Entangling Properties of Quantum Channels

Dariusz Chruściński

Nicolaus Copernicus University, Toruń

Quantum States

Quantum system $\mathcal{S}$ $\longrightarrow$ Hilbert space $\mathcal{H}$

Definition:

A **Quantum State** in $\mathcal{H}$ is represented by a density operator ρ

$$\rho \geq 0 \quad \& \quad \operatorname{Tr} \rho = 1$$

$\mathcal{S}(\mathcal{H})$ — Space of Quantum States

Quantum States

Quantum system $\mathcal{S}$ $\longrightarrow$ Hilbert space $\mathcal{H}$

Definition:

A **Quantum State** in $\mathcal{H}$ is represented by a density operator ρ

$$\rho \geq 0 \quad \& \quad \mathrm{Tr}\,\rho = 1$$

$\mathcal{S}(\mathcal{H})$ — Space of Quantum States

Quantum States

Quantum system $\mathcal{S}$ $\longrightarrow$ Hilbert space $\mathcal{H}$

Definition:

A **Quantum State** in $\mathcal{H}$ is represented by a density operator ρ

$$\rho \geq 0 \quad \& \quad \mathrm{Tr}\, \rho = 1$$

$\mathcal{S}(\mathcal{H})$ — Space of Quantum States

Quantum Channel — Physics

1) A Quantum Channel maps quantum states of $\mathcal{S}_1$ into quantum states of $\mathcal{S}_2$

$$\Phi \ : \ \mathcal{S}(\mathcal{H}_1) \longrightarrow \mathcal{S}(\mathcal{H}_2)$$

2) $\mathcal{H}_E$ —Hilbert space of the Environment

$$\mathbb{1}_E \otimes \Phi \ : \ \mathcal{S}(\mathcal{H}_E \otimes \mathcal{H}_1) \longrightarrow \mathcal{S}(\mathcal{H}_E \otimes \mathcal{H}_2)$$

for any $\mathcal{H}_E$!

Quantum Channel — Physics

1) A Quantum Channel maps quantum states of $\mathcal{S}_1$ into quantum states of $\mathcal{S}_2$

$$\Phi \; : \; \mathcal{S}(\mathcal{H}_1) \longrightarrow \mathcal{S}(\mathcal{H}_2)$$

2) $\mathcal{H}_E$ —Hilbert space of the Environment

$$\mathbb{1}_E \otimes \Phi \; : \; \mathcal{S}(\mathcal{H}_E \otimes \mathcal{H}_1) \longrightarrow \mathcal{S}(\mathcal{H}_E \otimes \mathcal{H}_2)$$

for any $\mathcal{H}_E$!

Quantum Channel — Mathematics

$\mathcal{A}_1$, $\mathcal{A}_2$ — $\mathbb{C}^*$-algebras

Definition:

A linear map $\Phi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$

i) is positive if

$$a \geq 0 \implies \Phi(a) \geq 0$$

ii) is k-positive if

$$\mathbb{1}_k \otimes \Phi : M_k(\mathbb{C}) \otimes \mathcal{A}_1 \longrightarrow M_k(\mathbb{C}) \otimes \mathcal{A}_2$$

is positive

iii) is Completely Positive if it is k-positive for all $k = 1, 2, \ldots$

Quantum Channel — Mathematics

$\mathcal{A}_1$, $\mathcal{A}_2$ — $\mathbb{C}^*$-algebras

Definition:

A linear map $\Phi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$
i) is positive if

$$a \geq 0 \implies \Phi(a) \geq 0$$

ii) is k-positive if

$$\mathbb{1}_k \otimes \Phi : M_k(\mathbb{C}) \otimes \mathcal{A}_1 \longrightarrow M_k(\mathbb{C}) \otimes \mathcal{A}_2$$

is positive
iii) is Completely Positive if it is k-positive for all $k = 1, 2, \ldots$

Quantum Channel — Mathematics

$\mathcal{A}_1$, $\mathcal{A}_2$ — $\mathbb{C}^*$-algebras

Definition:

A linear map $\Phi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$

i) is positive if

$$a \geq 0 \implies \Phi(a) \geq 0$$

ii) is k-positive if

$$\mathbb{1}_k \otimes \Phi : M_k(\mathbb{C}) \otimes \mathcal{A}_1 \longrightarrow M_k(\mathbb{C}) \otimes \mathcal{A}_2$$

is positive

iii) is Completely Positive if it is k-positive for all $k = 1, 2, \ldots$

Quantum Channel — Mathematics

$$\mathcal{A}_1 \;,\; \mathcal{A}_2 \quad \text{— } \mathbb{C}^*\text{-algebras}$$

