On the problem of optimal bond portfolio choice.

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Bedlewo, May 1, 2007
The problem:

Find $\sup_{\phi}\mathbb{E}[U(V_T(x))]$, where $V_T(x)$ is the value at time $T$ of a bond portfolio, with initial endowment $x$, $U$ is an utility function who satifies usual Inada’s type conditions, and $\phi_s$ is an admissible, self-financing portfolio strategy.

If the term structure model is a finite factor model, the problem is reduced to the usual Utility Maximization Problem in a stock market:
Let $df(t, T) = \alpha(t, T)\, dt + \sigma(t, T)\, dW_t$, (with $W_t$ $n$–dimensional), suppose $T \leq T_1 < \ldots < T_n$ be such that the matrix

$$
\left[ P(t, T_i) \cdot \int_t^{T_i} \sigma_j(t, s) \, ds \right]_{i,j=1,\ldots,n}
$$

is invertible (for $t \leq T$) : then the whole Bond Market until time $T$ is equivalent to the market given by $P(t, T_1), \ldots, P(t, T_n)$.

The problem of optimal Bond Portfolio choice is interesting (from a mathematical point of view) only in true infinite–dimensional models.
Three papers:

- **Ekeland–Taflin, (A.A.P. 2005)**: H.J.B. equation in infinite dimension


Main difference among these papers:

- in E.T. and C.T. the Bond Market model $P_t$ is taken as a stoch. process with values in a suitable space of functions.

- in DD.P a stochastic integration theory specialized for an infinite family of semimartingales is built.
Problems:

- existence results for the optimal strategy
- a Mutual Fund Theorem (generalization of Merton’s M.F.T.)

Merton’s Mutual Fund Theorem: the optimal investor’s allocation decision can be separated in two steps:

- an efficient portfolio of risky assets is determined (the M.F.)
- the investor decides the allocation between this efficient portfolio and the riskless asset.
An outline of our approach:

Elementary strategy:

\[ \Phi_t = \sum_{i=1}^{n} \Phi^i_t \delta_{T_i} \]

and

\[ \int_0^T \Phi_s \, dP_s = \int_0^T \sum_{i=i}^{n} \Phi^i_s \, dP(s, T_i) \]

Generalized strategies: processes with values unbounded linear operators.
Definition: $\Phi_t$ is integrable w.r.t. $P_t$ if there exists a sequence of elementary strategies $(\Phi^n_t)$ such that

- $\Phi^n_t \to \Phi$ a.s.
- $\int \Phi^n_t \, dP \to Y$ in the semimartingale topology

By definition, $Y = \int \Phi \, dP$. 
**Good results** with this approach:

- the **Memin’s** theorem is satisfied (limit of stochastic integrals is still a stochastic integral)

- the **Ansel–Stricker’s** lemma is satisfied (with a suitable definition of admissible strategies)
As a consequence, the following **Delbaen-Schachermayer’s** result is satisfied:

\[ 0 \leq f \leq x + \int \Phi_s d\tilde{P}_s \quad \text{for some suitable } \Phi \]

\[ \sup_Q \mathbb{E}^Q[f] \leq x, \quad Q \text{ equiv. martingale measure.} \]
In particular, the **Kramkov–Schachermayer (A.A.P. 2003)** results can be extended in this framework, thus obtaining:

- an *existence result* of the optimal portfolio $\hat{V}_T(x)$

- if $Q$ is unique, a characterization

$$
\hat{V}_T(x) = (U')^{-1} \left( y \frac{dQ}{d\mathbb{P}} \right)
$$

($x$ and $y$ are linked by the duality relation like in **K.S. 2003**)
Drawbacks with this approach:

- the \textit{generalized strategies} are not easy to characterize in practice

- the \textbf{self-financing condition} is not clear with this approach

**Self-financing condition:**

\[
V_t = \Phi^0_t B_t + \Phi_t P_t \\
\text{d}V_t = \Phi^0_t \text{d}B_t + \Phi_t \text{d}P_t
\]

or, equivalently,

\[
\text{d}\tilde{V}_t = \Phi_t \text{d}\tilde{P}_t
\]
**Problem** (with our approach): find conditions under which the optimal strategy can be characterized explicitly, the self-financing condition is satisfied, and a **Mutual Fund Theorem** can be determined.

Three steps towards a **M.F.T.**(under uniqueness of $Q$):

- $\frac{dQ}{dP}$ is measurable w.r.t to a smaller filtration $\mathcal{G}_t \subseteq \mathcal{F}_t$

- for the filtration $(\mathcal{G}_t)$ there exist a $Q$–martingale ($k$–dimensional) $(N_t)$ with the representation property

- $N_t = \int_0^t \Phi_s \, d\tilde{P}_s$ with an *explicit characterization* of $\Phi_t$ (the **mutual fund**)}
In this situation

\[
\hat{V}_T(x) = (U')^{-1} \left( y \frac{dQ}{dP} \right) = \int_0^T H_s dN_s = \int_0^T (H_s \Phi_s) d\tilde{P}_s
\]

Explicit results can be obtained if the noise is an infinite–dimensional Wiener process.
The noise:

**Ekeland–Taflin**

\[ d\tilde{P}_t = \sum_i \Gamma_i(t, \tilde{P}_t) dW_t^i \]

where \((W^i)\) is a sequence of independent Wiener processes.

