Classical and Quantum Systems Interacting with their Environment

Anthony M. Bloch

Work with Brockett, Ratiu, Marsden, Krishnaprasad, Zenkov, Hagerty, Weinstein, Rojo...

- **The Toda lattice and double bracket dissipation**
- **Dissipation in nonholonomic systems**
- **Dissipation and Squeezing of Phonons**
- **Spin Squeezing and Control**
- **Nonholonomic Systems and fields**

•Background

• Basic observation about Hamiltonian systems: satisfy Liouville's theorem, preserving volume in phase space, thus cannot exhibit asymptotic stability.

Reflection of this: spectrum of linearization about a fixed point symmetric about imaginary axis.

• However: Hamiltonian systems can exhibit asymptotic stability with respect to some of their variables.

Special case: system is integrable, the flow restricts to a level set of the integrals and the flow on this level set exhibits asymptotic stability.

See Toda: Moser [1974], Bloch, Brockett and Ratiu [1990, 92].

• Another class of systems exhibiting asymptotic stability behavior: nonholonomic systems – systems with nonintegrable constraints. In the absence of external dissipative forces, are always energy preserving.

Do not necessarily preserve volume in the phase space – – see for example Zenkov, Bloch and Marsden [1998], Zenkov and Bloch [2002], Kozlov, Jovanovich,

• Infinite Dimensions – oscillators interacting with fields. Hagerty, Bloch and Weinstein. Bloch, Hagerty, Rojo and Weinstein. Radiation Damping. Sofer and Weinstein.

•The Toda Lattice
Interacting particles on the line.
Non-periodic finite Toda lattice as analyzed by Moser [1974]:

$$H(x, y) = \frac{1}{2} \sum_{k=1}^{n} y_k^2 + \sum_{k-1}^{n-1} e^{(x_k - x_{k+1})} \, .$$

Hamiltonian equations:

$$\begin{aligned} \dot{x}_k &= \frac{\partial H}{\partial y_k} = y_k \\ \dot{y}_k &= -\frac{\partial H}{\partial x_k} = e^{x_{k-1} - x_k} - e^{x_k - x_{k-1}} \, , \end{aligned}$$

where assume $e^{x_0 - x_1} = e^{x_n - x_{n+1}} = 0$**.**
Flaschka:

$$a_k = \frac{1}{2} e^{(x_k - x_{k+1})/2} \quad b_k = -\frac{1}{2} y_k \, .$$

Get:

$$\begin{aligned} \dot{a}_k &= a_k(b_{k+1} - b_k)\,, \qquad k = 1, \cdots, n-1 \\ \dot{b}_k &= 2(a_k^2 - a_{k-1}^2)\,, \qquad k = 1, \cdots, n \end{aligned}$$

with the boundary conditions $a_0 = a_n = 0$ and where the $a_i > 0$.

Matrix form:

$$\frac{d}{dt}L = [B, L] = BL - LB,$$

where

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ & & \ddots & & \\ & & & b_{n-1} & a_{n-1} \\ 0 & & & a_{n-1} & b_n \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & 0 \\ & & \ddots & & \\ & & & 0 & a_{n-1} \\ 0 & & & -a_{n-1} & 0 \end{pmatrix}$$

Poisson matrix:

$$J = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}$$

where A **has entries** $a_{ii} = a_i$**,** $a_{ii+1} = -a_i$ **and all other entries are zero.** L **is assumed to be traceless.**

The flow is then given by

$$\dot{q} = J \mathrm{grad} H$$

where $q = [b_1, \cdots, b_{n-1}, a_1, \cdots, a_{n-1}]$ **and** $H = 1/2 \mathrm{Tr} L^2$**.**

• Eigenvalues of L **preserved along the flow. Enough to show system is integrable.**

Another basis for integrals: $1/2 \mathrm{Tr} L^k$**.**

If N is the matrix $\mathrm{diag}[1, 2, \cdots, n]$ the Toda flow can be written

$$\dot{L} = [L, [L, N]] \,.$$

Shows flow also gradient (on a level set of its integrals).

• Double bracket form of Brockett [1988] (see Bloch [1990], Bloch Brockett and Ratiu [1990, 1992]).

Gradient flow of the function $\mathrm{Tr}LN$ with respect to the normal metric.

• Explicit solution:

Let initial data be given by $L(0) = L_0$. Factorize a symmetric matrix L as $L = k(L)u(L)$ where $k(L)$ is orthogonal and $u(L)$ is upper triangular.

Toda flow given by

$$L(t) = (\mathrm{Ad}k(\exp(tL_0))L_0 \,.$$

•Double Brackets and Dissipation Double bracket flows: dissipative mechanism in otherwise energy conserving mechanical systems, Bloch, Krishnaprasad, Marsden and Ratiu [1996].

- **Simple example: rigid body equations:**

$$I\dot{\Omega} = (I\Omega) \times \Omega,$$

or, in terms of the body angular momentum $M = I\Omega$**,**

$$\dot{M} = M \times \Omega.$$

Energy equals the Lagrangian: $E(\Omega) = L(\Omega)$ **and energy is conserved.**

Add a term cubic in the angular velocity:

$$\dot{M} = M \times \Omega + \alpha M \times (M \times \Omega),$$

where α **is a positive constant.**

• Related example is the Landau-Lifschitz equations for the magnetization vector M in a given magnetic field B:

$$\dot{M} = \gamma M \times B + \frac{\lambda}{\|M\|^2}(M \times (M \times B)),$$

where γ is the magneto-mechanical ratio (so that $\gamma\|B\|$ is the Larmour frequency) and λ is the damping coefficient due to domain walls.

• The equations are Hamiltonian with the rigid body Poisson bracket:

$$\{F, K\}_{\mathrm{rb}}(M) = -M \cdot [\nabla F(M) \times \nabla K(M)]$$

with Hamiltonians given respectively by $H(M) = (M \cdot \Omega)/2$ and $H(M) = \gamma M \cdot B$.

Dissipation in these systems is not induced by *any* Rayleigh dissipation function in the *literal* sense

However, it is induced by a dissipation function in the following restricted sense: It is a gradient when restricted to each momentum sphere,

Have:

$$\frac{d}{dt}\|M\|^2 = 0$$

$$\frac{d}{dt}E = -\alpha\|M \times \Omega\|^2,$$

for the rigid body,

- **Interesting feature of these dissipation terms is that they can be derived from a symmetric bracket. in much the same way that the Hamiltonian equations can be derived from a skew symmetric Poisson bracket. For the case of the rigid body, this bracket is**

$$\{\{F, K\}\} = \alpha(M \times \nabla F) \cdot (M \times \nabla K).$$

(For more on symmetric brackets see Crouch [1981] and Lewis and Murray [1999].)

•The Two-dimensional Toda Lattice

In two-dimensional case matrices in the Lax pair are

$$L = \begin{pmatrix} b_1 & a_1 \\ a_1 & -b_1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}.$$

Equations of motion:

$$\begin{aligned} \dot{b_1} &= 2a_1^2 \\ \dot{a_1} &= -2a_1 b_1 \end{aligned}$$

For initial data $b_1 = 0$, $a_1 = c$, explicitly carrying out the factorization yields explicit solution

$$b_1(t) = -c\frac{\sinh 2ct}{\cosh 2ct}, \qquad a_1(t) = \frac{c}{\cosh 2ct}$$

•The Chaplygin Sleigh

Here we describe the Chaplygin sleigh, perhaps the simplest mechanical system which illustrates the possible dissipative nature of energy preserving nonholonomic systems.

Nonholonomic: subject to nonintegrable constraints – satisifes Lagrange D'Alembert equations.

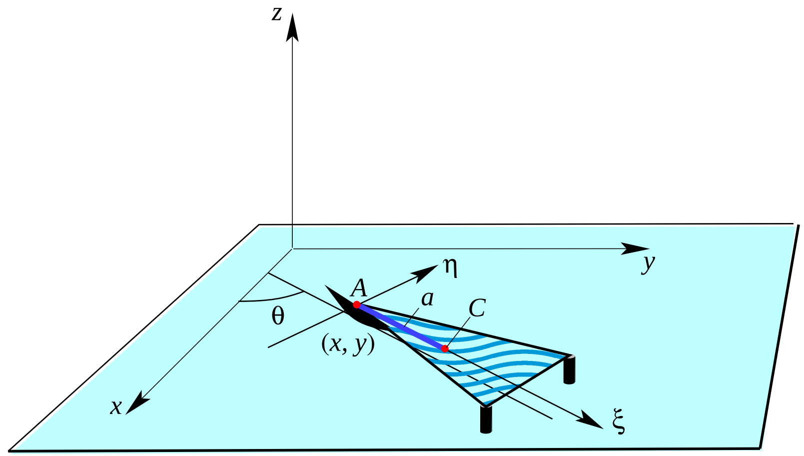

Figure 0.1: The Chaplygin sleigh is a rigid body moving on two sliding posts and one knife edge.