Definition:

A linear map $\Phi \; : \; \mathcal{A}_1 \longrightarrow \mathcal{A}_2$

i) is positive if

$$a \geq 0 \quad \Longrightarrow \quad \Phi(a) \geq 0$$

ii) is k-positive if

$$\mathbb{1}_k \otimes \Phi \; : \; M_k(\mathbb{C}) \otimes \mathcal{A}_1 \longrightarrow M_k(\mathbb{C}) \otimes \mathcal{A}_2$$

is positive

iii) is Completely Positive if it is k-positive for all $k = 1, 2, \ldots$

Positive maps — Example

$$\Phi_\mu \ : \ M_d \longrightarrow M_d$$

$$\Phi_\mu(a) \ = \ \mu\,\mathbb{I}_d\,\mathrm{Tr}\,a - a\ ; \qquad \mu \geq 1$$

Theorem:

$$k+1 > \mu \geq k$$

Φ_μ is k-positive but NOT $(k+1)$-positive

$\mu \geq d\,, \quad \Phi_\mu$ is CP

Positive maps — Example

$$\Phi_\mu \ : \ M_d \longrightarrow M_d$$

$$\Phi_\mu(a) \ = \ \mu \, \mathbb{I}_d \, \mathrm{Tr}\, a - a \ ; \qquad \mu \geq 1$$

Theorem:

$$k + 1 > \mu \geq k$$

Φ_μ is k-positive but NOT $(k+1)$-positive

$\mu \geq d$, Φ_μ is CP

Remark: 17th Hilbert Problem

Homogeneous polynomials in $(x_1, \ldots, x_n)$

PSD $\quad f(x_1, \ldots, x_n) \geq 0$

SOS $\quad f = f_1^2 + f_2^2 + \ldots$

SOS $\implies$ PSD

PSD $\implies$ SOS ????

$f(x)$ — YES

$f(x_1, \ldots, x_n)$ & f – quadratic form — YES

Remark: 17th Hilbert Problem

Homogeneous polynomials in $(x_1, \ldots, x_n)$

PSD $\quad f(x_1, \ldots, x_n) \geq 0$

SOS $\quad f = f_1^2 + f_2^2 + \ldots$

SOS $\Longrightarrow$ PSD

PSD $\Longrightarrow$ SOS ????

$f(x)$ — YES

$f(x_1, \ldots, x_n)$ & f – quadratic form — YES

Remark: 17th Hilbert Problem

Homogeneous polynomials in $(x_1, \ldots, x_n)$

PSD $\quad f(x_1, \ldots, x_n) \geq 0$

SOS $\quad f = f_1^2 + f_2^2 + \ldots$

SOS $\Longrightarrow$ PSD

PSD $\Longrightarrow$ SOS ????

$f(x)$ — YES

$f(x_1, \ldots, x_n)$ & f – quadratic form — YES

Remark: 17th Hilbert Problem

Homogeneous polynomials in $(x_1, \ldots, x_n)$

PSD $\quad f(x_1, \ldots, x_n) \geq 0$

SOS $\quad f = f_1^2 + f_2^2 + \ldots$

SOS $\implies$ PSD

PSD $\implies$ SOS ????

$f(x)$ — YES

$f(x_1, \ldots, x_n)$ & f – quadratic form — YES

Remark: 17th Hilbert Problem

Homogeneous polynomials in $(x_1, \dots, x_n)$

PSD $\quad f(x_1, \dots, x_n) \geq 0$

SOS $\quad f = f_1^2 + f_2^2 + \dots$

SOS $\implies$ PSD

PSD $\implies$ SOS ????

$f(x)$ — YES

$f(x_1, \dots, x_n)$ & f – quadratic form — YES

Remark: 17th Hilbert Problem

Homogeneous polynomials in $(x_1, \ldots, x_n)$

PSD $\quad f(x_1, \ldots, x_n) \geq 0$

SOS $\quad f = f_1^2 + f_2^2 + \ldots$

SOS $\Longrightarrow$ PSD

PSD $\Longrightarrow$ SOS ????

$f(x)$ — YES

$f(x_1, \ldots, x_n)$ & f – quadratic form — YES

Remark: 17th Hilbert Problem

Homogeneous polynomials in $(x_1, \ldots, x_n)$

PSD $\quad f(x_1, \ldots, x_n) \geq 0$

SOS $\quad f = f_1^2 + f_2^2 + \ldots$

SOS $\implies$ PSD

PSD $\implies$ SOS ????

