Control of Some Stochastic Systems with a Fractional Brownian Motion

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Outline

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  - Definition and some properties
- Fractional calculus (Riemann-Liouville)
  - Relation to FBM
  - Some linear operators
- Optimal control
  - Controlled stochastic equations with a FBM
  - Absolute continuity of measures
  - Weak solutions: existence and uniqueness
  - Randon-Nikodym derivatives and properties
  - Existence of an optimal control
Some Development of Fractional Brownian Motion

- I. J. Schonberg 1937
- A. N. Kolmogorov 1940
- H. E. Hurst 1951
- B. Mandelbrot 1965
Some Applications

- Hydrology
- Economic data
- Telecommunications
- Device noise
- Medicine
Fractional Brownian motion (FBM)

A standard FBM \((B^H(t), t \geq 0)\) is a Gaussian process with continuous sample paths such that

\[
\mathbb{E}[B^H(t)] = 0
\]

\[
\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}[|t|^{2H} + |s|^{2H} - |t - s|^{2H}]
\]

\(H \in (0, 1)\) is called the Hurst parameter

\(H = \frac{1}{2} \implies\) standard Brownian motion
Some Properties of FBM

1. Self-similarity

\[ (B^H(\alpha t), t \geq 0) \overset{L}{\sim} (\alpha^H B^H(t), t \geq 0) \]

for \( \alpha > 0 \)

2. Long range dependence for \( H \in (\frac{1}{2}, 1) \)

\[
  r(n) = \mathbb{E}[B^H(1)(B^H(n + 1) - B^H(n))] \\
  \sum_{n=0}^{\infty} r(n) = \infty
\]
3. A sample path property

\((B^H(t), t \geq 0)\) is of unbounded variation so the sample paths are not differentiable a.s.

\[
\sum_i |B^H(t^{(n)}_{i+1}) - B^H(t^{(n)}_i)|^p \rightarrow \begin{cases} 
0 & pH > 1 \\
c(p) & pH = 1 \\
+\infty & pH < 1
\end{cases}
\]

where \(c(p) = E|B^H(1)|^p\) and \((t^{(n)}_i, i = 0, 1, \ldots, n)\) is a sequence of nested partitions of \([0, 1]\)
Remark

- $p = 2$ and $H \in (\frac{1}{2}, 1) \implies$ zero quadratic variation
- $p = 2$ and $H \in (0, \frac{1}{2}) \implies$ infinite quadratic variation
- FBM is not a semimartingale for $H \neq \frac{1}{2}$
Fractional Calculus

Let \((\mathcal{V}, \| \cdot \|, \langle \cdot, \cdot \rangle)\) be a separable Hilbert space and let \(\alpha \in (0, 1)\). If \(\varphi \in L^1([0, T], \mathcal{V})\) then the left-sided and the right-sided fractional (Riemann-Liouville) integrals of \(\varphi\) are defined (for almost all \(t \in [0, T]\)) by

\[
(I_{0+}^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \varphi(s) \, ds
\]

and

\[
(I_{T-}^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s - t)^{\alpha-1} \varphi(s) \, ds
\]

respectively, where \(\Gamma(\cdot)\) is the gamma function.
The inverse operators of these fractional integrals are called fractional derivatives and can be given by their respective Weyl representations

\[
(D_{0+}^\alpha \psi)(t) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{\psi(t)}{t^\alpha} + \alpha \int_0^t \frac{\psi(t) - \psi(s)}{(t - s)^{\alpha+1}} \, ds \right)
\]

and

\[
(D_{T-}^\alpha \psi)(t) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{\psi(t)}{(T - t)^\alpha} + \alpha \int_t^T \frac{\psi(s) - \psi(t)}{(s - t)^{\alpha+1}} \, ds \right)
\]

where \( \psi \in I_{0+}^\alpha (L^1([0, T], V)) \) and \( \psi \in I_{T-}^\alpha (L^1([0, T], V)) \) respectively.
An Elementary Itô Formula for $H \in \left( \frac{1}{2}, 1 \right)$