Equations:

$$\begin{aligned}\dot{v} &= a\omega^2 \\ \dot{\omega} &= -\frac{ma^2}{I+ma^2}v\omega\end{aligned}$$

Equations have a family of relative equilibria given by $(v,\omega)|v = \mathrm{const}, \quad \omega = 0$**.**

Linearizing about any of these equilibria one finds one zero eigenvalue and one negative eigenvalue.

In fact the solution curves are ellipses in $v - \omega$ plane with the positive v-axis attracting all solutions.

Normalizing, we have the equations

$$\begin{aligned} \dot{v} &= \omega^2 \\ \dot{\omega} &= -v\omega \,. \end{aligned}$$

Scaling time by a factor of two have: Chaplygin sleigh equations are equivalent to the two-dimensional Toda lattice equations except for the fact that there is no sign restriction on the variable ω. Hence can be written in Lax pair form and solved by the method of factorization.

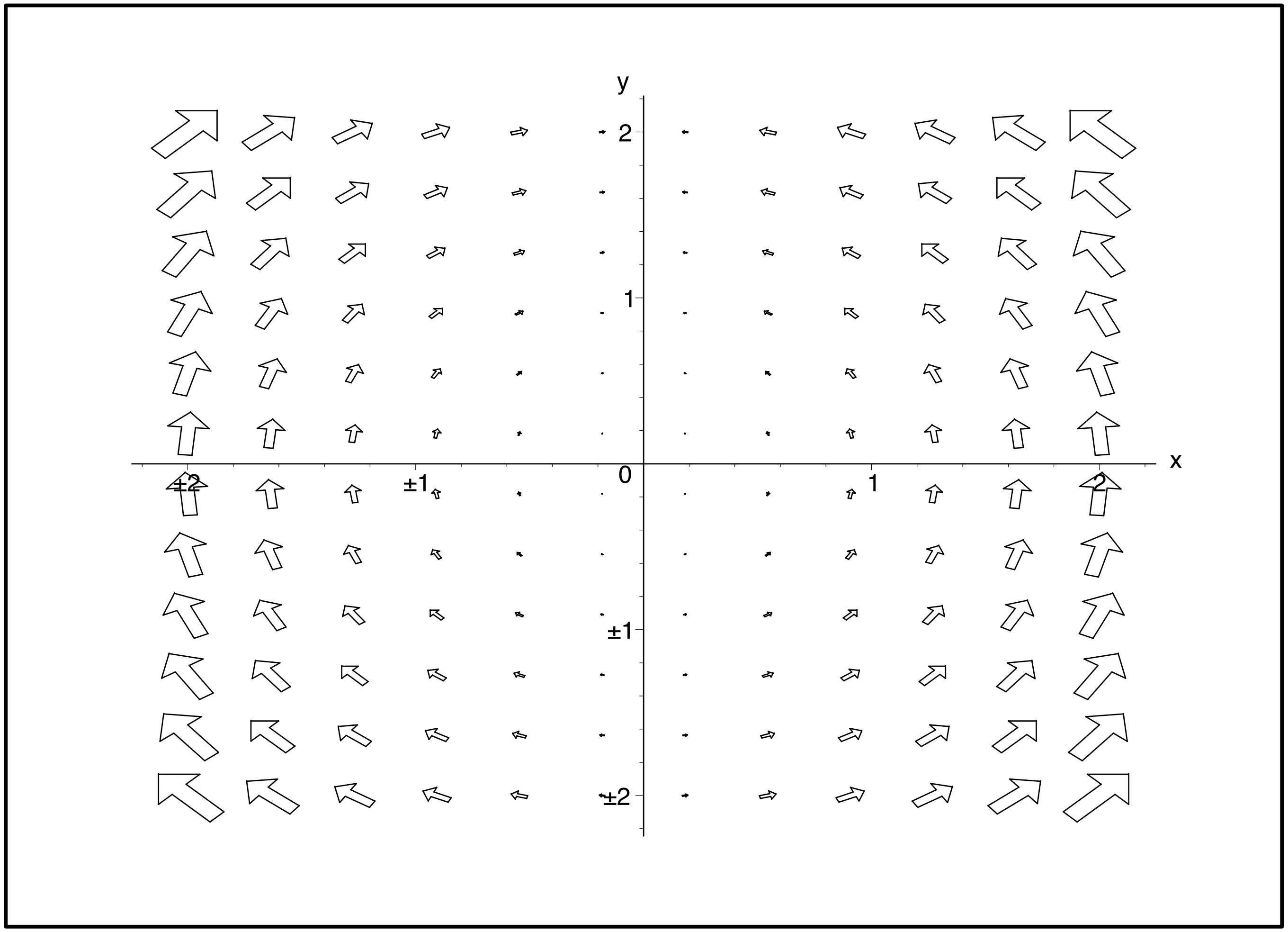

Figure 0.2: Chaplygin Sleigh/2d Toda phase portrait.

•Almost Poisson Systems

Recall:

Definition 0.1 ***An almost Poisson manifold is a pair*** $(M, \{,\})$ ***where*** M ***is a smooth manifold and (i)***$\{,\}$ ***defines an almost Lie algebra structure on the*** C^∞ ***functions on*** M***, i.e. the bracket satisfies all conditions for a Lie algebra except that the Jacobi identity is not satisfied and (ii)*** $\{,\}$ ***is a derivation in each factor.***

If in addition Jacobi satisfied, Poisson manifold.

An almost Poisson structure on M **will be Poisson if its** *Jacobiator*, **defined by**

$$J(f,g,h) = \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\}g\}$$

vanishes.

• "Hamiltonian" Formulation of Nonholonomic Systems

Nonholonomic systems are almost Poisson.

Start on the Lagrangian side with a configuration space Q and a Lagrangian L (possibly of the form kinetic energy minus potential energy, *i.e.*,

$$L(q, \dot{q}) = \frac{1}{2}\langle\langle \dot{q}, \dot{q} \rangle\rangle - V(q),$$

As above, our nonholonomic constraints are given by a distribution $\mathcal{D} \subset TQ$. We also let $\mathcal{D}^0 \subset T^*Q$ denote the annihilator of this distribution. Using a basis ω^a of the annihilator $\mathcal{D}^{\mathrm{o}}$, we can write the constraints as

$$\omega^a(\dot{q}) = 0,$$

where $a = 1, \ldots, k$.

Recall that the cotangent bundle T^*Q is equipped with a canonical Poisson bracket and is expressed in the canonical coordinates (q, p) as

$$\{F, G\}(q, p) = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} = \left(\frac{\partial F^T}{\partial q}, \frac{\partial F^T}{\partial p}\right) J \begin{pmatrix} \frac{\partial G}{\partial q} \\ \frac{\partial G}{\partial p} \end{pmatrix}.$$

Here J is the canonical Poisson tensor

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

A constrained phase space $\mathcal{M} = \mathbb{F}L(\mathcal{D}) \subset T^*Q$ is defined so that the constraints on the Hamiltonian side are given by $p \in \mathcal{M}$. In local coordinates,

$$\mathcal{M} = \left\{ (q, p) \in T^*Q \;\middle|\; \omega_i^a \frac{\partial H}{\partial p_i} = 0 \right\}.$$

Let $\{X_\alpha\}$ be a local basis for the constraint distribution $\mathcal{D}$ and let $\{\omega^a\}$ be a local basis for the annihilator $\mathcal{D}^0$. Let $\{\omega_a\}$ span the complementary subspace to $\mathcal{D}$ such that $\langle \omega^a, \omega_b \rangle = \delta_b^a$, where δ_b^a is the usual Kronecker delta. Here $a = 1, \dots, k$ and $\alpha = 1, \dots, n-k$. Define a coordinate transformation $(q, p) \mapsto (q, \tilde{p}_\alpha, \tilde{p}_a)$ by

$$\tilde{p}_\alpha = X_\alpha^i p_i, \qquad \tilde{p}_a = \omega_a^i p_i.$$

In the new (generally not canonical) coordinates $(q, \tilde{p}_\alpha, \tilde{p}_a)$, the Poisson tensor becomes

$$\tilde{J}(q, \tilde{p}) = \begin{pmatrix} \{q^i, q^j\} & \{q^i, \tilde{p}_j\} \\ \{\tilde{p}_i, q^j\} & \{\tilde{p}_i, \tilde{p}_j\} \end{pmatrix}.$$