$f(x)$ — YES

$f(x_1, \ldots, x_n)$ & f – quadratic form — YES

Remark: 17th Hilbert Problem

Hilbert 1888 — counterexample !!!

Motzkin $\longrightarrow$ $M(x,y,z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$

Hilbert 1900 — 17th Problem

PSD $\ni f \implies f = g_1^2 + g_2^2 + \ldots$

g_i rational functions

Proof — Artin 1927

Remark: 17th Hilbert Problem

Hilbert 1888 — counterexample !!!

Motzkin $\longrightarrow$ $M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$

Hilbert 1900 — 17th Problem

PSD $\ni f \implies f = g_1^2 + g_2^2 + \dots$

g_i rational functions

Proof — Artin 1927

Remark: 17th Hilbert Problem

Hilbert 1888 — counterexample !!!

Motzkin $\longrightarrow$ $M(x,y,z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$

Hilbert 1900 — 17th Problem

PSD $\ni f \implies f = g_1^2 + g_2^2 + \dots$

g_i rational functions

Proof — Artin 1927

Remark: 17th Hilbert Problem

Hilbert 1888 — counterexample !!!

Motzkin $\longrightarrow$ $M(x,y,z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$

Hilbert 1900 — 17th Problem

PSD $\ni f \implies f = g_1^2 + g_2^2 + \ldots$

g_i rational functions

Proof — Artin 1927

Quantum Channel

$$\dim \mathcal{H}_k = d_k < \infty \ , \quad k = 1, 2$$

$$\mathcal{B}(\mathcal{H}_k) \ \text{— } \mathbb{C}^*\text{-algebra}$$

Definition:

A Quantum Channel is represented by a Completely Positive Map

$$\Phi \ : \ \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_2)$$

Schrödinger vs. Heisenberg

States $\longleftrightarrow$ Observables

$$A \in \mathcal{O}(\mathcal{H})\ ,\ \rho \in \mathcal{S}(\mathcal{H}) \quad \longrightarrow \quad \mathrm{Tr}(A\rho)$$

$$\mathrm{Tr}\,[A\Phi(\rho)] = \mathrm{Tr}\left[\widetilde{\Phi}(A)\rho\right]$$

$\mathrm{Tr}\,\Phi(\rho) = \mathrm{Tr}\rho$ — trace preserving channel

$\widetilde{\Phi}(\mathbb{I}) = \mathbb{I}$ — unital channel

Schrödinger vs. Heisenberg

States $\longleftrightarrow$ Observables

$$A \in \mathcal{O}(\mathcal{H}) \ , \ \ \rho \in \mathcal{S}(\mathcal{H}) \quad \longrightarrow \quad \mathrm{Tr}(A\rho)$$

$$\mathrm{Tr}\left[A\Phi(\rho)\right] = \mathrm{Tr}\left[\widetilde{\Phi}(A)\rho\right]$$

$\mathrm{Tr}\,\Phi(\rho) = \mathrm{Tr}\rho$ — trace preserving channel

$\widetilde{\Phi}(\mathbb{I}) = \mathbb{I}$ — unital channel

Schrödinger vs. Heisenberg

States $\longleftrightarrow$ Observables

$$A \in \mathcal{O}(\mathcal{H}) \ , \ \rho \in \mathcal{S}(\mathcal{H}) \quad \longrightarrow \quad \mathrm{Tr}(A\rho)$$

$$\mathrm{Tr}\left[A\Phi(\rho)\right] = \mathrm{Tr}\left[\widetilde{\Phi}(A)\rho\right]$$

$\mathrm{Tr}\,\Phi(\rho) = \mathrm{Tr}\rho$ — trace preserving channel

$\widetilde{\Phi}(\mathbb{I}) = \mathbb{I}$ — unital channel

Stinespring–Kraus–Choi

A CP map

$$\Phi \ : \ \mathcal{B}(\mathcal{H}_1) \ \longrightarrow \ \mathcal{B}(\mathcal{H}_2)$$

may be represented as follows:

$$\Phi(a) = \sum_{\alpha} K_\alpha \, a \, K_\alpha^*$$

$$K_\alpha \ : \ \mathcal{H}_1 \ \longrightarrow \ \mathcal{H}_2$$

$$\sum_{\alpha} K_\alpha K_\alpha^* = \mathbb{I}_1 \quad \Longleftrightarrow \quad \text{trace preserving}$$

