# Controllability of the Schrödinger Equation via Intersection of Eigenvalues

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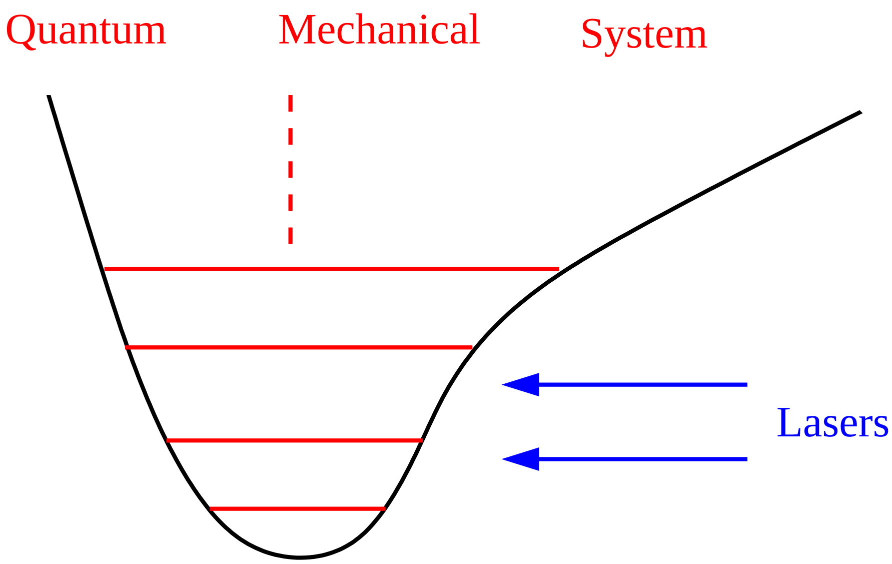
## Plan of the talk

- the controllability question in quantum mechanics (finite and infinite dim)

- A technique for infinite dim. systems based on adiabatic approximation
("try not to use brackets")

- two toy models

## The controllability question




(roughly) Given a system coupled with some external fields, can we design the external fields so that to reach every point of the state space?

A Quantum mechanical system in interaction with external fields

$$i\dot{\psi}(t) = (H_0 + \sum_{j=1}^{m} u_m(t) H_J)\psi$$

- for every $t$, $\psi(t) \in$ Hilbert space (finite or infinite dimensional)
- $H_0$ is the "drift Hamiltonian" (discrete spectrum)
- $u_j(t)$ are the external fields (controls) belonging to some functional space ($L^\infty$)
- $H_j$ are the the " coupling Hamiltonian"

$H_0, ....H_m$ are (essentially) self adjoint operators $\Rightarrow \psi(t) \in$ Hilbert sphere

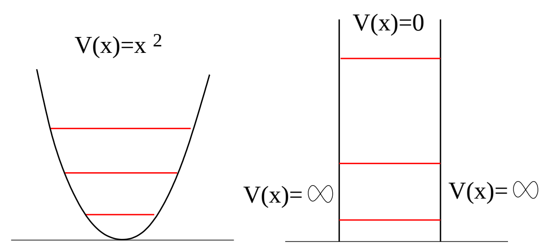
## The finite dimensional case

$\psi(t) \in \mathbf{C}^n$, $iH_0, ..., iH_m \in su(n)$.

## example the infinite dimensional case

$$i\frac{\partial\psi(x,t)}{\partial t} = \underbrace{\left(\underbrace{-\frac{d^2}{dx^2} + V(x)}_{H_0 \ \ (drift)} + \underbrace{\sum_{j=1}^{m} u_j(t) H_i}_{control \ \ term}\right)}_{H(u_1(t),...,u_m(t))} \psi(x,t)$$

- $x \in \mathbf{R}$,
- fixed $t$, $\psi(.,t) \in H^1(\mathbf{R})$, $\quad \int_R |\psi(x,t)|^2 dx = 1$
- $V(x) \in L^1_{loc}$, s.t. $H_0$ has discr. spectr.

