then have that $\mu_{n,x}$ converges weakly to μ_ϕ for μ_ϕ-almost every x.

We consider the energies $I_s(\mu_{n,x})$ and their expectations $\int I_s(\mu_{n,x})\,\mathrm{d}\mu_\phi$ just as in the proof of Theorem 1 and carry out the same estimates. When we use Lemma 3 we need to know that

$$\int |x-y|^{-s}\,\mathrm{d}\mu_\phi(y)$$

is uniformly bounded in x. This follows from Assumption 6 according to Remark 1.

In this way we are able to conclude that for μ_ϕ-almost all x, there is a sub-sequence along which the energies

$$\iint |x-y|^{-s}\,\mathrm{d}\mu_\phi(x)\mathrm{d}\mu_\phi(y)$$

are uniformly bounded provided

$$s < \sup\{\, t : \exists c, \forall n : n^{-2} \sum_{j=1}^{n} r_j^{-t} < c \,\}.$$

We can now apply Lemmata 4, 5, 6 and 7 to get the desired result on the dimension of the set $E(x,r)$. □

Proof of Theorem 3. In the case that T is a Markov map, then we can use Lemma 5 to conclude that for $t < s$ there is a constant K such that

$$\iint_{I\times I} |x-y|^{-t}\,\mathrm{d}\mu_\phi(x)\mathrm{d}\mu_\phi(y) < K|I|^{-t}\mu_\phi(I)^2$$

holds for any interval $I \subset [0,1]$. Together with Lemma 4 and what was proved in the proof of Theorem 2, we can conclude that for μ_ϕ-almost all x, whenever I is an interval and n is sufficiently large, then any cover $\{U_k\}$ of $E_n \cap I$ satisfies

$$\sum |U_k|^t \geq \frac{1}{2K}|I|^t.$$

This implies, according to Falconer [3], that $E(x,r) \in \mathscr{G}^t$. □

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Centre for Mathematical Sciences, Lund University, Box 118, 22100 Lund, Sweden
E-mail address: tomasp@maths.lth.se
URL: http://www.maths.lth.se/~tomasp

Instytut Matematyczny, Polska Akademia Nauk, ul. Sniadeckich 8, 00-956 Warszawa, Poland
E-mail address: rams@impan.pl
URL: http://www.impan.pl/~rams