Use $(q, \tilde{p}_\alpha)$ as induced local coordinates for $\mathcal{M}$. It is easy to show that

$$\frac{\partial \tilde{H}}{\partial q^j}(q, \tilde{p}_\alpha, \tilde{p}_a) = \frac{\partial H_{\mathcal{M}}}{\partial q^j}(q, \tilde{p}_\alpha),$$
$$\frac{\partial \tilde{H}}{\partial \tilde{p}_\beta}(q, \tilde{p}_\alpha, \tilde{p}_a) = \frac{\partial H_{\mathcal{M}}}{\partial \tilde{p}_\beta}(q, \tilde{p}_\alpha),$$

where $H_{\mathcal{M}}$ is the constrained Hamiltonian on $\mathcal{M}$ expressed in the induced coordinates. We can also truncate the Poisson tensor $\tilde{J}$ by leaving out its last k columns and last k rows and then describe the constrained dynamics on $\mathcal{M}$ expressed in the induced coordinates $(q^i, \tilde{p}_\alpha)$ as follows:

$$\begin{pmatrix} \dot{q}^i \\ \dot{\tilde{p}}_\alpha \end{pmatrix} = J_{\mathcal{M}}(q, \tilde{p}_\alpha) \begin{pmatrix} \frac{\partial H_{\mathcal{M}}}{\partial q^j}(q, \tilde{p}_\alpha) \\ \frac{\partial H_{\mathcal{M}}}{\partial \tilde{p}_\beta}(q, \tilde{p}_\alpha) \end{pmatrix}, \qquad \begin{pmatrix} q^i \\ \tilde{p}_\alpha \end{pmatrix} \in \mathcal{M}.$$

Here $J_{\mathcal{M}}$ is the $(2n-k) \times (2n-k)$ truncated matrix of $\tilde{J}$ restricted to $\mathcal{M}$ and is expressed in the induced coordinates.

The matrix $J_{\mathcal{M}}$ defines a bracket $\{\cdot\,,\cdot\}_{\mathcal{M}}$ on the constraint submanifold $\mathcal{M}$ as follows:

$$\{F_{\mathcal{M}}, G_{\mathcal{M}}\}_{\mathcal{M}}(q, \tilde{p}_\alpha) := \begin{pmatrix} \frac{\partial F_{\mathcal{M}}^T}{\partial q^i} & \frac{\partial F_{\mathcal{M}}^T}{\partial \tilde{p}_\alpha} \end{pmatrix} J_{\mathcal{M}}(q^i, \tilde{p}_\alpha) \begin{pmatrix} \frac{\partial G_{\mathcal{M}}}{\partial q^j} \\ \frac{\partial G_{\mathcal{M}}}{\partial \tilde{p}_\beta} \end{pmatrix},$$

for any two smooth functions $F_{\mathcal{M}}, G_{\mathcal{M}}$ on the constraint submanifold $\mathcal{M}$. Clearly, this bracket satisfies the first two defining properties of a Poisson bracket, namely, skew symmetry and the Leibniz rule, and one can show that it satisfies the Jacobi identity if and only if the constraints are holonomic. Furthermore, the constrained Hamiltonian $H_{\mathcal{M}}$ is an integral of motion for the constrained dynamics on $\mathcal{M}$ due to the skew symmetry of the bracket.

Following e.g. van der Schaft and Maschke [1994] and Koon and Marsden [1997] we can write the nonholonomic equations of motion as follows:

$$\begin{pmatrix} \dot{s}^a \\ \dot{r}^\alpha \\ \dot{\tilde{p}}_\alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 & -A^a_\beta \\ 0 & 0 & \delta^\alpha_\beta \\ (A^b_\alpha)^T & -\delta^\beta_\alpha & -p_c B^c_{\alpha\beta} \end{pmatrix} \begin{pmatrix} \frac{\partial H_{\mathcal{M}}}{\partial s^b} \\ \frac{\partial H_{\mathcal{M}}}{\partial r^\beta} \\ \frac{\partial H_{\mathcal{M}}}{\partial \tilde{p}_\beta} \end{pmatrix}$$

Jacobiator of the Poisson tensor vanishes precisely when the curvature of the nonholonomic constraint distribution is zero or the constraints are holonomic.

•Euler-Poincaré-Suslov Equations

Important special case of the reduced nonholonomic equations.

•Example: Euler-Poincaré-Suslov Problem on $SO(3)$ **In this case the problem can be formulated as the standard Euler equations**

$$I\dot{\omega} = I\omega \times \omega$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ **are the system angular velocities in a frame where the inertia matrix is of the form** $I = \mathrm{diag}(I_1, I_2, I_3)$ **and the system is subject to the constraint**

$$a \cdot \omega = 0$$

where $a = (a_1, a_2, a_3)$**.**

The nonholonomic equations of motion are then given by

$$I\dot{\omega} = I\omega \times \omega + \lambda a$$

subject to the constraint. Solve for λ:

$$\lambda = -\frac{I^{-1}a \cdot (I\omega \times \omega)}{I^{-1}a \cdot a}\,.$$

If a is an eigenvector of the moment of inertia tensor flow is measure preserving.

•Radiation Damping

See Hagerty, Bloch and Weinstein [1999], [2002].

Important early work: Lamb [1900]. Related recent work may be found in Soffer and Weinstein [1998a,b] [1999] and Kirr and Weinstein [2001].

• Original Lamb model an oscillator is physically coupled to a string. The vibrations of the oscillator transmit waves into the string and are carried off to infinity. Hence the oscillator loses energy and is effectively damped by the string.

• Lamb model

$w(x,t)$ **displacement of the string. with mass density** ρ, **tension** T. **Assuming a singular mass density at** $x = 0$, **we couple dynamics of an oscillator,** q, **of mass** M**:**

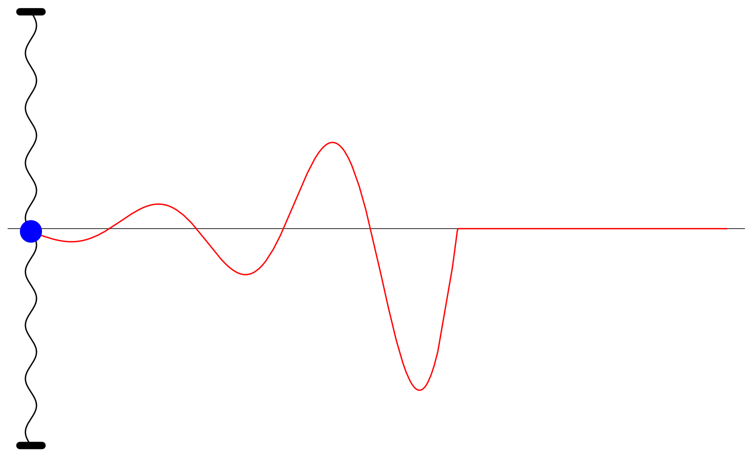

Figure 0.3: **Lamb model of an oscillator coupled to a string.**

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$
$$M\ddot{q} + Vq = T[w_x]_{x=0}$$
$$q(t) = w(0,t).$$

$[w_x]_{x=0} = w_x(0+,t) - w_x(0-,t)$ **is the jump discontinuity of the slope of the string. Note that this is a Hamiltonian system.**

Can solve for w **and reduce:**

• **Obtain a reduced form of the dynamics describing the explicit motion of the oscillator subsystem,**

$$M\ddot{q} + \frac{2T}{c}\dot{q} + Vq = 0.$$

The coupling term arises explicitly as a Rayleigh dissipation term $\frac{2T}{c}\dot{q}$ in the dynamics of the oscillator.