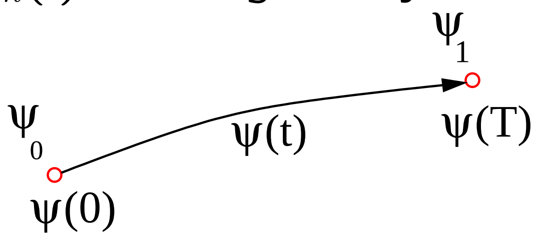



- $H_i$ are essentially self-adjoint operators

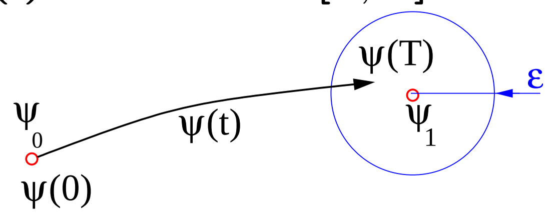
# Notions of Controllability

Fix a functional class for the controls (e.g essentially bounded) and an Hilbert space for $\psi$.

- We have exact controllability if for every $\psi_0$, $\psi_1$, there exist controls $u_1(.), ..., u_m(.)$ steering the system from $\psi_0$ to $\psi_1$ in finite time.




- we have approximate controllability if for every $\psi_0$, $\psi_1$, $\varepsilon > 0$, there exist $T > 0$, controls $u_1(.), ..., u_m(.)$ defined in $[0, T]$ such that $\psi(0) = \psi_0$, $\|\psi(T) - \psi_1\| \leq \varepsilon$.




- we have exact state to state controllability if we have exact controllability for every pairs of eigenstates of $H_0$ (here I am assuming that they are not degenerate)

- we have approximate state to state controllability $\rightarrow$ similarly

**Finite dimensional case** $\rightarrow$ completely understood

generically we have **exact controllability**

(generically $Lie\{iH_0, iH_1, ..., iH_m\} = su(n)$ +compactness )

(Jurdjevic, Kupka, Sussmann, Gauthier, see the review by Yuri Sachkov)

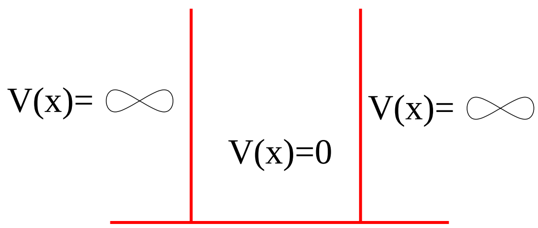
**Infinite dimensional case** $\longrightarrow$ few results

- in general one does not expect exact controllability for an infinite dim. systems with a finite number of controls.

- up to 2003 the community believed that in general the Schrödinger equation is not controllable. Many noncontrollability results: linearization, harmonic oscillator, non exact controllability (Rouchon).

- **(surprise)** in 2003 Beauchard Coron proved exact controllability for a 1d well of potential controlled by $u(t)x$

$$i\frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{d^2}{dx^2} + V(x) + u(t)x\right)\psi(x,t)$$




$\longrightarrow$ for every initial and final state in $H^7$. ($L^2$ functions with seven derivatives in $L^2$).

$\longrightarrow$ by density $\Rightarrow$ approximate controllability in $H^1$

$\longrightarrow$ but eigenstates are analitic. There is exact state to state-exact controllability.

# Very recent results

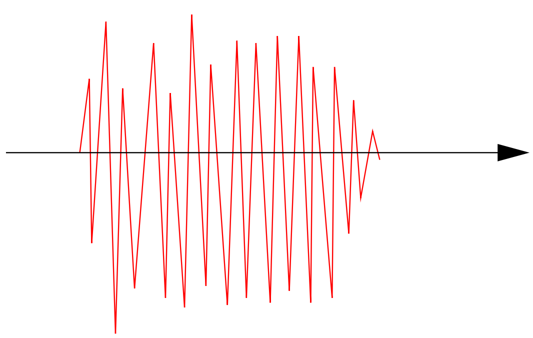
- Mirrahimi: approximate controllability between eigenstates for systems having a continuous part of the spectrum

- approximate controllability for generic systems by Thomas Chambrion, Mario Sigalotti, Paolo Mason (Agrachev School)

In all these results controls are

- not explicit

- even if in principle it is possible to find them, they are highly oscillating (unusable)



## A Method based on Intersection of Eigenvalues and Adiabatic Theory (using slow varying controls)

- it works only in some special cases
  (eigenvalues intersections, at least two controls)

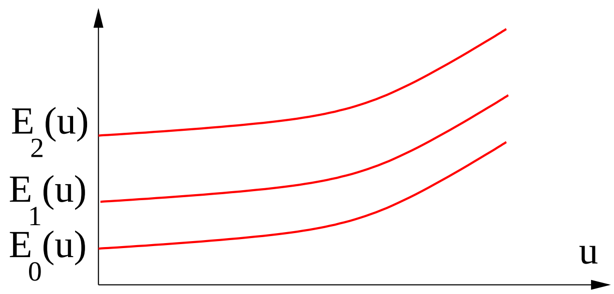
- it provides approximate state to state controllability