Stinespring–Kraus–Choi

A CP map

$$\Phi \ : \ \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_2)$$

may be represented as follows:

$$\Phi(a) = \sum_\alpha K_\alpha \, a \, K_\alpha^*$$

$$K_\alpha \ : \ \mathcal{H}_1 \longrightarrow \mathcal{H}_2$$

$$\sum_\alpha K_\alpha K_\alpha^* = \mathbb{I}_1 \quad \Longleftrightarrow \quad \text{trace preserving}$$

Example 1

$$\Phi(a) = \sum_{\alpha} K_\alpha \, a \, K_\alpha^*$$

$$\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$$

$U : \mathcal{H} \longrightarrow \mathcal{H}$; — unitary

$$\Phi(a) = U \, a \, U^*$$

Example 1

$$\Phi(a) = \sum_{\alpha} K_\alpha \, a \, K_\alpha^*$$

$$\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$$

$$U \; : \; \mathcal{H} \longrightarrow \mathcal{H} \; ; \qquad \text{— unitary}$$

$$\Phi(a) = U \, a \, U^*$$

Example 2

$$\Phi(a) = \sum_{\alpha} K_\alpha \, a \, K_\alpha^*$$

$$\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$$

$$K_\alpha = \frac{1}{\sqrt{d}} U_\alpha \; ; \quad U_\alpha \in U(d) \; ; \quad \mathrm{Tr}(U_\alpha U_\beta^*) = \delta_{\alpha\beta}$$

$$\alpha, \beta = 1, 2, \ldots, d^2$$

$$\Phi(a) = \mathbb{I}$$

Example 2

$$\Phi(a) = \sum_{\alpha} K_\alpha \, a \, K_\alpha^*$$

$$\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$$

$$K_\alpha = \frac{1}{\sqrt{d}} U_\alpha \; ; \quad U_\alpha \in U(d) \; ; \quad \mathrm{Tr}(U_\alpha U_\beta^*) = \delta_{\alpha\beta}$$

$$\alpha, \beta = 1, 2, \ldots, d^2$$

$$\Phi(a) = \mathbb{I}$$

Example 2

$$\Phi(a) = \sum_{\alpha} K_\alpha \, a \, K_\alpha^*$$

$$\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$$

$$K_\alpha = \frac{1}{\sqrt{d}} U_\alpha \; ; \quad U_\alpha \in U(d) \; ; \quad \mathrm{Tr}(U_\alpha U_\beta^*) = \delta_{\alpha\beta}$$

$$\alpha, \beta = 1, 2, \ldots, d^2$$

$$\Phi(a) = \mathbb{I}$$

Channels vs. States

$$\mathcal{H}_1 = \mathcal{H}_2 =: \mathcal{H} \ , \qquad \dim \mathcal{H} = d < \infty$$

$$\Phi \ : \ \mathcal{B}(\mathcal{H}) \ \longrightarrow \ \mathcal{B}(\mathcal{H})$$

$$\Phi \quad \longleftrightarrow \quad \rho_\Phi \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$$

$$\Psi^+ = \frac{1}{\sqrt{d}} \sum_{\alpha=1}^{d} e_\alpha \otimes e_\alpha \qquad \longrightarrow \qquad P^+ = |\Psi^+\rangle\langle\Psi^+|$$

$$\rho_\Phi = (\mathbb{1} \otimes \Phi) P^+$$

Channels vs. States

$$\mathcal{H}_1 = \mathcal{H}_2 =: \mathcal{H} \,, \qquad \dim \mathcal{H} = d < \infty$$

$$\Phi \;:\; \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$$

$$\Phi \quad \longleftrightarrow \quad \rho_\Phi \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$$

$$\Psi^+ = \frac{1}{\sqrt{d}} \sum_{\alpha=1}^{d} e_\alpha \otimes e_\alpha \qquad \longrightarrow \qquad P^+ = |\Psi^+\rangle\langle\Psi^+|$$

$$\rho_\Phi = (\mathbb{1} \otimes \Phi) P^+$$

Channels vs. States

$$\mathcal{H}_1 = \mathcal{H}_2 =: \mathcal{H} \;, \qquad \dim \mathcal{H} = d < \infty$$

$$\Phi \;:\; \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$$

$$\Phi \quad \longleftrightarrow \quad \rho_\Phi \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$$