Gyroscopic systems:
See Bloch, Krishnaprasad, Marsden and Ratiu [1994].
Linear systems of the form

$$M\ddot{q} + S\dot{q} + \Lambda q = 0$$

where $q \in \mathbb{R}^n$, M **is a positive definite symmetric** $n \times n$ **matrix,** S **is skew, and** Λ **is symmetric.**

This system Hamiltonian with $p = M\dot{q}$**, energy function**

$$H(q,p) = \frac{1}{2}pM^{-1}p + \frac{1}{2}q\Lambda q$$

and the bracket

$$\{F, K\} = \frac{\partial F}{\partial q^i}\frac{\partial K}{\partial p_i} - \frac{\partial K}{\partial q^i}\frac{\partial F}{\partial p_i} - S_{ij}\frac{\partial F}{\partial p_i}\frac{\partial K}{\partial p_j}.$$

Systems of this form arise from simple mechanical systems via reduction; normal form of the linearized equations when one has an *abelian* **group.**

Theorem 0.2 Dissipation induced instabilities—abelian case ***Under the above conditions, if we modify the equation to***

$$M\ddot{q} + (S + \epsilon R)\dot{q} + \Lambda q = 0$$

for small $\epsilon > 0$***, where*** R ***is symmetric and positive definite, then the perturbed linearized equations***

$$\dot{z} = L_\epsilon z,$$

where $z = (q, p)$ ***are spectrally unstable,*** *i.e.,* ***at least one pair of eigenvalues of*** L_ϵ ***is in the right half plane.***

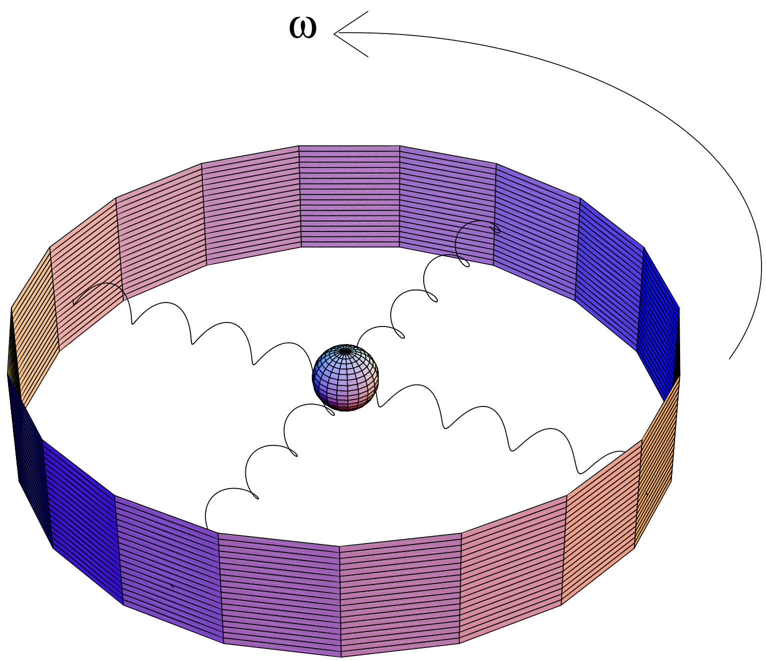

Figure 0.4: **Rotating plate with springs.**

- **Gyroscopic systens connected to wave fields.**

In Hagerty, Bloch and Weinstein [2002] we describe a gyroscopic version of the Lamb model coupled to a standard non-dispersive wave equation and to a dispersive wave equation. Show that instabilities will arise in certain mechanical systems.

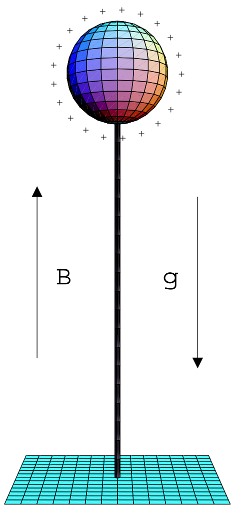

Figure 0.5: **Inverted spherical pendulum.**

In the dispersionless case, the system is of the form

$$\begin{aligned} \frac{\partial^2 \mathbf{w}}{\partial t^2}(z,t) &= c^2 \frac{\partial^2 \mathbf{w}}{\partial z^2}(z,t), \\ M\ddot{\mathbf{q}}(t) + S\dot{\mathbf{q}}(t) + V\mathbf{q}(t) &= T\left[\frac{\partial \mathbf{w}}{\partial z}\right]_{z=0} \\ \mathbf{w}(0,t) &= \mathbf{q}(t), \end{aligned}$$

$\mathbf{w} = \begin{bmatrix} w_1(z,t) & \cdots & w_n(z,t) \end{bmatrix}^T$ **is the displacement of the string in the first** n **dimensions and** $[\frac{\partial \mathbf{w}}{\partial z}]_{z=0}$ **is the jump discontinuity in the slope of the string.**

- **Can reduce dynamics to essentially:**

$$M\ddot{\mathbf{q}}(t) = -S\dot{\mathbf{q}}(t) - V\mathbf{q}(t) - \frac{2T}{c}\dot{\mathbf{q}}(t),$$

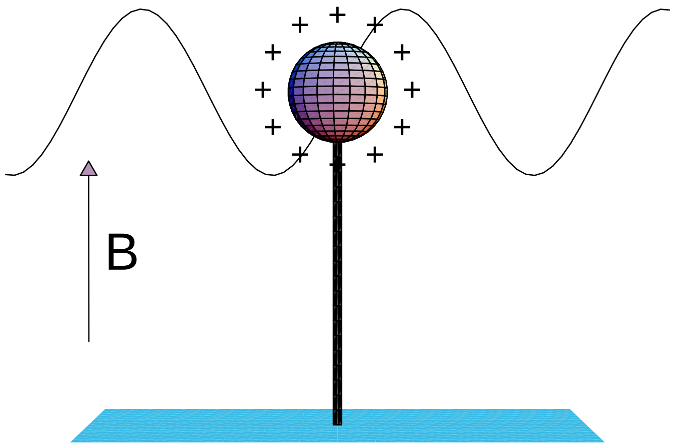

Figure 0.6: **Gyroscopic Lamb coupling to a spherical pendulum.**

•Non-local field coupling

$$M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} = \kappa \int_{\mathbb{R}} \chi(z)w(z,t)dz \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

$$\ddot{w} - c^2\frac{\partial^2 w}{\partial z^2} = \kappa\chi(z) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T \mathbf{q}$$

where κ is s coupling parameter and $\chi(\xi)$; is a suitable distribution.

• Squeezing and Control
Bloch and Rojo, EJC 2004.

Squeezing: method for reducing noise in quantum systems below the standard quantum limit.

• Laser pulses, quantum contro

• Classical Squeezing: interested in reducing noise induced by random perturbations.

Here: in classical case, system subject to thermal noise while in quantum case consider a system at zero temperature and in the presence of noise. In both cases the control is given by an external electromagnetic field and enters the control equations multiplicatively. In this sense the setting is similar to the NMR control problems.

• Key feature of squeezing: results in a redistribution of un-

certainty between observables.

Here consider a model for phonon squeezing in solids following the work of Garret at. al. [1997] (see also Hu and Nori [1996] and references therein for interesting related work), but one can equally well consider the case of photons in quantum optics.

Control is via a single pulse on a large ensemble of oscillators and this sense we are considering under-actuated control systems in both the classical and quantum case.

We also model the effect of dissipation on the classical system and the effect of coupling to a heat bath in the quantum setting. This causes the squeezing effect to gradually moderate.