- it provides **explicit expressions** of controls, that are nice and easy to implement

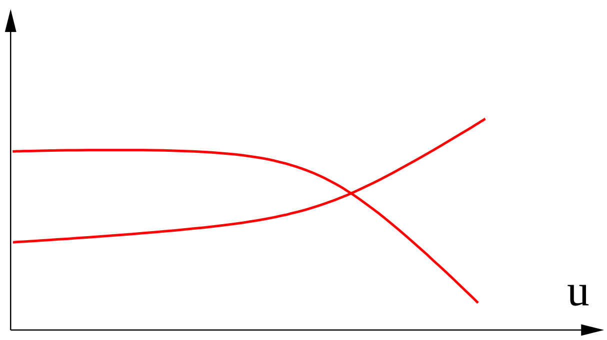
- it is a NON-BRACKET method

Consider an Hamiltonian depending on one control: $H(u(t))$. Assume that $\psi(x,0) = \Phi_n$

- Adiabatic Theory asserts that if we use slow varying controls then $\psi(x,t) \sim \Phi_n(u(t))$ (in the $L^2$ norm, up to phases)



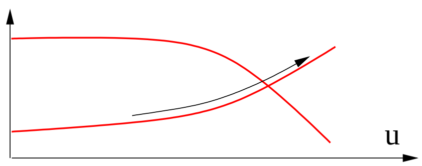

- if eigenvalues intersects as functions of of controls, in some cases it is possible to jump to the intersected state.



# 3 Difficulties

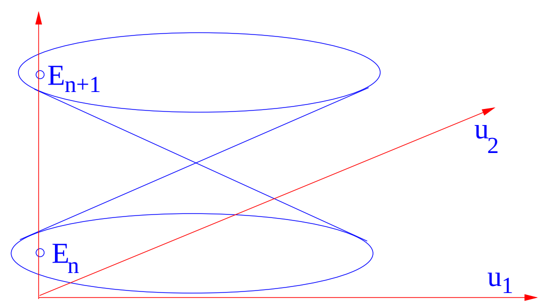
1. Existence of Eigenvalues intersections
   (in dim 1 if $V(x)+\sum_{j=1}^{m} u_j(t)H_i(x) \in L^1_{loc}$ then the spectrum is never degenerate)
   (in dim $d$ if $V(x)+\sum_{j=1}^{m} u_j(t)H_i(x) \in L^1_{loc}$ then the ground state is never degenerate)
   $\rightarrow$ Relax the Hp. that $V(x)+\sum_{j=1}^{m} u_j(t)H_i(x) \in L^1_{loc}$ or use a $d$ dim model with $d > 1$ and forget about the ground state

2. for reversibility reason:
   number of controls must be $>$ dim. of intersections $+1$




   $\rightarrow$ not a problem, using two controls, generically, intersections are CONICAL (codimension 2)

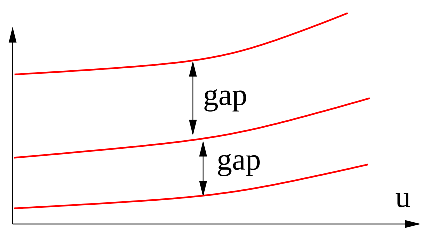
   generically if $H_0 + u_1 H_1 + u_2 H_2$ has an eigenvalue inters., then




(conical inters. have been studied by Hagedorn, Teufel, Lasser,for other purposes)

3. Adiabatic Theory need the gap condition

$\longrightarrow$ the Adiabatic Theorem must be rewritten in a neighborhood of a singularity



**Theorem 1** *Consider a family $H(t)$ of self adjoint operators on a Hilbert space $\mathcal{H}$, with $t$ in the possibly unbounded interval $(t_1, t_2)$. Suppose that:*

- *all $H(t)$'s have a common dense domain $\mathcal{D}$.*
- *$H(\cdot) \in \mathcal{C}_b^2((t_1, t_2), \mathcal{L}(\mathcal{D}, \mathcal{H}))$.*
- *for every $t$, the spectrum $\sigma(H(t))$ of $H(t)$ is discrete and non degenerate, i.e. $\sigma(H(t)) = \{\lambda_j(t), j = 0, \ldots, n, \ldots, \ \lambda_i(t) < \lambda_k(t) \ \text{ if } \ i < k\}$.*
- *Fixed $j \in \mathbb{N}$, the following gap condition is satisfied:*

$$g := \inf_{t \in (t_1, t_2)} \min \left( \lambda_{j+1}(t) - \lambda_j(t), \lambda_j(t) - \lambda_{j-1}(t) \right) > 0$$