$$\Psi^+ = \frac{1}{\sqrt{d}} \sum_{\alpha=1}^{d} e_\alpha \otimes e_\alpha \qquad \longrightarrow \qquad P^+ = |\Psi^+\rangle\langle\Psi^+|$$

$$\rho_\Phi = (\mathbb{1} \otimes \Phi) P^+$$

Schmidt number

Schmidt decomposition

$$\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$e_1, e_2, \ldots \in \mathcal{H}_1 \quad \& \quad f_1, f_2, \ldots \in \mathcal{H}_2$$

$$\psi = \sum_\alpha \lambda_\alpha \, e_\alpha \otimes f_\alpha$$

$$\lambda_\alpha \geq 0 \,, \qquad \sum_\alpha \lambda_\alpha^2 = 1$$

Schmidt number

$$\mathrm{SN}(\psi) = \text{ number of nonvanishing } \lambda_\alpha$$

$$\psi \text{ is separable} \iff \mathrm{SN}(\psi) = 1$$

Schmidt number

Schmidt decomposition

$$\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$e_1, e_2, \ldots \in \mathcal{H}_1 \quad \& \quad f_1, f_2, \ldots \in \mathcal{H}_2$$

$$\psi = \sum_\alpha \lambda_\alpha \, e_\alpha \otimes f_\alpha$$

$$\lambda_\alpha \geq 0 \,, \qquad \sum_\alpha \lambda_\alpha^2 = 1$$

Schmidt number

$$\mathrm{SN}(\psi) = \text{ number of nonvanishing } \lambda_\alpha$$

$$\psi \text{ is separable} \iff \mathrm{SN}(\psi) = 1$$

Schmidt number

Schmidt decomposition

$$\psi \;\in\; \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$e_1, e_2, \ldots \;\in\; \mathcal{H}_1 \quad \& \quad f_1, f_2, \ldots \;\in\; \mathcal{H}_2$$

$$\psi \;=\; \sum_\alpha \lambda_\alpha \, e_\alpha \otimes f_\alpha$$

$$\lambda_\alpha \geq 0 \;, \qquad \sum_\alpha \lambda_\alpha^2 = 1$$

Schmidt number

$$\mathrm{SN}(\psi) = \text{ number of nonvanishing } \lambda_\alpha$$

$$\psi \;\text{ is separable } \;\Longleftrightarrow\; \mathrm{SN}(\psi) = 1$$

Schmidt number

ρ — density operator in $\mathcal{H} \otimes \mathcal{H}$

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$$

$$\psi_k \in \mathcal{H} \otimes \mathcal{H}$$

Schmidt number

$$\mathrm{SN}(\rho) = \min_{p_k, \psi_k} \left\{ \max_k \mathrm{SN}(\psi_k) \right\}$$

$$1 \leq \mathrm{SN}(\rho) \leq d$$

ρ is separable $\iff$ $\mathrm{SN}(\rho) = 1$

Schmidt number

ρ — density operator in $\mathcal{H} \otimes \mathcal{H}$

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$$

$$\psi_k \in \mathcal{H} \otimes \mathcal{H}$$

Schmidt number

$$\mathrm{SN}(\rho) = \min_{p_k, \psi_k} \left\{ \max_k \mathrm{SN}(\psi_k) \right\}$$

$$1 \le \mathrm{SN}(\rho) \le d$$

ρ is separable $\iff$ $\mathrm{SN}(\rho) = 1$

Schmidt number

ρ — density operator in $\mathcal{H} \otimes \mathcal{H}$

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$$

$$\psi_k \in \mathcal{H} \otimes \mathcal{H}$$

Schmidt number

$$\mathsf{SN}(\rho) = \min_{p_k, \psi_k} \left\{ \max_k \mathsf{SN}(\psi_k) \right\}$$

$$1 \leq \mathsf{SN}(\rho) \leq d$$

ρ is separable $\iff$ $\mathsf{SN}(\rho) = 1$

Schmidt number

ρ — density operator in $\mathcal{H} \otimes \mathcal{H}$

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$$

$$\psi_k \in \mathcal{H} \otimes \mathcal{H}$$

Schmidt number

$$\mathrm{SN}(\rho) = \min_{p_k, \psi_k} \left\{ \max_k \mathrm{SN}(\psi_k) \right\}$$

$$1 \ \leq \ \mathrm{SN}(\rho) \ \leq d$$

ρ is separable $\iff$ $\mathrm{SN}(\rho) = 1$

$$S_k = \{ \rho \mid \mathsf{SN}(\rho) \leq k \}$$

(k-separable states)

