Controllability of the Schrödinger equation in the case of discrete spectrum

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Physical model: the controlled Schrödinger equation

Schrödinger equation

$$i\dot{\psi} = -\Delta\psi + V\psi + uW\psi$$

Ω domain of $\mathbf{R}^d$
$\psi(t,x)$ wave function
$V : \Omega \to \mathbf{R}$ potential of the Schrödinger operator
$u = u(t)$ real-valued control
$W : \Omega \to \mathbf{R}$ controlled potential

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Two cases:

- $\Omega \subset \mathbf{R}^d$ bounded regular domain, $\psi|_{\partial\Omega} \equiv 0$,
- $\Omega = \mathbf{R}^d$.

Controllability results for the Schrödinger equation

Infinite dimension

- Adami, Boscain
- Beauchard, Coron
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- Zuazua
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Eigenvalues of the Schrödinger operator

Theorem

Let Ω be bounded and smooth and $V \in L^\infty(\Omega)$. Then $-\Delta + V$ admits a family of eigenfunctions in $H^2(\Omega) \cap H_0^1(\Omega)$ which form an orthonormal basis of $L^2(\Omega)$

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Let $\Omega = \mathbf{R}^d$ and $V \in L^1_{\mathrm{loc}}$ be bounded from below and such that

$$\lim_{|x| \to \infty} V(x) = +\infty.$$

Then $-\Delta + V$ admits a family of eigenfunctions in $H^2(\mathbf{R}^d)$ which form an orthonormal basis of $L^2(\mathbf{R}^d)$.
If $V \geq 0$ then all eigenfunctions have exponential decay at infinity

Mathematical framework

We will consider control systems of the form

$$\frac{d\psi}{dt} = A(\psi) + uB(\psi), \qquad u \in U \qquad\qquad (A, B, U)$$

with the following ingredients

- H complex Hilbert space;
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The hypotheses above guarantee that

$$\forall u \in U, \exists\ e^{t(A+uB)} : H \to H \text{ group of unitary transformations}$$

Approximate controllability

We call $e^{t_k(A+u_k B)} \circ \cdots \circ e^{t_1(A+u_1 B)}(\psi_0)$ the solution of the control system (A, B, U) starting from ψ_0 associated to the piecewise constant control $u_1\chi_{[0,t_1]} + u_2\chi_{[t_1,t_1+t_2]} + \cdots$

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We say that (A, B, U) is approximatively controllable if for every $\psi_0, \psi_1 \in \mathcal{S}$ and every $\varepsilon > 0$ there exist $k \in \mathbf{N}$, $t_1, \ldots, t_k > 0$ and $u_1, \ldots, u_k \in U$ such that

$$\|\psi_1 - e^{t_k(A+u_kB)} \circ \cdots \circ e^{t_1(A+u_1B)}(\psi_0)\| < \varepsilon.$$

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Ball–Marsden–Slemrod [1982] (adapted by Turinici to the skew-symmetric case) proved that exact controllability (in $\mathcal{S} \cap H^2 \cap H^1_0$) does not hold in infinite dimension.

We look for conditions that guarantee the approximate controllability.

Main result

$(\lambda_n)_{n\in\mathbf{N}}$ eigenvalues of A corresponding to $(\phi_n)_{n\in\mathbf{N}}$.

Theorem

If $(\lambda_{n+1}-\lambda_n)_{n\in\mathbf{N}}$ are $\mathbf{Q}$-linearly independent and if $\langle B\phi_j,\phi_{j+1}\rangle \neq 0$ for every $j\in\mathbf{N}$, then $(A,B,(0,\delta))$ is approximatively controllable for every $\delta>0$.

Recall that $(\lambda_{n+1}-\lambda_n)_{n\in\mathbf{N}}$ is $\mathbf{Q}$-linearly independent (or non-resonant) if for every $N\in\mathbf{N}$ and $(q_1,\dots,q_N)\in\mathbf{Q}^N\setminus\{0\}$ one has

$$\sum_{n=1}^{N} q_n(\lambda_{n+1}-\lambda_n)\neq 0.$$

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As in works by Agrachev, Sarychev and Rodrigues on the control of Navier-Stokes equation, we follow the following pattern

- time reparameterization;
- controllability of the Galerkyn approximations;
- controllability in observed projections;
- approximate controllability of the original system.