- *at time $t_0 \in (t_1, t_2)$ the system lies in an eigenstate of $H(\varepsilon t_0)$ associated to the eigenvalue $\lambda_j(t_0)$.*

*Then, for any $t$ and $t_0$ in $(t_1, t_2)$,*

$$\|\psi_\varepsilon(t) - \psi_a^\varepsilon(t)\| < C\varepsilon \left(1 + \varepsilon |t - t_0|\right) \tag{1}$$

*where $\psi_\varepsilon(t)$ represents the actual state of the system and $\psi_a^\varepsilon(t)$ is eigenvector of $H(\varepsilon t)$ relative to the eigenvalue $\lambda_j(\varepsilon t)$.*

Notice that the constant $C$ diverges for vanishing $g$.

$\rightarrow$ passing inside the singularities the adiabatic approximation does not work, but it is possible to show the existence of a path, along which we have the transition at the same order of the adiabatic approximation.

$\longrightarrow$ inspired by works in finite dimension by Jauslin, Guérin, Yatsenko (for STIRAP process)

I will present two toy models that show how 1,2,3 can be solved

The first toy model

→ already in the base of eigenvectors of $H_0$.

→ generalization of 3-level problems used for STIRAP

$$H(u,v) = \begin{pmatrix} E_0 & \alpha_0 u & 0 & 0 & 0 & \cdots \\ \alpha_0 u & E_1 & \beta_0 v & 0 & 0 & \cdots \\ 0 & \beta_0 v & E_2 & \alpha_1 u & 0 & \cdots \\ 0 & 0 & \alpha_1 u & E_3 & \beta_1 v & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \qquad (2)$$

$\alpha_j, \beta_j > 0$ coupling constants

Assume that: $E_j$'s diverges and $\alpha_j/|E_{2j}|^\mu$ and $\beta_j/|E_{2j}|^\mu$ vanish as $j$ goes to infinity for some $0 < \mu < 1$. Then $H(u,v)$ defines a self adjoint operator with purely discrete spectrum on $\ell^2$.

→ in this case it is very easy to prove exact-SSC using classical control theory (but I will try to implement our method).

Problems:

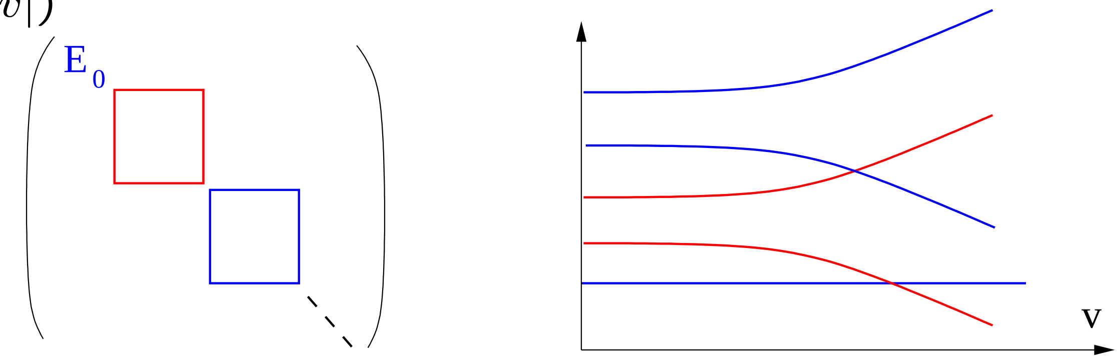
1. Classification of Eigenvalues intersections
2. Check that number of controls is > than dim of singularities+1
3. Application of adiabatic theory