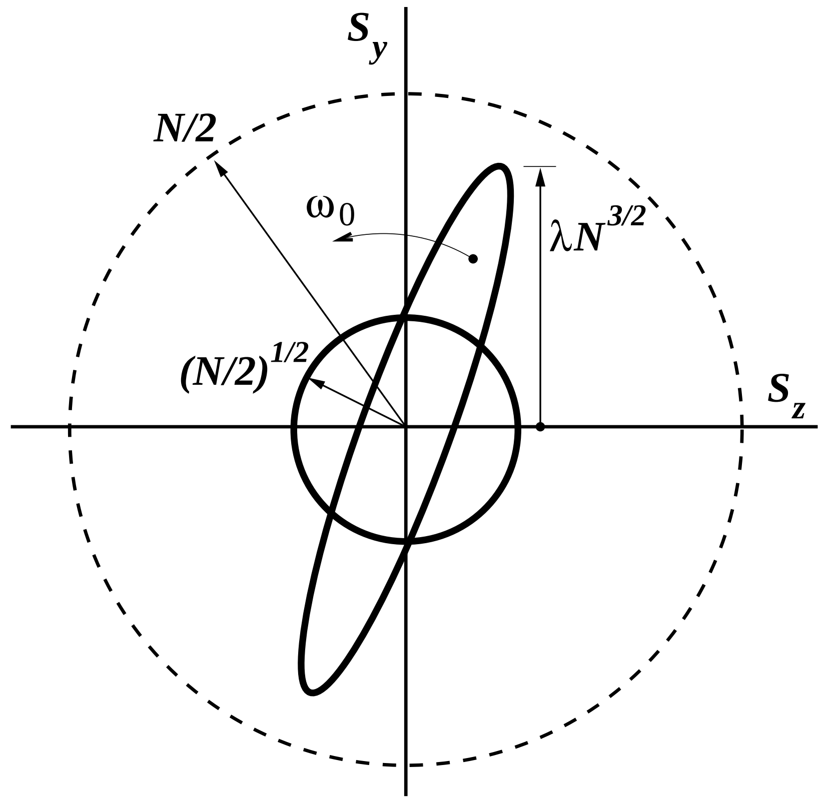

Figure 0.7: Quasiprobability distribution in the (S_z, S_y) plane for N spins, before and after a pulse $H' = \delta(t)\lambda S_z^2$ is applied on the lowest eigenstate of $H_0 = \omega_0 S_x$. The response is equivalent to the harmonic oscillator case, with the proviso that the distribution is bounded by a circle of radius $N/2$.

• Squeezing of the Quantum Harmonic Oscillator

Consider the following Hamiltonian

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2} Q^2 + \lambda\delta(t) Q^2,$$

which reflects an impulsive change in the spring constant and where $\omega = \sqrt{K/m}$, K being the original spring constant.

The variables P and Q, operators in the quantum case, obey canonical commutation rules $[P, Q] = i\hbar$.

• Rewrite Hamiltonian in terms of creation operators a and $a^\dagger$ defined by

$$Q = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger),\ \ P = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a),$$

with $[a, a^\dagger] = 1$.

Hamiltonian becomes

$$H = \hbar\omega(a^\dagger a + 1/2) + \lambda\delta(t)(a + a^\dagger)^2.$$

The ground state of the system, for $t \neq 0$, $|0\rangle$, corresponds to the vacuum of a, ($a|0\rangle = 0$), and the excited states are of the form $(a^\dagger)^2|0\rangle$.

Now want to study the behavior of the system at $t > 0$, given that the system is in its ground state at $t < 0$.

The wave function at $t = 0^+$ **is of the form** $|\psi(t = 0^+)\rangle = \exp(-i\lambda Q^2)|0\rangle$, **and for longer times the system evolves with the "unperturbed" Hamiltonian:** $|\psi(t > 0)\rangle = \exp(-iH_0 t)e^{-i\lambda Q^2}|0\rangle$.

First quantity of interest is $\langle\psi(t)|Q^2|\psi(t)\rangle \equiv \langle Q^2(t)\rangle$**.**

- **Find**

$$\langle Q^2(t)\rangle = \langle 0|e^{i\lambda Q^2}(ae^{-i\omega t} + a^\dagger e^{-i\omega t})^2 e^{-i\lambda Q^2}|0\rangle,$$

where we have used the fact that $e^{iH_0 t}ae^{-iH_0 t} = ae^{-i\omega t}$, **which states that** $a^\dagger$ **and** a **respectively destroy and create eigenstates of** H_0, **and where** Q **is defined in units of** $\sqrt{\hbar/(2m\omega)}$**.**

Now we introduce a basis of coherent states $|z\rangle$, **which satisfy** $a|z\rangle = z|z\rangle, \qquad \langle z|a^\dagger = \langle z|z^*$, **and form an overcomplete set of states:**

$$1 = \frac{1}{2\pi i}\int dz\, dz^* e^{-zz^*}|z\rangle\langle z|.$$

Find

$$\langle Q^2(t)\rangle = \frac{1}{2\pi i}\int\int dz\; dz^* e^{-zz^*}$$
$$(z^2 e^{-2i\omega t} + z^{*2}e^{2i\omega t} + 2zz^* - 1)|\langle 0|e^{i\lambda x^2}|z\rangle|^2.$$

To evaluate the last term we need the position representation of the ground state (note that at this point Q is a real number)

$$\langle 0|Q\rangle = \frac{1}{\pi^{\frac{1}{4}}}e^{-Q^2/2}$$

and that of the coherent state

$$\langle Q|z\rangle = \frac{1}{\pi^{\frac{1}{4}}}e^{-Q^2/2+\sqrt{2}zQ-z^2/2}.$$

Integration gives

$$\begin{aligned}\langle 0|e^{i\lambda Q^2}|z\rangle &= \int dx \langle 0|Q\rangle\langle Q|z\rangle e^{i\lambda Q^2}\\ &= \frac{1}{\sqrt{1-i\lambda}}e^{i\lambda z^2/2(1-i\lambda)}.\end{aligned}$$

Changing to the variables $z = u + iv$ **we have**

$$e^{-zz^*}|\langle 0|e^{i\lambda Q^2}|z\rangle|^2 = \frac{1}{\sqrt{1+\lambda^2}}e^{-[v^2+(2\lambda^2+1)u^2+2\lambda uv]/(1+\lambda^2)},$$

and

$$\langle Q^2(t)\rangle = \frac{4}{\pi\sqrt{1+\lambda^2}}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv$$

$$\left(u^2\cos^2\omega t + v^2\sin^2\omega t + uv\sin 2\omega t - \frac{1}{4}\right)$$

$$\times e^{-[v^2+(2\lambda^2+1)u^2+2\lambda uv]/(1+\lambda^2)}$$

$$= 1 + 4\lambda^2\sin^2\omega t + 2\lambda\sin 2\omega t$$

• Compare this with an ensemble of classical oscillators with initial conditions taken from a heat bath.

For simplicity take $\omega = m = k_B = T = 1$ (k_B is Boltzman's constant).

Arbitrary oscillator evolves as

$$Q(t) = u \cos t + v \sin t,$$

with u and v its initial position and velocity. If a pulse is applied at $t = 0$ of the form treated above:

$$Q(t) = u \cos t + (v + 2\lambda u) \sin t.$$

Now average over initial conditions taken from a measure given by (a thermal bath):

$$\begin{aligned} \langle Q^2(t) \rangle &\sim \int du\, dv [u \cos t + (v + 2\lambda u) \sin t]^2 e^{-(u^2+v^2)} \\ &= 1 + 4\lambda^2 \sin^2 t + 2\lambda \sin 2t. \end{aligned}$$

• **Note: the two expressions for, respectively, the quantum oscillator at zero temperature and the classical oscillator at finite temperature, are exactly the same.**

The general time dependence of the variance for a squeezed harmonic oscillator with frequency ω can thus be written in the following form:

$$\langle [Q(t)]^2 \rangle = \frac{\epsilon_0}{K} \left[1 + \left(\frac{2\lambda}{\omega} \right) \sin 2\omega t + \left(\frac{2\lambda}{\omega} \right)^2 \sin^2 \omega t \right]$$

with $\epsilon_0 = \hbar\omega/2$ for the quantum case and $\epsilon_0 = k_B T$ for the classical oscillator at a temperature T.

• Method of coherent states above has the advantage of being suitable for calculating other quantities. For example, if the oscillators are atoms within a solid, the scattering amplitude for an X-ray is decreased by a factor (called the Debye-Waller factor – see Ziman [1972]) $\sim \langle \exp ikQ(t) \rangle$, with k the wave-vector of the X-ray.

Now ask what is the time evolution of the Debye-Waller factor for a squeezed phonon. Need to compute

$$I(\lambda, t) = \langle 0| e^{i\lambda Q^2} e^{(ae^{-i\omega t} + a^\dagger e^{-i\omega t})} e^{-i\lambda Q^2} |0\rangle$$
$$= \frac{1}{\sqrt{e}} \frac{1}{\sqrt{1+\lambda^2}} \frac{1}{\pi} \int du\, dv$$
$$e^{2u\cos\omega t + 2v \sin\omega t - \frac{-[v^2 + (2\lambda^2+1)u^2 + 2\lambda uv]}{(1+\lambda^2)}}$$
$$= e^{1+4\lambda^2 \sin^2 \omega t + 2\lambda \sin 2\omega t}.$$

For the Debye-Waller factor, we obtain the following time dependence

$$\langle e^{ikQ(t)} \rangle = e^{-k^2 \langle Q^2(t) \rangle}$$

Measurement of the Debye-Waller factor may provide a practical method of detecting the squeezing phenomenon experimentally.