Separable states $= S_1 \subset S_2 \subset \ldots \subset S_d =$ all states

Theorem

If

$$\Phi : M_d \longrightarrow M_d \quad \text{— } k\text{-positive} ,$$

then

$$(\mathbb{1} \otimes \Phi)(S_k) \geq 0 .$$

$$S_k = \{ \ \rho \mid \mathsf{SN}(\rho) \leq k \ \}$$

(k-separable states)

Separable states $=$ $S_1 \subset S_2 \subset \ldots \subset S_d$ $=$ all states

Theorem

If

$$\Phi \ : \ M_d \longrightarrow M_d \qquad \text{— } k\text{-positive} \ ,$$

then

$$(\mathbb{1} \otimes \Phi)(S_k) \geq 0 \ .$$

$$S_k = \{\ \rho \mid \mathsf{SN}(\rho) \le k\ \}$$

(k-separable states)

Separable states $= S_1 \subset S_2 \subset \ldots \subset S_d =$ all states

Theorem

If

$$\Phi \ :\ M_d \longrightarrow M_d \qquad \text{— } k\text{-positive}\ ,$$

then

$$(\mathbb{1} \otimes \Phi)(S_k) \ge 0\ .$$

$$S_k = \{ \rho \mid \mathsf{SN}(\rho) \le k \}$$

(k-separable states)

$$\text{Separable states} \;=\; S_1 \;\subset\; S_2 \;\subset \ldots \subset\; S_d \;=\; \text{all states}$$

Theorem

If

$$\Phi \;:\; M_d \;\longrightarrow\; M_d \qquad \text{— } k\text{-positive} \;,$$

then

$$(\mathbb{1} \otimes \Phi)(S_k) \ge 0 \;.$$

$$\Phi \;:\; M_d \longrightarrow M_d \qquad \text{— Quantum Channel}$$

Schmidt number of Φ

$$\text{SN}(\Phi) := \text{SN}(\rho_\Phi)$$

r-Partially Entanglement Breaking Channel

$$\text{SN}\Big((\mathbb{1} \otimes \Phi)\rho\Big) \leq r\ , \quad \text{for all states } \rho$$

Theorem

$$\Phi \;\text{ is }\; r\text{-PEB} \quad \Longleftrightarrow \quad \text{SN}(\Phi) \leq r$$

$$\text{EB} \;=\; 1\text{-PEB} \;\subset\; 2\text{-PEB} \;\subset \ldots \subset\; d\text{-PEB} \;=\; \text{all quantum channels}$$

$$\Phi \; : \; M_d \longrightarrow M_d \qquad \text{— Quantum Channel}$$

Schmidt number of Φ

$$\mathrm{SN}(\Phi) := \mathrm{SN}(\rho_\Phi)$$

r-Partially Entanglement Breaking Channel

$$\mathrm{SN}\Big((\mathbb{1} \otimes \Phi)\rho\Big) \leq r \,, \quad \text{for all states } \rho$$

Theorem

$$\Phi \text{ is } r\text{-PEB} \iff \mathrm{SN}(\Phi) \leq r$$

$$\mathrm{EB} = 1\text{-PEB} \subset 2\text{-PEB} \subset \ldots \subset d\text{-PEB} = \text{all quantum channels}$$

$$\Phi \ : \ M_d \longrightarrow M_d \qquad \text{— Quantum Channel}$$

Schmidt number of Φ

$$\mathrm{SN}(\Phi) := \mathrm{SN}(\rho_\Phi)$$

r-Partially Entanglement Breaking Channel

$$\mathrm{SN}\Big((\mathbb{1} \otimes \Phi)\rho\Big) \le r\ , \quad \text{for all states } \rho$$

Theorem

$$\Phi \ \text{is} \ r\text{-PEB} \iff \mathrm{SN}(\Phi) \le r$$

$$\mathrm{EB} = 1\text{-PEB} \subset 2\text{-PEB} \subset \ldots \subset d\text{-PEB} = \text{all quantum channels}$$

$$\Phi \; : \; M_d \longrightarrow M_d \qquad \text{— Quantum Channel}$$

Schmidt number of Φ

$$\mathsf{SN}(\Phi) := \mathsf{SN}(\rho_\Phi)$$

r-Partially Entanglement Breaking Channel

$$\mathsf{SN}\Big((\mathbb{1} \otimes \Phi)\rho\Big) \leq r \; , \quad \text{for all states } \rho$$