# Classification of Eigenvalues Intersections

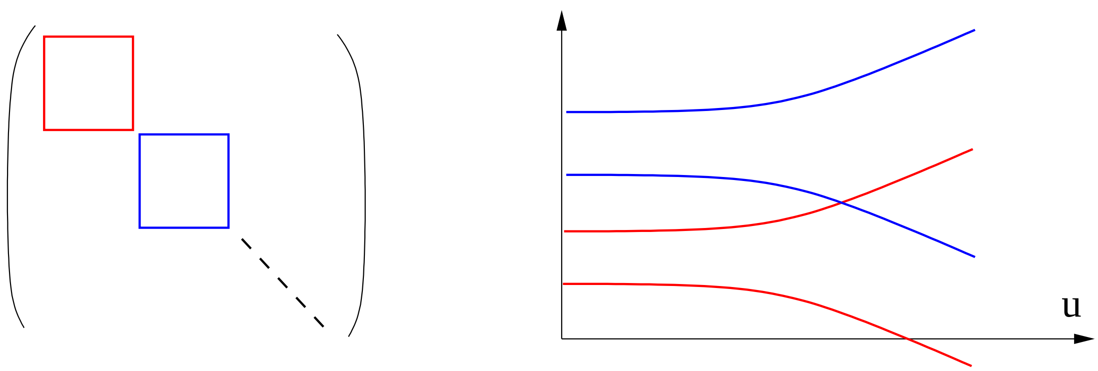
- if $u \neq 0$ and $v \neq 0$ then all eigenvalues are not degenerate

- the spectrum of $H(u,v)$ is the same as the spectrum of $H(|u|,|v|)$

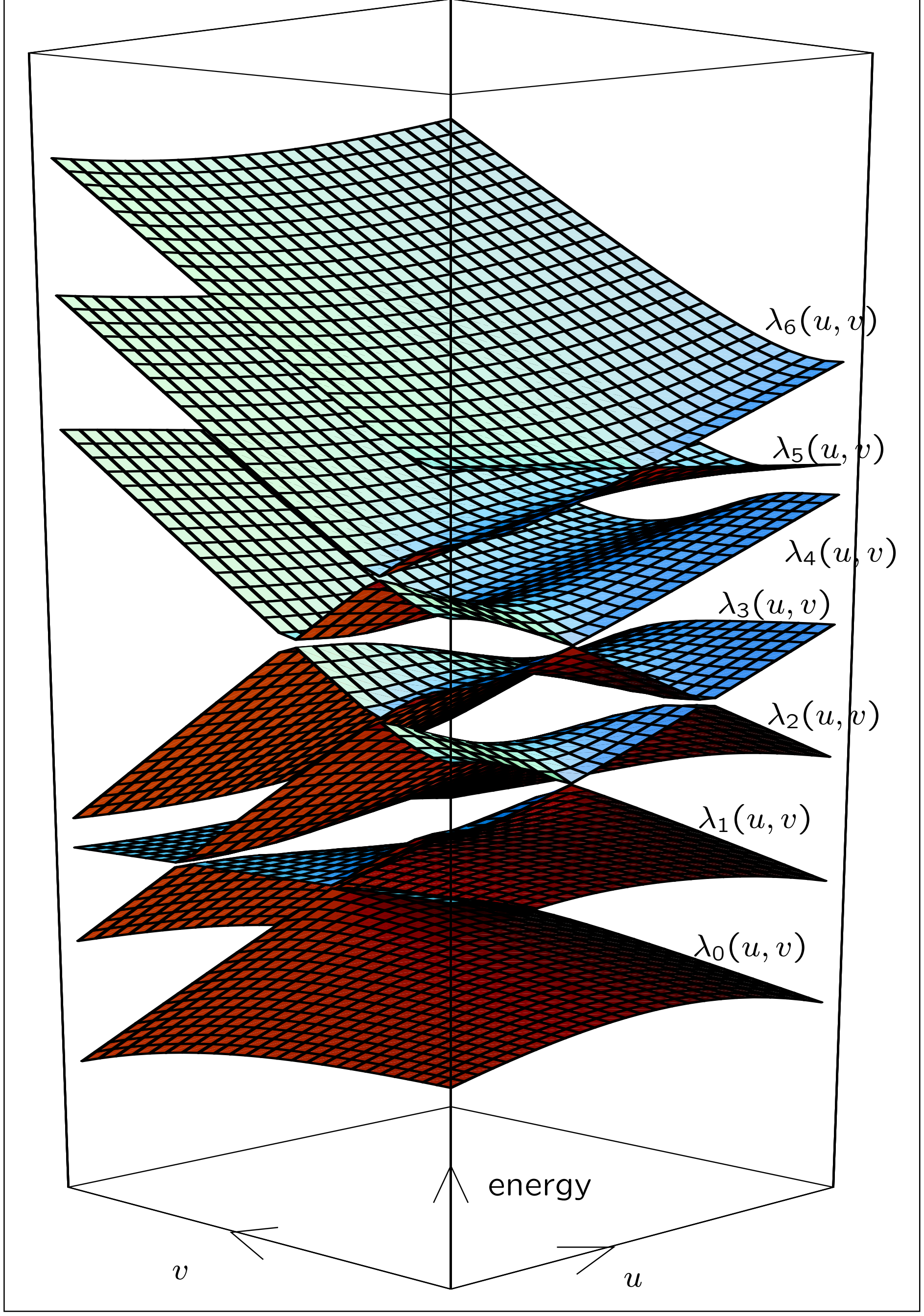
If u=0 then:



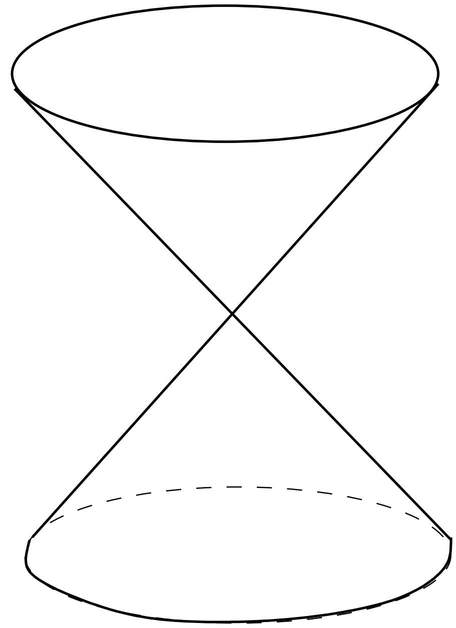

If v=0 then:



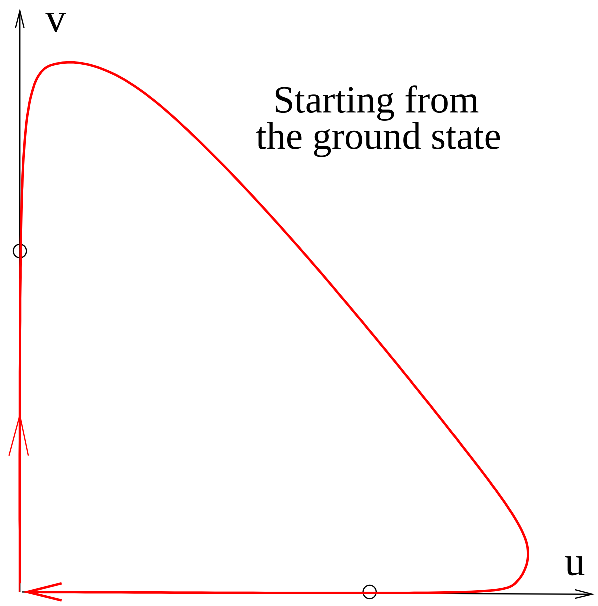

$\longrightarrow$ Eigenvalues intersections has dim zero
$\longrightarrow$ Ground State can become degenerate $\Rightarrow$ this model cannot be in the form: $H(u,v) = -\Delta + V(x) + uB_1(x) + vB_2(x)$ with $V, B_1, B_2 \in L^1_{loc}$.