Time reparameterization

For $u \neq 0$, clearly,

$$e^{t(A+uB)} = e^{tu\left(\left(\frac{1}{u}\right)A+B\right)}.$$

Theorem 3 is therefore equivalent (under the same hypotheses) to

$\forall \delta, \varepsilon > 0$, $\forall \psi_0, \psi_1 \in \mathcal{S}$, there exist $k \in \mathbf{N}$, $t_1, \ldots, t_k > 0$ and $u_1, \ldots, u_k > \delta$ such that

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> $$\|\psi_1 - e^{t_k(u_k A+B)} \circ \cdots \circ e^{t_1(u_1 A+B)}(\psi_0)\| < \varepsilon.$$

I.e., the system

$$\frac{d\psi}{dt} = u(t)A(\psi) + B(\psi), \qquad u \in U \qquad (\Sigma)$$

is approximatively controllable provided that the control set U contains a half-line.

Galerkyn approximations

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For $j, k \in \mathbf{N}$, let a_{jk} and b_{jk} be the components of A and B in the base $(\phi_m)_{m\in\mathbf{N}}$.
Galerkyn approximation of order n:

$$\frac{dx}{dt} = uA^{(n)}x + B^{(n)}x, \qquad x \in \mathcal{S}_n, \qquad u > \delta, \qquad (\Sigma_n)$$

where $\mathcal{S}_n$ denotes the unit sphere of $\mathbf{C}^n$.

Controllability of the Galerkyn approximations

$\lambda_2 - \lambda_1, \dots, \lambda_n - \lambda_{n-1}$ $\mathbf{Q}$-linearly independent $+$ $b_{j,j+1} \neq 0$

$$\Downarrow$$

the Lie algebra generated by $A^{(n)}$ and $B^{(n)}$ has max. dimension

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(Σ_n) is controllable. (see Jurdjevic, Sussmann, Sachkov,...)

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(Σ_n) is controllable. (see Jurdjevic, Sussmann, Sachkov,...)

Let $u : [0, T] \to (\delta, \infty)$ be a piecewise constant control driving $\xi_0/\|\xi_0\|$ to $\xi_1/\|\xi_1\|$ where

$$\xi_j = \overline{\Pi}_n(\psi_j), \quad j = 1, 2$$

and

$$\overline{\Pi}_n : H \to \mathbf{C}^n$$

associates the first n coordinates.

Higher-order Galerkyn approximation and elimination of the drift

$\mu > 0$ small constant to be chosen later (depending on T)
$\phi_j \in D(B) \Longrightarrow (b_{jk})_{k\in\mathbf{N}}$ is in l^2
Choose $N \geq n$ such that $\sum_{k>N} |b_{jk}|^2 < \mu$ for every $j = 1, \ldots, n$

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ELIMINATION OF THE DRIFT

If $t \mapsto X(t)$ is a solution of (Σ_N) corresponding to a control function U, then $t \mapsto e^{-V(t)A^{(N)}}X(t) = Y(t)$, where $V(t) = \int_0^t U(\tau)d\tau$, is a solution of

$$\dot{Y} = e^{-V(t)A^{(N)}}B^{(N)}e^{V(t)A^{(N)}}Y. \qquad (\Theta_N)$$

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We want to track the solution of (Θ_n) corresponding to u (chosen above) by a solution of (Θ_N).