Theorem

$$\Phi \;\text{ is }\; r\text{-PEB} \quad \Longleftrightarrow \quad \mathsf{SN}(\Phi) \leq r$$

$$\mathsf{EB} = 1\text{-PEB} \subset 2\text{-PEB} \subset \ldots \subset d\text{-PEB} = \text{all quantum channels}$$

$$\Phi \ : \ M_d \longrightarrow M_d \quad \text{— Quantum Channel}$$

Schmidt number of Φ

$$\mathrm{SN}(\Phi) := \mathrm{SN}(\rho_\Phi)$$

r-Partially Entanglement Breaking Channel

$$\mathrm{SN}\Big((\mathbb{1} \otimes \Phi)\rho\Big) \leq r \ , \quad \text{for all states } \rho$$

Theorem

$$\Phi \ \text{ is } \ r\text{-PEB} \iff \mathrm{SN}(\Phi) \leq r$$

$$\mathrm{EB} = 1\text{-PEB} \subset 2\text{-PEB} \subset \ldots \subset d\text{-PEB} = \text{all quantum channels}$$

Φ CP map & Λ positive map

$\Lambda \circ \Phi$ is not a quantum channel

Theorem

Φ r-PEB & Λ r-positive map

$\Lambda \circ \Phi$ is a quantum channel!!!

Φ CP map & Λ positive map

$\Lambda \circ \Phi$ is not a quantum channel

Theorem

Φ r-PEB & Λ r-positive map

$\Lambda \circ \Phi$ is a quantum channel!!!

$$\mathsf{SN}(\Phi) := \mathsf{SN}(\rho_\Phi)$$

$$\Phi(a) = \sum_\alpha K_\alpha \, a \, K_\alpha^*$$

Theorem

$$\mathsf{SN}(\Phi) = \min_{K_\alpha} \left\{ \max_\alpha \operatorname{rank} K_\alpha \right\}$$

$$\mathsf{SN}(\Phi) := \mathsf{SN}(\rho_\Phi)$$

$$\Phi(a) = \sum_\alpha K_\alpha \, a \, K_\alpha^*$$

Theorem

$$\mathsf{SN}(\Phi) = \min_{K_\alpha} \left\{ \max_\alpha \operatorname{rank} K_\alpha \right\}$$

$$\mathrm{SN}(\Phi) := \mathrm{SN}(\rho_\Phi)$$

$$\Phi(a) = \sum_\alpha K_\alpha \, a \, K_\alpha^*$$

Theorem

$$\mathrm{SN}(\Phi) = \min_{K_\alpha} \left\{ \max_\alpha \operatorname{rank} K_\alpha \right\}$$

Example 1

$$\Phi(a) = U\,a\,U^*$$

$$\operatorname{rank} U = d$$

$$\Phi \quad \text{— } d\text{-PEB} \quad \text{and NOT} \quad (d-1)\text{-PEB}$$

Example 1

$$\Phi(a) = U\, a\, U^*$$

$$\operatorname{rank} U = d$$

Φ — d-PEB and NOT $(d-1)$-PEB

Example 2

$$\Phi(a) = \sum_{\alpha} K_\alpha \, a \, K_\alpha^*$$

$$K_\alpha = \frac{1}{\sqrt{d}} U_\alpha \ ; \quad U_\alpha \in U(d) \ ; \quad \mathrm{Tr}(U_\alpha U_\beta^*) = \delta_{\alpha\beta}$$

$$\mathrm{rank}\, U_\alpha = d$$

$$\Phi(a) = \mathbb{I} \quad \Longrightarrow \quad (\mathbb{1} \otimes \Phi)\rho \ = \ \text{separable state !!!}$$

$$\Phi \quad \text{— 1-PEB = EB}$$

separable channel

Example 2

$$\Phi(a) = \sum_{\alpha} K_\alpha \, a \, K_\alpha^*$$

$$K_\alpha = \frac{1}{\sqrt{d}} U_\alpha \; ; \quad U_\alpha \in U(d) \; ; \quad \mathrm{Tr}(U_\alpha U_\beta^*) = \delta_{\alpha\beta}$$

$$\mathrm{rank}\, U_\alpha = d$$

$$\Phi(a) = \mathbb{I} \implies (\mathbb{1} \otimes \Phi)\rho \; = \; \text{separable state !!!}$$