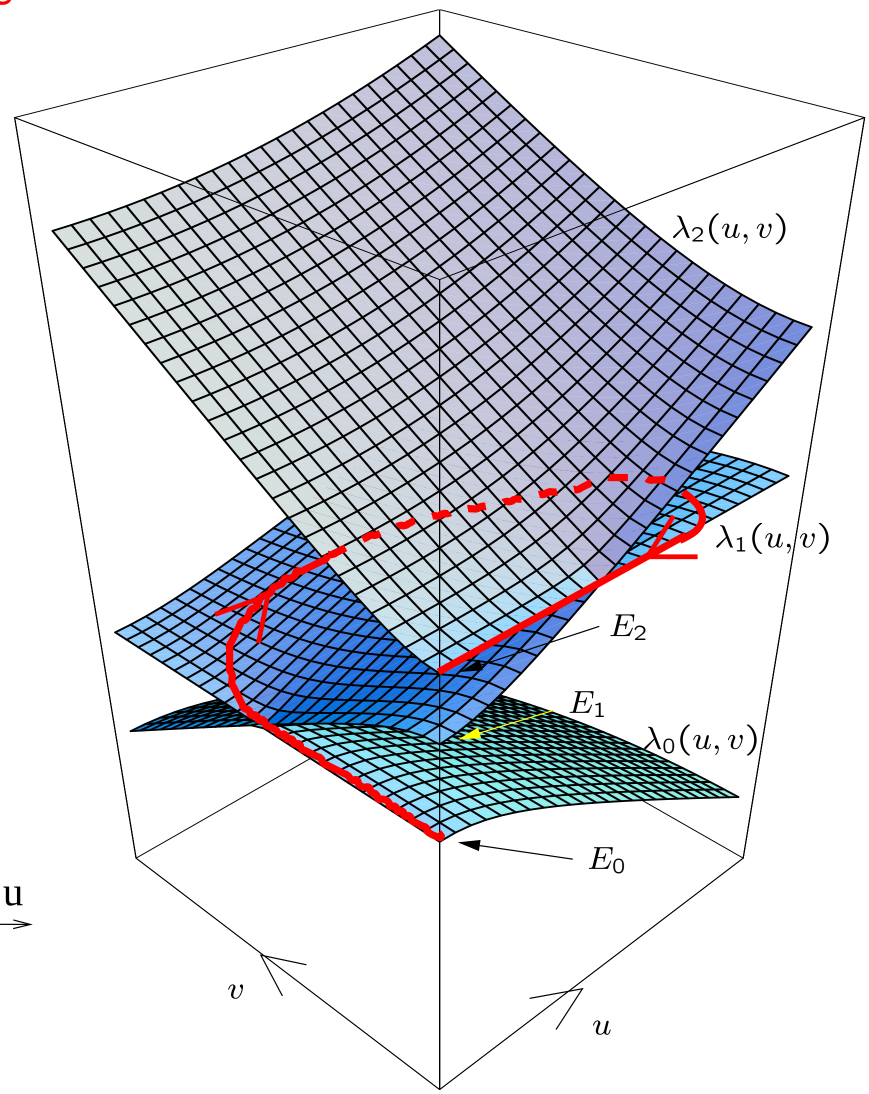


$\lambda_6(u,v)$
$\lambda_5(u,v)$
$\lambda_4(u,v)$
$\lambda_3(u,v)$
$\lambda_2(u,v)$
$\lambda_1(u,v)$
$\lambda_0(u,v)$
energy
$v$
$u$

# The Adiabatic Theory




This happens because when u=0, $E_0$ is decoupled and $E_1$ is coupled only with $E_2$

There are two kind of decoupling: 1) far from singularities (adiabatic decoupling)

2) close to the singularities (due to the symmet.)

**Definition 1** *Consider a map* $\gamma(\cdot) := (u(\cdot), v(\cdot), p(\cdot)) : [0, \tau] \to \mathcal{S} \subset \mathbb{R}^3$. *We say that this map is a* <u>*climbing path*</u> *if:*

- *it is a* $\mathcal{C}^2$ *map from* $[0, \tau]$ *to* $\mathbb{R}^3$;
- $\gamma(0) = (u(0), v(0), p(0)) = (0, 0, E_A)$ *and* $\gamma(\tau) = (u(\tau), v(\tau), p(\tau)) = (0, 0, E_B)$ *for some* $A, B \in \mathbb{N}$;
- *it passes through a finite number of singularities. i.e.* $\text{Supp}(\gamma) \cap \mathcal{Z}$ *is finite.*
- *if* $\tau_1, ..., \tau_n$ *are the values of the parameter at which the singularities are met, namely* $\gamma(\tau_i) \in \mathcal{Z}$ *for any* $i$, *then there exist intervals* $[a_i, b_i]$ *such that* $\tau_i \in ]a_i, b_i[$ *and* $u$ *or* $v$ *constantly vanishes on* $[a_i, b_i]$.

**Theorem 2** *Consider the family of Hamiltonians* $H(u, v)$ *and a climbing path* $\gamma$. *Given* $\varepsilon \ll 1$ *consider the following parametrization of* $\gamma$: $\gamma(\varepsilon t) = (u(\varepsilon t), v(\varepsilon t), p(\varepsilon t))$, *with* $t \in [0, T]$ *and* $T := \varepsilon^{-1}\tau$. *Let* $\Phi_j(u, v)$ *be the eigenvector corresponding to the eigenvalue* $\lambda_j(u, v)$. *Let* $t_1, ..., t_n$ *be the times at which the singularities are met, namely* $\gamma(\varepsilon t_i) \in \mathcal{Z}$ *for any* $i$. *Let* $j_i$ *be defined by* $p(\varepsilon t) = \lambda_{j_i}(u(\varepsilon t), v(\varepsilon t))$, $t \in ]t_i, t_{i+1}]$. *Then, for every* $t \in ]t_i, t_{i+1}]$, *we have*

$$\| \exp\left( i \int_0^{\varepsilon t} ds\, \lambda_{j_i}(u(s), v(s)) \right) \Phi_{j_i}(u(\varepsilon t), v(\varepsilon t)) - \psi(\varepsilon t) \| < C\varepsilon(1 + \varepsilon|t|) \leq C\varepsilon(1 + \tau)$$