If $f \in C^2$, then

$$f(B^H(T)) - f(B^H(0)) = \int_0^T f'(B^H(s))dB^H(s)$$

$$+ H \int_0^T s^{2H-1}f''(B^H(s))ds$$

Remark

- Formally letting $H = \frac{1}{2}$ recovers the usual Itô formula.
- Recall that an FBM with $H \in \left( \frac{1}{2}, 1 \right)$ has zero quadratic variation.
Let $K_H(t, s)$ for $0 \leq s \leq t \leq T$ be the real-valued kernel function

$$K_H(t, s) = c_H (t - s)^{H - \frac{1}{2}}$$

$$+ c_H \left( \frac{1}{2} - H \right) \int_s^t (u - s)^{H - \frac{3}{2}} \left( 1 - \left( \frac{s}{u} \right)^{\frac{1}{2} - H} \right) du$$

where

$$c_H = \left[ \frac{2H\Gamma(H + \frac{1}{2})\Gamma(\frac{3}{2} - H)}{\Gamma(2 - 2H)} \right]^{\frac{1}{2}}$$

and $H \in (0, 1)$. If $H \in (1/2, 1)$, then $K_H$ has a simpler form as

$$K_H(t, s) = c_H \left( H - \frac{1}{2} \right) s^{\frac{1}{2} - H} \int_s^t (u - s)^{H - \frac{3}{2}} u^{H - \frac{1}{2}} du .$$
Define the integral operator $\mathbb{K}_H$ induced from the kernel $K_H$ by

$$\mathbb{K}_H \varphi(t) = \int_0^t K_H(t, s)h(s) \, ds$$

for $h \in L^2([0, T], V)$. It is well-known that

$$\mathbb{K}_H : L^2([0, T], V) \to \mathbb{I}_{0+}^{H+\frac{1}{2}} (L^2([0, T], V))$$

is a bijection.
$K_H$ can be described as

$$K_H h(s) = \tilde{c}_H \int_0^{2H} \left( u_{1/2-H}^{1/2-H} \left( u_{H-1/2} \right) \right)(s)$$

for $H \in (0, 1/2]$ and

$$K_H h(s) = c_H \int_0^1 \left( u_{H-1/2}^{H-1/2} \left( u_{1/2-H} \right) \right)(s)$$

for $H \in [1/2, 1)$ where $u_a(s) = s^a$ for $s \geq 0$ and $a \in \mathbb{R}$. 
The inverse operator, $\mathbb{K}_H^{-1}$ for the two cases is given by

$$\mathbb{K}_H^{-1} \varphi(s) = \overline{c}_H^{-1} s^{\frac{1}{2} - H} D_{0+}^{\frac{1}{2} - H} \left( u_{H - \frac{1}{2}} D_{0+}^{2H} \varphi \right)(s)$$

for $H \in (0, 1/2]$ and

$$\mathbb{K}_H^{-1} \varphi(s) = c_H^{-1} s^{H - \frac{1}{2}} D_{0+}^{H - \frac{1}{2}} \left( u_{\frac{1}{2} - H} D_{0+}^{2H} \varphi \right)(s)$$

for $H \in [1/2, 1)$ and $\varphi \in I_{0+}^{H + \frac{1}{2}} (L^2([0, T], V))$.

If $\varphi(s) \in H^1([0, T], V)$ and $H \in (0, 1/2)$ then

$$\mathbb{K}_H^{-1} \varphi(s) = \overline{c}_H^{-1} s^{H - \frac{1}{2}} I_{0+}^{\frac{1}{2} - H} \left( u_{H - \frac{1}{2}} \varphi' \right)(s)$$
\( K_H \) and the Covariance of a FBM

\[
\int_0^{s \wedge t} K_H(t, r)K_H(s, r)dr = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H})
\]

\[
B(t) = \int_0^t K_H(t, s)dW(s)
\]