• Squeezing and dissipation

• Now consider the squeezing of a quantum oscillator coupled to a an infinite number of oscillators representing a "heat" bath.

• Show that this causes a decay in the squeezing oscillation for small time and true damping in the limit of a continuum of oscillators.

Damping effect of the heat bath is similar to that analyzed classically in Lamb [1900], Komech [1995], Sofer and Weinstein [1999] and Hagerty, Bloch and Weinstein [1999].

• Considering here a zero temperature case, and the damping effects appear due to a) the coupling of a single variable with a continuum of variables and b) an "asymmetry" in the initial conditions.

The Hamiltonian of the system consists of three parts: H_0

describing the original oscillator:

$$H_0 = \frac{p_0^2}{2m} + \frac{m\omega_0^2}{2} q_0^2 \,,$$

the Hamiltonian H_e of the environment:

$$H_e = \sum_\alpha \left[\frac{p_\alpha^2}{2m} + \frac{m\omega_\alpha^2}{2} q_\alpha^2 \right] \,,$$

and a linear coupling between the two

$$H_{\rm int} = \sum_\alpha \; \xi_\alpha q_\alpha q_0 \,.$$

Formally, the total Hamiltonian

$$H = H_0 + H_e + H_{\rm int}$$

can be written in terms of its normal mode coordinates X_ν **and** P_ν**:**

$$H = \sum_\nu \left[\frac{P_\nu^2}{2m} + \frac{m\omega_\nu^2}{2} X_\nu^2 \right] ,$$

and we will consider a situation in which the initial (before the pulse) wave function corresponds to all the modes in the ground state:

$$\Psi_0 = \prod_\nu \left(\frac{\omega_\nu}{\pi\hbar} \right)^{1/4} e^{-\omega_\nu X_\nu^2/2\hbar}.$$

At $t = 0$ a pulse is applied to the (original) oscillator, the wave function immediately after the pulse given by:

$$\begin{aligned}\Psi_0(t = 0^+) &= e^{i\lambda q_0^2}\Psi_0 \\ &= e^{i\lambda \sum_{\mu\nu} U_{0\mu}U_{0\nu}X_\mu X_\nu}\Psi_0\,,\end{aligned}$$

where $U_{\mu\nu}$ is the matrix transforming from the original (uncoupled) modes to the coupled system ($q_0 = \sum_\nu U_{0\nu}X_\nu$).

- **Interested in the fluctuations of the variance of q_0, given in this case by**

$$\langle q_0^2(t)\rangle = \sum_{\mu\nu} U_{0\mu}U_{0\nu}\langle X_\mu X_\nu\rangle(t)\,.$$

Compute by solving the equation of motion obeyed by the correlations $\langle X_\mu X_\nu\rangle(t)$.

Since X_μ and X_ν correspond to harmonic coordinates, using the quantum mechanical commutation relations compute the equations of motion:

$$\frac{d}{dt}\langle X_\mu X_\nu\rangle = \frac{1}{m}\langle (P_\mu X_\nu + P_\nu X_\mu)\rangle$$
$$\frac{d^2}{dt^2}\langle X_\mu X_\nu\rangle = -(\omega_\mu^2 + \omega_\nu^2)\langle X_\mu X_\nu\rangle + \frac{2}{m^2}\langle P_\mu P_\nu\rangle$$
$$\frac{d}{dt}\langle P_\mu P_\nu\rangle = -m\left(\omega_\mu^2\langle X_\mu P_\nu\rangle + \omega_\nu^2\langle X_\nu P_\mu\rangle\right)$$
$$\frac{d^2}{dt^2}\langle P_\mu P_\nu\rangle = -(\omega_\mu^2 + \omega_\nu^2)\langle P_\mu P_\nu\rangle + 2m^2\omega_\mu^2\omega_\nu^2\langle X_\mu X_\nu\rangle\ .$$

• **Note that the above equations are identical to those of classical harmonic oscillators for the quantities $X_\mu(t)X_\nu(t)$ etc., with initial conditions given by the values of the correlations evaluated for the quantum wave function:**

$$\langle X_\mu X_\nu\rangle(0^+) = \delta_{\mu\nu}\frac{\hbar}{2m\omega_\mu},$$

$$\langle P_\mu P_\nu\rangle(0^+) = \delta_{\mu\nu}\frac{\hbar m\omega_\mu}{2}$$

$$+2\hbar^2\lambda^2(1+\delta_{\mu\nu})\frac{U_{0\mu}}{m\omega_\mu}\frac{U_{0\nu}}{m\omega_\nu}q_0^2$$

$$\langle(X_\mu P_\nu + P_\nu X_\mu)\rangle(0^+) = 4\lambda\hbar U_{0\mu}U_{0\nu}\frac{\hbar}{2m}(\frac{1}{\omega_\mu}+\frac{1}{\omega_\nu})$$

with $q_0^2 \equiv \langle q_0^2(0^-)\rangle = \sum_\alpha \hbar U_{0\alpha}^2/2m\omega_\alpha$**.**

Collecting the above equations we obtain

$$\langle q_0^2(t)\rangle = q_0^2\left\{1+4\lambda^2 S^2(t)+\frac{\lambda}{q_0^2}C(t)S(t)\right\},$$

with

$$S(t) = \sum_\mu \frac{\hbar U_{0\mu}^2}{m\omega_\mu}\sin\omega_\mu t \;\; C(t) = \sum_\mu \frac{\hbar U_{0\mu}^2}{m\omega_\mu}\cos\omega_\mu t.$$

• All the information of the evolution of the variance is contained in the function $J(\omega)$, the physical interpretation of which is that of a local density of states of the oscillator, defined as

$$J(\omega) = \sum_{\mu} \frac{\hbar U_{0\mu}^2}{m\omega_\mu} \delta(\omega - \omega_\mu),$$

from which

$$S(t) = \int d\omega J(\omega) \sin \omega t, \quad C(t) = \int d\omega J(\omega) \cos \omega t.$$

• $J(\omega)$ is a sum over delta functions, leading to a superposition of oscillations with frequencies ω_ν for both $S(t)$ and $C(t)$. In the infinite limit when the modes are spatially extended over all space $J(\omega)$ becomes a continuous function. The oscillatory behavior acquires a damped component, the time dependence being given by the frequency spectrum of $J(\omega)$. A lorenzian shape for $J(\omega)$ gives an exponentially damped oscillation for both $S(t)$ and $C(t)$.

• Illustration: consider a model for which $J(\omega)$ can be computed explicitly – see the classical analysis in Lamb [1900] Komech [1995].

Consider a one-dimensional string coupled to our oscillator. The string is described by a "transverse" displacement $u(x,t)$. The classical equations of motion of the system are

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t) \\ M d^2 q_0(t)/dt^2 &= -V q_0(t) + T[u_x(0+,t) - u_x(0-,t)] \\ q_0(t) &= u(0,t). \end{aligned}$$

The normal modes consist of even and odd (in x) solutions. The odd solutions do not involve q_0 and are of the form $u_{q,o}(x,t) = e^{icqt}\sin qx$, whereas the even solutions are of the form $u_{q,e}(x,t) = e^{icqt}\cos(q|x| + \delta_q)$, with δ_q a phase shift (to be found).

The wave vectors q label the normal modes, and play the role of the index μ in the above discussion: $\omega_\mu = cq$, and $U^2_{\mu 0} = \cos^2(\delta_q)$ (up to a normalization constant) here.