**Carmona–Tehranchi**

\[ d\tilde{P}_t = \sigma(t, \tilde{P}_t) dW_t \]

where \(W\) is a cylindrical Wiener process on a separable Hilbert space \(G\) and \(\sigma(.,.,.)\) with values \(L_{HS}(G, H^2_w)\), \(H^2_w\) is a suitable weighted Sobolev space of functions.
In both papers

\[
\frac{dQ}{dP} = \exp \left( \int_0^T \lambda_s dW_s - \frac{1}{2} \int_0^T \| \lambda_s \|^2_G ds \right)
\]

Consider the continuous martingale \( M_t = \int_0^t \lambda_s dW_s \) (a time–changed Wiener process): when \( (M_t) \) has the stochastic integral representation property? (Revuz–Yor).

A sufficient condition: \( \langle M \rangle_t = \int_0^t \| \lambda_s \|^2_G ds \) is deterministic.
Ekeland–Taflin obtain an existence result and a Mutual Fund Theorem under a condition satisfied if $\|\lambda_s\|_G$ is deterministic.

This result, obtained with an accurate investigation of an infinite-dimensional H.J.B. equation, can be derived as an exercise in the framework of DD.P. results.
Carmona–Tehranchi’s point of view.

Let $F^i_w$ the weighted Sobolev space of functions $f$ defined on $[0, +\infty[$, $i$-times derivable (sense of distributions) such that $f^j( +\infty) = 0$ for $j \leq i - 1$, with norm $\|f\|^2 = \int_0^{+\infty} f^i(s)^2 w(s) \, ds$; C.T. consider $P_t$ as a process with values $F^2_w$.

On this space $\delta_T$ and $\delta'_T$ are defined (as elements of $(F^2_w)'$) and hence:

$$P(t, T) = \delta_T P_t$$

$$f(t, T) = -\frac{\partial}{\partial T} \log P(t, T) = -\frac{\delta'_T P_t}{\delta_T P_t}$$
Strategies are processes $\Phi_t$ with values in $(F^1_u)'$ where $u$ and $w$ are chosen in such a way that $F^2_w \hookrightarrow F^1_u$ and therefore $(F^1_u)' \hookrightarrow (F^2_w)'$.

**Why these conditions?** A parsimonious choice of allowed strategies gives uniqueness of strategies under suitable assumptions ($\delta'_t$ in not allowed, observe that $\delta'_t = r_t \delta_t$).

**Why Malliavin Calculus?** This family of allowed strategies is evidently restricted (only approximate completeness) but every contingent claim a time $T$ of the form

$$g(P(t,T_1),\ldots,P(t,T_n))$$

, with $T \leq T_1 < \ldots < T_n$ and $g$ lipschitz can be hedged with these strategies.
Idea of the proof:

- in the C.T. model every $P(t,T)$ has a Malliavin derivative
- therefore $H = g(P(t,T_1), \ldots, P(t,T_n))$ has a derivative
- Clark–Ocone–Karatzas formula

$$H = \mathbb{E}^Q[H] + \int_0^T \mathbb{E}^Q[D_t H | \mathcal{F}_t] \, dW_t$$
and
\[ \int_0^T \mathbb{E}^Q[D_tH|\mathcal{F}_t] \, dW_t = \int_0^T (\sigma^*(t, \tilde{P}_t)\mathbb{E}^Q[D_tH|\mathcal{F}_t]) \, d\tilde{P}_t \]

Ideas of the paper Ringer-Tehranchi:

- the model is (more or less) the C.T. model with an extra (ad hoc) hypothesis:

\[ \frac{dQ}{dP} = \exp \left( \int_0^T \lambda_s dW_s - \frac{1}{2} \int_0^T \|\lambda_s\|^2_G \, ds \right) \]

where \( \lambda_s = \lambda(s, \tilde{P}_s) = \sigma^*(s, \tilde{P}_s)\Gamma(s, \tilde{P}_s) \) with a suitable \( \Gamma(.,.,.) \).
• the **existence** of optimal solution is obtained with a method similar to the paper **Karatzas–Ocone 1991**

• the **optimal strategy** is of the form $\Phi^1_t + \Phi^2_t + \Phi^3_t$

where $\Phi^3_t$ is essentially the component with respect to the *riskless asset* (money–market account), $\Phi^2_t = 0$ if $\lambda_t$ is deterministic and $\Phi^1_t$ is proportional to a fixed *universal* strategy (the **M.F.**).
Where the terms $\Phi_1^t$ and $\Phi_2^t$ come from?

- the value of the optimal portfolio is
  \[
  \left( U' \right)^{-1} \left( y \exp \left( \int_0^T \lambda_s dW_s - \frac{1}{2} \int_0^T \|\lambda_s\|^2 ds \right) \right)
  \]
- chain rule for Malliavin derivative
- Clark–Ocone–Karatzas formula
Moreover, if $\lambda_t$ is deterministic:

$$D_t\left(\int_0^T \lambda_s dW_s\right) = \lambda_t$$

If $\lambda_t$ is stochastic (adapted):

$$D_t\left(\int_0^T \lambda_s dW_s\right) = \lambda_t + \int_t^T D_t \lambda_s \, dW_s$$