\( W \) is a Wiener process
Theorem 1

Let $H \in (0, 1)$ and $T > 0$ be fixed and let $(u(t), t \in [0, T])$ be a $V$-valued, $(\mathcal{F}_t)$-adapted process such that

$$\int_0^T \| u(t) \| \, dt < \infty \quad \text{a.s. } \mathbb{P}$$

and

$$U(t) := \int_0^t u(s) \, ds \in l_0^{H+\frac{1}{2}} (L^2 ([0, T], V)) \quad \text{a.s. } \mathbb{P}.$$
Furthermore, it is assumed that

$$E\xi(T) = 1$$

where

$$\xi(T) = \exp \left[ \int_0^T \left( \langle K_H^{-1}(U)(t), dW(t) \rangle - \frac{1}{2} \int_0^T \| K_H^{-1}(U)(t) \|^2 dt \right) \right]$$

where \((W(t), t \in [0, T])\) is a standard cylindrical Wiener process in \(V\).
Then the process \( \left( \tilde{B}(t), t \in [0, T] \right) \) given by

\[
\tilde{B}(t) := B(t) - U(t)
\]

is a standard cylindrical fractional Brownian motion in \( V \) with the Hurst parameter \( H \) on the probability space \( (\Omega, \mathcal{F}, \tilde{P}) \) where

\[
\frac{d\tilde{P}}{dP} = \xi(T) \quad \text{a.s.}
\]
Definition

A weak solution is a triple $(X(t), B(t), (\Omega, \mathcal{F}, \mathbb{P}), t \geq 0)$ where $(B(t), t \geq 0)$ is a standard cylindrical fractional Brownian motion in $V$ that is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(X(t), t \geq 0)$ satisfies the stochastic differential equation.
Definition

The equation has a unique weak solution if for any two weak solutions, \((X(t), B(t), (\Omega, \mathcal{F}, \mathbb{P}), t \geq 0)\), and \((\tilde{X}(t), \tilde{B}(t), (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), t \geq 0)\), the processes \((X(t), t \geq 0)\) and \((\tilde{X}(t), t \geq 0)\) have the same probability law.
\[ dX(t) = f(X(t))dt + \Phi dB^H(t) \]
\[ X(0) = x \]

where \( f : \mathbb{R}^n \to \mathbb{R}^n \), \( \Phi \in L(\mathbb{R}^n) \), \( \Phi \Phi^* > 0 \).

\( (B^H(t), t \geq 0) \) is a \( \mathbb{R}^n \)-valued standard FBM. \( E = V = \mathbb{R}^n \).

\( S(t) = I \) for \( t \in \mathbb{R}_+ \).

\( (\int_0^t \Phi dB^H, t \in [0, T]) \) has \( C^\beta([0, T], \mathbb{R}^n) \) sample paths for \( \beta \in (0, H) \).
Let $H \in (0, \frac{1}{2})$.

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is Borel measurable and

$$
\|f(x)\| \leq k_1(1 + \|x\|)
$$

for all $x \in \mathbb{R}^n$, then there is one and only one weak solution.

Let $H \in (\frac{1}{2}, 1)$.

Assume additionally that

$$
\|f(x) - f(y)\| \leq k\|x - y\|^\gamma
$$

for all $x, y \in \mathbb{R}^n$ and some $\gamma > 1 - \frac{1}{2H}$, then there is one and only one weak solution.
\[
\left| K_H^{-1} \left( \int_0^t G(Z) \right) \right|_{L^2([0,T])}^2 \\
\leq c_T \left( 1 + \|X\|^2 + \|\tilde{Z}\|^2_{C([0,T])} + |\tilde{Z}|^2_{C^\beta([0,T])} \right)
\]
\[
\tilde{Z}(t) = \int_0^t f(s, X(s)) ds
\]
Controlled Stochastic Equation

\[ dX(t) = f(t, X(\cdot), u(t, X(\cdot)))dt + dB^H(t) \]
\[ X(0) = X_0 \]  

where \( X_0 \in \mathbb{R}^n \) is a constant, \( X(t) \in \mathbb{R}^n \), \( t \in [0, T] \), \( T > 0 \) is fixed, \( (B^H(t), t \in [0, T]) \) is an \( \mathbb{R}^n \)-valued FBM with \( H \in (0, \frac{1}{2}) \), \( u(t) \in U \subset \mathbb{R}^m \) (compact), \( f(t, z, U) \) is closed and convex for each \( t \in [0, T] \) and \( z \in C([0, T]) \).
Let $S(T)$ be the Borel $\sigma$-algebra on $C[0, T] = C$ with the topology of uniform convergence. The following conditions are used subsequently.