Substituting in the above we obtain $(\omega_0^2 = V/M)$

$$\tan\delta_q = \frac{Mc}{2T}\frac{(\omega_0^2 - \omega_q^2)}{\omega_q},$$

from which $U_{\mu 0}^2 = \cos^2\delta_q$ **is given by**

$$U_{\mu 0}^2 = \frac{\alpha^2\omega_q^2}{\alpha^2\omega_q^2 + (\omega_q^2 - \omega_0^2)^2} \equiv U_q^2,$$

where we have defined $\alpha = 2T/Mc$**.**

Note that U_q **represents the transformation matrix that has to be normalized and since the frequencies form a continuum we normalize** $U_q(\omega_q)$ **to its integral over** ω_q**. Omitting the index** q **in** ω_q**, we obtain**

$$U(\omega) = \frac{2\alpha}{\pi}\frac{\omega^2}{\alpha^2\omega^2 + (\omega^2 - \omega_0^2)^2} = \frac{m\omega}{\hbar}J(\omega).$$

We obtain

$$S(t) = \frac{\hbar}{m\omega_0}e^{-\Gamma t}\sin\Omega_0 t,\ \ C(t) = \frac{\hbar}{m\omega_0}e^{-\Gamma t}\cos\Omega_0 t,$$

with

$$\Omega_0 \;=\; \omega_0 \left(1+[\alpha/\omega_0]^2\right)^{1/4} \cos\delta/2,$$
$$\Gamma \;=\; \omega_0 \left(1+[\alpha/\omega_0]^2\right)^{1/4} \sin\delta/2,$$

where $\delta = \tan^{-1}\alpha/\omega_0$**.**

• **In the realistic limit** $\alpha \ll \omega_0$ **which corresponds to a "weak" coupling to the environment) these expressions take the form:**

$$S(t) \cong (\hbar/(m\omega_0)\exp(-Tt/Mc)\sin\omega_0 t$$

$$C(t) \cong (\hbar/(m\omega_0)\exp(-Tt/Mc)\cos\omega_0 t.$$

Note that in this model, and in the limit of weak coupling, the initial variance q_0^2 **of the reference oscillator is unchanged due to the coupling to the environment, and is given by** $q_0^2 = \hbar/2m\omega_0$**.**

Final result then:

$$\langle q_0^2(t)\rangle \cong q_0^2 \left\{1 + e^{-2(T/Mc)t} \right.$$
$$\left. \left[\left(\frac{2\lambda\hbar}{m\omega_0}\right)\sin 2\omega_0 t + \left(\frac{2\lambda\hbar}{m\omega_0}\right)^2 \sin \omega_0^2 t\right]\right\}.$$

• Spin squeezing

The mechanism of squeezing by the application of non-linear pulses extends to spin systems, where the quantum nature of the spatial components S_i is reflected in the commutation relations $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$. Squeezing for spin systems is of topical interest in quantum information, where quantum processing protocols require manipulation of entangled systems.

One realization of a string of quantum bits is an ensemble of two-level atoms, where each atom can be treated as a spin $1/2$. Wineland et. al. showed that the resolution in spectroscopic experiments on N two-level atoms is determined by the factor

$$\xi = \frac{\Delta S_\perp}{|\langle \mathbf{S} \rangle|},$$

which measures the quantum noise in a direction perpendicular to the mean value of the total spin.

Note that ξ measures the precision of a measurement on the

rotation of a spin. We establish a parallel between the squeezed states of the harmonic oscillator and those of spin systems.

Depending on the context, other definitions of spin squeezing can also be used: starting from the uncertainty relation $\Delta S_x \Delta S_y \geq |\langle S_z \rangle/2|$ **(and cyclic permutations), a possibility is to define states satisfying** $\Delta^2 S_i < |\langle S_j \rangle/2|$ **as spin-squeezed. However, these states don't have in general a noise reduced in the direction perpendicular to the mean spin, and therefore are not relevant to quantum information.**

Consider an ensemble of identical N two-level atoms with energy splitting $\hbar\omega_0$. We define the corresponding spin quantization axis in the x direction so that

$$H_0 = \omega_0 \sum_{i=1}^{N} S_{x,i} = \omega_0 S_x,$$

where $\mathbf{S}_i$ is the spin of atom i. The equations of motion for $S_z(t)$ and $S_y(t)$ are very similar to those of x and p for a harmonic oscillator of frequency ω_0:

$$\begin{aligned} \dot{S}_z(t) &= \omega_0 S_y \\ \dot{S}_y(t) &= -\omega_0 S_z, \end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}S_z^2(t) &= \omega_0(S_yS_z + S_zS_y)\\ \frac{d}{dt}(S_yS_z + S_zS_y) &= 2\omega_0(S_y^2 - S_z^2)\\ \frac{d}{dt}S_y^2(t) &= -\omega_0(S_yS_z + S_zS_y)\end{aligned}$$

which have also the same structure as the corresponding operators for the harmonic oscillator. Note here that $S_y^2(t) + S_z^2(t)$ is conserved along the flow.

The solutions for the expectation values are:

$$\begin{aligned}\langle S_z^2(t)\rangle &= \frac{\langle S_z^2\rangle_0 + \langle S_y^2\rangle_0}{2}\\ &+ \left[\frac{\langle S_z^2\rangle_0 - \langle S_y^2\rangle_0}{2}\right]\cos 2\omega_0 t - \frac{X_0}{2}\sin 2\omega_0 t,\end{aligned}$$

with $X_0 = \langle S_zS_y\rangle_0 + \langle S_xS_y\rangle_0$. If the initial state $|\Psi\rangle$ is an eigenstate of S_x, for example $S_x|\Psi\rangle = -(N/2)|\Psi\rangle$, then $\langle S_z^2\rangle_0 = \langle S_y^2\rangle_0 = N/4$,

$X_0 = 0$ **and**

$$\xi(t) = \frac{\sqrt{\langle S_z^2(t)\rangle}}{\langle S_x\rangle} = \frac{1}{\sqrt{N}}.$$

This time independent value of ξ **corresponds to the unsqueezed state, and we are interested in decreasing its value, bringing it as close as possible to the Heisenberg limit** $\xi = 1/N$**. Proceeding in analogy with the harmonic oscillator, we consider the effect of a pulse acting on the ground state of** H_0

$$H' = \delta(t)\lambda S_z^2.$$

The wave function right after the pulse is

$$|\Psi(t=0^+)\rangle = e^{i\lambda S_z^2}|\Psi\rangle_0,$$

and the quasiprobability distribution in the (S_z, S_y) **plane is modified.**

In order to compute the modified initial conditions we consider the case of large N**.**

We define boson creation operators a and $a^\dagger$, with $[a, a^\dagger] = 1$, in terms of which

$$H_0 = \hbar\omega_0 \left(\frac{N+1}{2} - a^\dagger a \right),$$

in such a way that the spin projections in the x direction correspond to the occupation number of the new bosons. The transformation to the S^+ and S^- operators from these bosons is the well known Holstein-Primakov transformation

$$\begin{aligned} S^+ &= S_z + iS_y \\ &= N^{1/2}(1 - a^\dagger a/N)^{1/2} a \simeq N^{1/2} a \\ S^- &= S_z - iS_y \\ &= N^{1/2} a^\dagger (1 - a^\dagger a/N)^{1/2} a^\dagger \simeq N^{1/2} a^\dagger. \end{aligned}$$

where the approximation is valid as long as the relative variations of the spin projection are small:

$$\langle a^\dagger a \rangle / N \ll 1.$$

The operator equivalence between the bosons and spins implies the following correspondence:

$$\begin{aligned}
S_z &\rightarrow \sqrt{\frac{N}{2}}x, \\
S_y &\rightarrow i\sqrt{\frac{N}{2}}\frac{d}{dx} \\
S_x &\rightarrow -\frac{N+1}{2} + \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right),
\end{aligned}$$

where x **is a new variable, in terms of which the ground state of** H_0 **is** $|\Psi\rangle_0 = \pi^{1/4}\exp -x^2/2$**, and**

$$\Psi(t = 0^+)\rangle \equiv |\Psi_\lambda\rangle = \frac{1}{\pi^{1/4}} e^{iN\lambda x^2/2} e^{-x^2/2}.$$

The mapping allows us to compute the initial values:

$$\langle S_z^2\rangle_0 = \frac{N}{2}\langle\Psi_\lambda|x^2|\Psi_\lambda\rangle = \frac{N}{4}$$
$$\langle S_y^2\rangle_0 = -\frac{N}{2}\langle\Psi_\lambda|\frac{d^2}{dx^2}|\Psi_\lambda\rangle = \frac{N}{4}\left(1+(\lambda N)^2\right)$$
$$\langle S_z S_y\rangle_0 + \langle S_x S_y\rangle_0 = \frac{N}{2}\langle\Psi_\lambda|\left(ix\frac{d}{dx}+i\frac{d}{dx}x\right)|\Psi_\lambda\rangle$$
$$= -\frac{\lambda N^2}{2}$$
$$\langle S_x\rangle_0 = \frac{N}{2}\left(1-\frac{N\lambda^2}{2}\right),$$

where we stress that these values are exact provided $\lambda < 1/\sqrt{N}$ (for larger λ the response is periodic in λ). Notice that the quasiprobability distribution, which before the pulse is a circle of radius $\sqrt{N/2}$ in the (S_z, S_y) plane, now becomes an ellipse as shown in Figure 0.8. With the above initial values, for $t > 0$ the distorted distribution rotates at frequency ω_0 and the squeezing

factor evolves as

$$\xi(t) = \frac{1}{\sqrt{N}} \frac{\left[1 + (\lambda N \sin \omega_0 t)^2 - \lambda N \sin 2\omega_0 t\right]^{1/2}}{1 - N\lambda^2/2}.$$