The tracking procedure

Claim

There exists a sequence $u_k : [0, T] \to (\delta, \infty)$ *piecewise constants such that the sequence*

$$t \mapsto M_k(t) = e^{-v_k(t)A^{(N)}} B^{(N)} e^{v_k(t)A^{(N)}},$$

where $v_k(t) = \int_0^t u_k(\tau)d\tau$, *converges to*

$$t \mapsto M(t) = \begin{pmatrix} e^{-v(t)A^{(n)}} B^{(n)} e^{v(t)A^{(n)}} & 0_{n\times(N-n)} \\ 0_{(N-n)\times n} & G(t) \end{pmatrix},$$

where $v(t) = \int_0^t u(\tau)d\tau$, $G(t)$ *is continuous and* $M_k \to M$ *in the following integral sense,*

$$\int_0^t M_k(\tau)d\tau \to \int_0^t M(\tau)d\tau$$

The tracking procedure

Therefore, the resolvent $R_k(t,s) : \mathbf{C}^N \to \mathbf{C}^N$ of the linear time-varying equation

$$\dot{Y} = M_k(t)Y,$$

converges, uniformly with respect to (t,s), to the resolvent $R(t,s) : \mathbf{C}^N \to \mathbf{C}^N$ of

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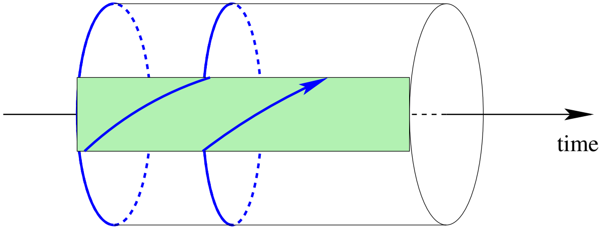

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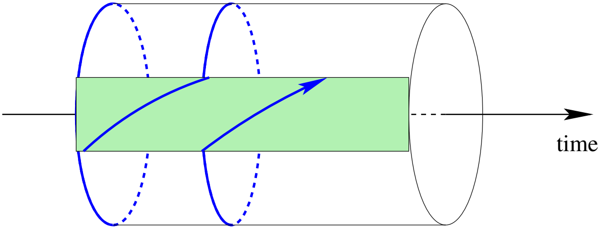
Notice that $R(t,s)$ preserves the norm of both the vector formed by the first n coordinates and the one formed by the last $N-n$.



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The curves $P^k(t) = (q_1^k(t), \dots, q_n^k(t))^T$ and
$Q^k(t) = (q_{n+1}^k(t), \dots, q_N^k(t))^T$ satisfy

$$\begin{pmatrix} \dot{P}^k(t) \\ \dot{Q}^k(t) \end{pmatrix} = M_k(t) \begin{pmatrix} P^k(t) \\ Q^k(t) \end{pmatrix} + \begin{pmatrix} H^k(t) \\ I^k(t) \end{pmatrix}$$

with $\|H^k\|_\infty < \sqrt{n\mu}$ and $\|I^k\|_\infty \leq C$ for $C = C(N)$ large enough.
Hence

$$\begin{pmatrix} P^k(t) \\ Q^k(t) \end{pmatrix} = R_k(t,0)\overline{\Pi}_N(\psi_0) + \int_0^t R_k(s,t) \begin{pmatrix} H^k(s) \\ I^k(s) \end{pmatrix} ds.$$

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$$\|\Pi_n(e^{-iv_k(T)A}\psi^k(T)) - \Pi_n(e^{-iv(T)A}\psi_1)\| < \frac{\varepsilon}{100}$$

and the components of $\psi^k(T)$ are in modulus $\varepsilon/2$-close to those of ψ_1.

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Claim

For every $\varepsilon > 0$ and $v_1 \in \mathbf{R}$ there exist $\tau > 0$ and a positive control function $u : [0, \tau] \to \mathbf{R}$ such that a trajectory $\psi(\cdot)$ corresponding to $u(\cdot)$ satisfies $\|\Pi_n(\psi(\tau)) - \Pi_n(e^{iv_1 A}\psi(0))\| \leq \varepsilon$.

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- We can not apply our result.

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- Use analyticity of the eigenvalues to show that for a dense set of ν, $-\Delta + V_\nu$ has the desired properties.
- apply the result for a good ν.

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- Use the same trick as for the 1-D box of potential:
- The first derivative of the kth eigenvalue is $C_k \pi^3 + D_k e^{\pi k}$.
- π and e^{π} are algebraically independent (Nesterenko, 1996)

Perspectives

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- Different models