C1. The function $f : [0, T] \times C \times U \rightarrow \mathbb{R}^n$ is jointly measurable;

C2. For each $t \in [0, T]$, the function $f(t, \cdot, \cdot) : C([0, t]) \times U \rightarrow \mathbb{R}^m$ is $S(t) \otimes B(U)$-measurable;

C3. The function $f(t, z, \cdot) : U \rightarrow \mathbb{R}$ is continuous for each $t \in [0, T]$ and $z \in C([0, t])$;
C4. The function $g : [0, T] \times C([0, T])$ given by
$g(t, z) = f(t, z, u(t, z))$ satisfies

$$|g(t, z)| \leq k(1 + \|z\|)$$

for $k > 0$ and each $z \in C([0, T])$ where $\| \cdot \|$ is the uniform norm. An admissible control is an $S(T)$-measurable map such that for each $t \in [0, T], u(t, \cdot)$ is $S(t)$-measurable. The family of admissible controls is denoted by $\mathcal{U}$. 
Proposition 2

Let $H \in (0, \frac{1}{2})$ be fixed. Let $g_u : [0, T] \times \mathbb{C}$ be the function given by

$$g_u(t, z) = f(t, z, u(t, z)).$$

for $u \in \mathcal{U}$. If C1, C2 and C4 are satisfied then

$$\mathbb{E}[\rho(g_u)] = 1$$

where

$$\rho(g_u) = \exp\left( \int_0^T K_H^{-1}(\int_0^t g_u(s))dW(s) \right)$$

$$- \frac{1}{2} \int_0^T \left| K_H^{-1}(\int_0^t g_u(s)) \right|^2 dt.$$
Proposition 3

Let \( u \in \mathcal{U} \) and let \( g_u \) be the associated drift term given above and \( H \in (0, \frac{1}{2}) \).

Then

\[
dX(t) = g_u(t, X(\cdot))dt + dB^H(t)
\]

\[
= f(t, X(\cdot), u(t, X(\cdot)))dt + dB^H(t)
\]  \hspace{1cm} (2)

\[ X(0) = X_0 \]

has one and only one weak solution. The solution can be obtained from \((B^H(t), t \in [0, T])\) by a transformation of its measure by the Radon-Nikodym derivative \( \rho(g_u) \) given above.
Proposition 4

Let C1-C4 be satisfied, $G = \{g_u(\cdot, \cdot) | g_u(t, z) = f(t, z, u(t, z)) \text{ for } f \text{ in (2) and } u \in \mathcal{U}\}$, and $\mathcal{D}(G) = \{\rho(g_u) | g_u \in G\}$. Then $\mathcal{D}(G)$ is a closed, convex, uniformly integrable subset of $L^1(\mathbb{P})$. 
Let $L : \mathcal{C} \to \mathbb{R}$ be a bounded, continuous function. For $u \in \mathcal{U}$, let $J(u)$ be given by

$$J(u) = \int_{\mathcal{C}} L(z) \rho(g_u) d\mu_0(z)$$

(3)

where $\mu_0$ is the measure on $(\mathcal{C}, \mathcal{S}(T))$ for the standard fractional Brownian motion $(B^H(t), t \in [0, T])$. 
Theorem 5

Consider the stochastic control problem given by (1) and (3). If $J^*$ is given by

$$J^* = \inf_{u \in \mathcal{U}} J(u),$$

then there is a control $u^* \in \mathcal{U}$ such that

$$J^* = J(u^*).$$