*where* $\psi(t)$ *is the solution of the Schrödinger equation*

$$i\partial_t \psi(t) = H(u(\varepsilon t), v(\varepsilon t))\psi(t), \quad \psi(0) = \Phi_{j_i}(0, 0). \tag{3}$$

## The second model

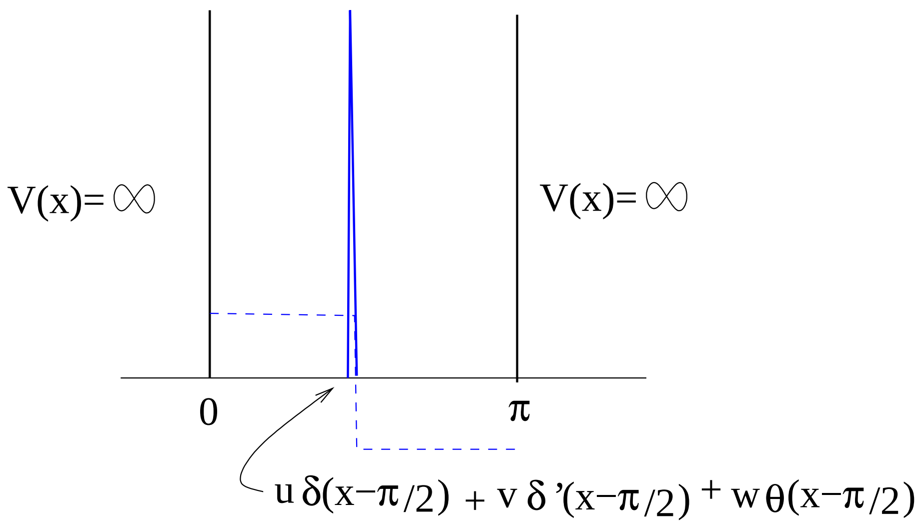
A model with potential $\notin L^1_{loc}$, having a similar behavior.

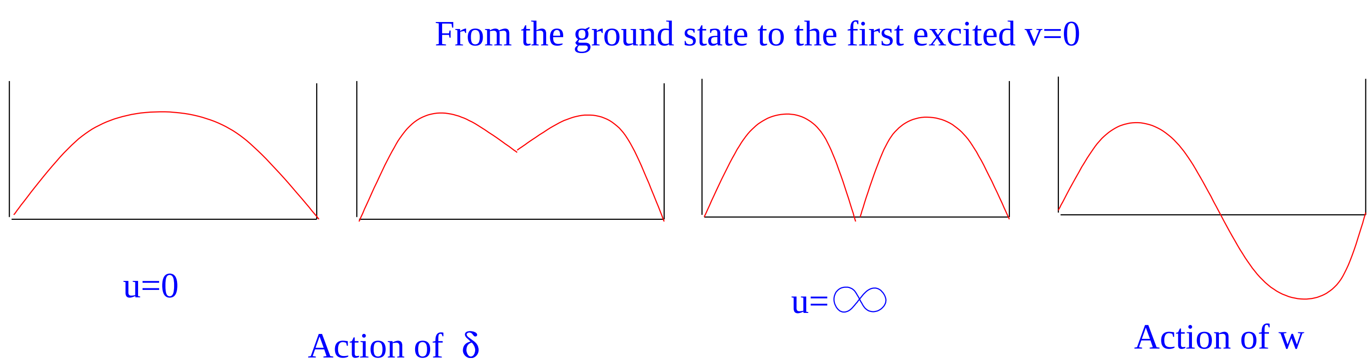
$$H(u,v,w) := -\partial_x^2 + u\delta(x-\pi/2) + v\delta'(x-\pi/2) + w\theta(x-\pi/2) \tag{4}$$

$\partial_x^2$ is the partial derivative with respect to $x$ with Dirichlet boundary conditions

$u, v$: $\mathbf{R} \to \mathbf{R} \cup \{\infty\}$

$w$: $\mathbf{R} \to [0,1]$



Conclusions:

this method:

$\longrightarrow$ provides explicit expression for controls
$\longrightarrow$ is very robust
$\longrightarrow$ can be applied to many other situations (e.g. to symmetric potentials)