$$\Phi \quad \text{— 1-PEB = EB}$$

separable channel

Example 2

$$\Phi(a) = \sum_{\alpha} K_\alpha \, a \, K_\alpha^*$$

$$K_\alpha = \frac{1}{\sqrt{d}} U_\alpha \; ; \quad U_\alpha \in U(d) \; ; \quad \mathrm{Tr}(U_\alpha U_\beta^*) = \delta_{\alpha\beta}$$

$$\mathrm{rank}\, U_\alpha = d$$

$$\Phi(a) = \mathbb{I} \implies (\mathbb{1} \otimes \Phi)\rho \; = \; \text{separable state !!!}$$

$$\Phi \quad \text{— 1-PEB = EB}$$

separable channel

Example 2

$$\Phi(a) = \sum_\alpha K_\alpha \, a \, K_\alpha^*$$

$$K_\alpha = \frac{1}{\sqrt{d}} U_\alpha \; ; \quad U_\alpha \in U(d) \; ; \quad \mathrm{Tr}(U_\alpha U_\beta^*) = \delta_{\alpha\beta}$$

$$\mathrm{rank}\, U_\alpha = d$$

$$\Phi(a) = \mathbb{I} \quad \Longrightarrow \quad (\mathbb{1} \otimes \Phi)\rho \;=\; \text{separable state !!!}$$

$$\Phi \quad \text{— 1-PEB = EB}$$

separable channel

Example 2

$$\Phi(a) = \sum_{\alpha} K_\alpha \, a \, K_\alpha^*$$

$$K_\alpha = \frac{1}{\sqrt{d}} U_\alpha \; ; \quad U_\alpha \in U(d) \; ; \quad \mathrm{Tr}(U_\alpha U_\beta^*) = \delta_{\alpha\beta}$$

$$\mathrm{rank}\, U_\alpha = d$$

$$\Phi(a) = \mathbb{I} \quad \Longrightarrow \quad (\mathbb{1} \otimes \Phi)\rho \;=\; \text{separable state !!!}$$

$$\Phi \quad \text{— 1-PEB = EB}$$

separable channel

Example 3 — controlling SN

$$\Phi_f(a) = \frac{d - df}{d^2 - 1} \operatorname{Tr} a \, \mathbb{I} \; + \; \frac{d^2 f - 1}{d^2 - 1} \, a$$

$0 \leq f \leq 1$ — control parameter

$$f = 1 \quad \longrightarrow \quad \Phi_1(a) = a$$

$$f = 0 \quad \longrightarrow \quad \Phi_0(a) = \frac{1}{d^2 - 1} \left[d \operatorname{Tr} a \, \mathbb{I} \; - \; a \right]$$

Example 3 — controlling SN

$$\Phi_f(a) = \frac{d - df}{d^2 - 1} \mathrm{Tr}\, a \, \mathbb{I} \; + \; \frac{d^2 f - 1}{d^2 - 1}\, a$$

$0 \leq f \leq 1$ — control parameter

$$f = 1 \quad \longrightarrow \quad \Phi_1(a) = a$$

$$f = 0 \quad \longrightarrow \quad \Phi_0(a) = \frac{1}{d^2 - 1}\left[d\, \mathrm{Tr}\, a \, \mathbb{I} \; - \; a \right]$$

Example 3 — controlling SN

$$\Phi_f(a) = \frac{d - df}{d^2 - 1} \,\mathrm{Tr}\, a \,\mathbb{I} \;+\; \frac{d^2 f - 1}{d^2 - 1}\, a$$

$0 \leq f \leq 1$ — control parameter

$$f = 1 \quad \longrightarrow \quad \Phi_1(a) = a$$

$$f = 0 \quad \longrightarrow \quad \Phi_0(a) = \frac{1}{d^2 - 1} \left[d \,\mathrm{Tr}\, a \,\mathbb{I} \;-\; a \right]$$

Example 3 — controlling SN

$$\Phi_f(a) = \frac{d - df}{d^2 - 1} \, \mathrm{Tr}\, a \, \mathbb{I} \; + \; \frac{d^2 f - 1}{d^2 - 1} \, a$$

$0 \leq f \leq 1$ — control parameter

Φ_f is r-PEB but NOT $(r-1)$-PEB

$$\frac{r-1}{d} < f \leq \frac{r}{d}$$

Conclusions

1. Quantum Channels are of primary importance in QM!
2. Schmidt number of a Quantum Channel
3. characterization of Quantum Channels
4. controlling entangling properties of Quantum Channels