If we call $\lambda = \alpha_0/\sqrt{N}$**, with** $\alpha_0 < 1$**, we obtain the minimum squeezing**

$$\xi_{\min} = \frac{1}{N} \frac{1}{[1 - \alpha_0^2/2]\alpha_0},$$

which scales as $1/N$ **and is reached twice during the cycle of the rotation of the ellipse of Figure 0.8.**

From the development above we can see that the analysis of dissipation above extends essentially without change to the spin setting provided the number of spins N **is large.**

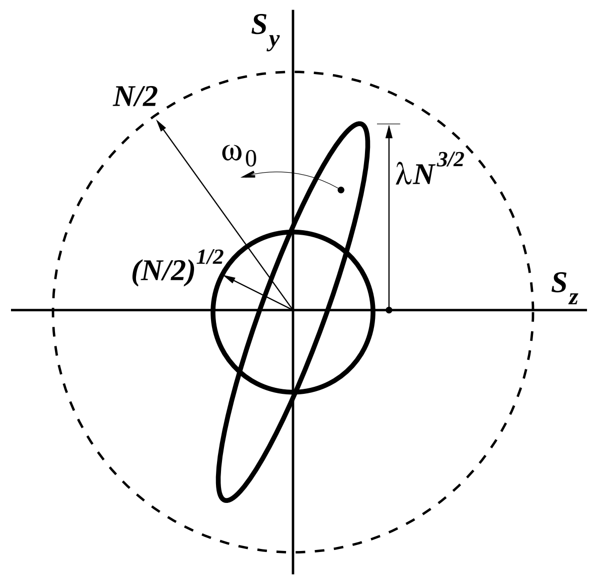

Figure 0.8: Quasiprobability distribution in the (S_z, S_y) plane for N spins, before and after a pulse $H' = \delta(t)\lambda S_z^2$ is applied on the lowest eigenstate of $H_0 = \omega_0 S_x$. The response is equivalent to the harmonic oscillator case, with the proviso that the distribution is bounded by a circle of radius $N/2$.

- **Cancellation of Squeezed States**

Here we compare the cancellation of squeezing to that of coherent states as generated by impulsively excited coherent phonons, as observed in the experiment by Hase *et al.*. **In the case of coherent states, a second pulse can cancel the coherent state if the separation time between pulses is matched to the period of the phonon oscillation. For squeezed states the separation between the pulses has to be adjusted to the intensity of the pulse. This can be easily seen graphically or from the the structure of the propagator of the harmonic oscillator:**

$$\Psi(x,t) = \int dx' G(x,x';t)\Psi(x',t=0^+)$$

with

$$\begin{aligned} G(x,x';t) &= \frac{1}{\sqrt{2\pi i\hbar\sin(\omega t)/m\omega}} \\ &\quad \exp\left\{\frac{im\omega}{2\hbar\sin\omega t}\left[(x^2+(x')^2)\cos\omega t - 2xx'\right]\right\} \end{aligned}$$

and where $\Psi(x', t = 0^+) = \exp(i\lambda(x')^2)\Psi_0(x')$ **and** $\Psi_0(x') = \Psi(x', t = 0^-)$ **is the initial (ground) state of the oscillator.**

Note that at times $t = t_n$ **with**

$$\frac{m\omega}{2\hbar}\cot\omega t_n = -\frac{\lambda}{2}, \quad \frac{m\omega}{2\hbar\sin\omega t_n} = \sqrt{\left(\frac{\lambda}{2}\right)^2 + \left(\frac{m\omega}{2\hbar}\right)^2}$$

we have

$$\Psi(x, t_n) \propto \exp\left(-i\lambda x^2/2\right) \int dx' \exp\left\{\left(i\frac{\lambda}{2} - \frac{m\omega}{2\hbar}\right)(x')^2 \right.$$
$$\left. -2ixx'\sqrt{\left(\frac{\lambda}{2}\right)^2 + \left(\frac{m\omega}{2\hbar}\right)^2}\right\},$$

which gives

$$\Psi(x, t_n) = \exp i\delta \exp\left(-i\lambda x^2\right)\Psi_0(x),$$

with δ an overall phase factor.

This means that a second pulse with the same intensity applied at t_n restores the wave function to the ground state.

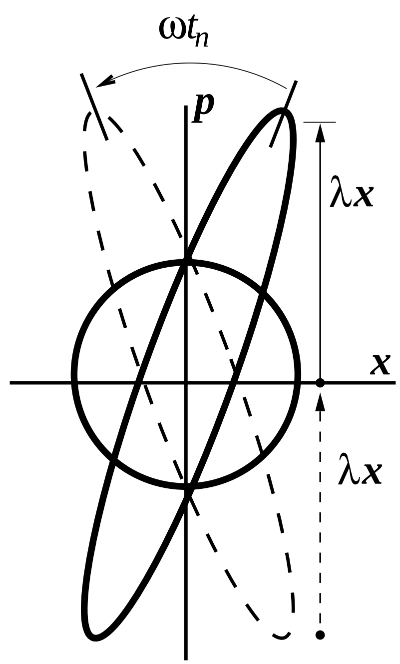

Figure 0.9: Graphical rendition of the squeezing cancellation produced by a second pulse, of equal amplitude as the first, applied at a time t_n such that $\frac{m\omega}{2\hbar}\cot\omega t_n = -\frac{\lambda}{2}$.

- **The Chaplygin Sleigh as as Particle in a Radiation Field**

See Bloch and Rojo [2007].

We now show that the sleigh equations can be obtained from a variational principle as reduced equations of motion after the system is coupled to an environment described by an $U(1)$ infinite field of the form $\mathbf{a}(\mathbf{z}, t) \equiv [\cos\alpha(\mathbf{z}, t), \sin\alpha(\mathbf{z}, t)]$. For the Lagrangian of the free field we choose

$$L_{\mathrm{F}} = \frac{K}{2} \int d^2\mathbf{z}\ \dot{\mathbf{a}}^2, \tag{0.1}$$

and we couple the sleigh and the field with a term of the form

$$L_1 = \int d^2\mathbf{z}\ \delta(\mathbf{z} - \mathbf{x})\ \left[\gamma \dot{\mathbf{x}} \cdot \mathbf{a} + \mu \cos\left(\alpha(\mathbf{z}, t) - \theta\right)\right]. \tag{0.2}$$

The first term in square brackets corresponds to a minimal coupling that favors ẋ in the direction of a; the second has the form of a potential coupling that favors an alignment of the internal variable θ with the local direction of a.

The total action is $S = \int dt(L_0 + L_F + L_1)$ where L_0 is the Lagrangian of the free sleigh

$$L_0 = \frac{m}{2}\left[\left(\dot{x} - a\dot{\theta}\sin\theta\right)^2 + \left(\dot{y} + a\dot{\theta}\cos\theta\right)^2\right] + \frac{I}{2}\dot{\theta}^2, \qquad (0.3)$$

and can be regarded as a full "microscopic" theory of the sleigh coupled to an environment.

At this point we take the limit $\mu \to \infty$ in the third equation above. This limit can be understood from the singular perturbation theory. We do similar for K.

As a result of taking both limits we have

$$\dot{x} \sin \alpha(\mathbf{x}, t) - \dot{y} \cos \alpha(\mathbf{x}, t) = \dot{x} \sin \theta - \dot{y} \cos \theta = 0, \tag{0.4}$$

which means that the constraint is satisfied. Replacing the constraint (and $\sin[\alpha(\mathbf{x}, t) - \theta] = 0$) in the first three equations we obtain the same flow as the nonholonomic equations.

The calculation shows that we have succeeded in deriving the nonholonomic equations for a system with one internal (compact) variable from a pure Lagrangian formalism. The classical trayectories are obtained from a variational principle and quantization can be introduced through the Path integral formalism: the propagator is $e^{iS/\hbar}$, where S is the complete action.

Details are in Bloch and Rojo [2007]