Effects of imperfect noise correlations on decoherence-free subsystems: SU(2) diffusion model

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Depolarizing channel

- **Random unitary rotation of a qubit:**

$$|\psi\rangle = \cos\theta|\leftrightarrow\rangle + \sin\theta e^{i\phi}|\updownarrow\rangle \qquad \mathcal{E}(|\psi\rangle\langle\psi|) = \int \mathrm{d}U\, U|\psi\rangle\langle\psi|U^\dagger = \mathbb{1}/2$$

- **In long fibers the output polarization of a photon is completely random**

Collectively depolarizing channel

- N qubit depolarizing channel, where each qubit experience the same disturbance

$SU(2)$ Haar measure

$$\mathcal{E}(\rho_N) = \int \mathrm{d}U U^{\otimes N} \rho_N U^{\dagger \otimes N}$$

N qubit state

- The model applies e.g. to:

  - photons transmitted through a long fiber
  - spins ½ being sent through a slowly varying magnetic field
  - communication in the absence of reference frames

Stucture of the output state

- **Irreducible subspaces under the action of $U^{\otimes N}$:**

$$\mathcal{H}^{\otimes N} = \bigoplus_{j=0}^{N/2} \underbrace{\mathcal{H}_j \oplus \ldots \oplus \mathcal{H}_j}_{d_j \text{ times}} = \bigoplus_{j=0}^{N/2} \mathcal{H}_j \otimes \mathbb{C}_{d_j}$$

multiplicity subspace (decoherence free subsystem)

$$\boxed{\mathcal{T}(\rho) := \mathcal{E}(\rho) = \int \mathrm{d}U U^{\otimes N} \rho U^{\dagger \otimes N} = \bigoplus_{j=0}^{N/2} \frac{p_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho_j}$$

$$p_j = \mathrm{Tr}(P_j \rho) \qquad \rho_j = \frac{1}{p_j} \mathrm{Tr}_{\mathcal{H}_j}(P_j \rho P_j)$$

P_j - projection on $\mathcal{H}_j \otimes \mathbb{C}_{d_j}$

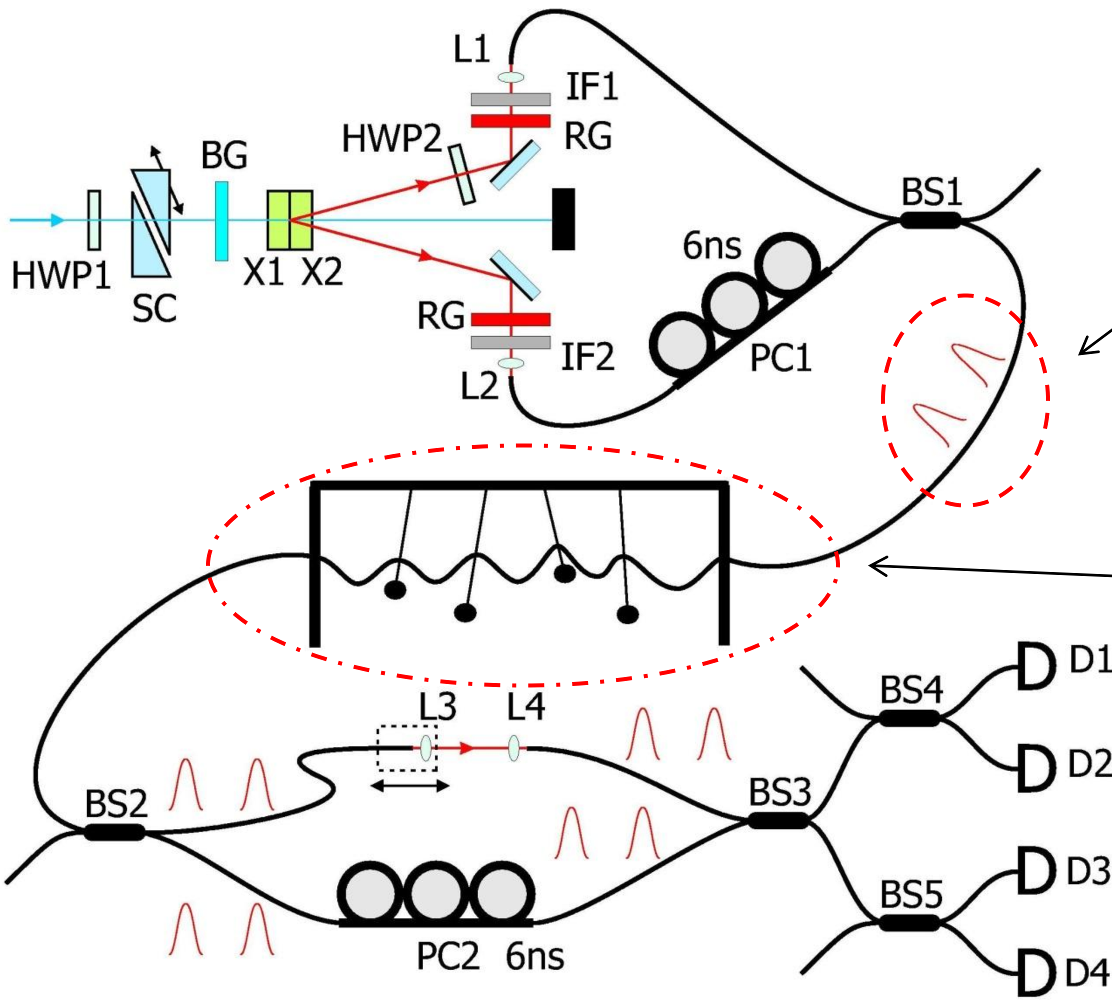
- **Faithfully transmitted states - allow for noiseless classical and quantum communication**

$$\rho = \bigoplus_{j=0}^{N/2} \frac{p_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho_j$$

twirled structure

Photons and salt bags

K. Banaszek, A. Dragan, W. Wasilewski, and C. Radzewicz, Phys. Rev. Lett. **92**, 257901 (2004)



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\updownarrow\leftrightarrow\rangle - |\leftrightarrow\updownarrow\rangle\right)$$

Salt bags = integrating with Haar measure

singlet state remains intact!

What happens if noise is not perfectly correlated?

In other words:

what happens if oscilation time of salt bags is comparable with photon separation time

Imperfectly correlated noise model

- Consecutive qubits experience slightly different rotations

$$\mathcal{E}(\rho) = \int \mathrm{d}U_1 \ldots \mathrm{d}U_N p(U_1, \ldots, U_N) U_1 \otimes \cdots \otimes U_N \ \rho \ U_1^\dagger \cdots \otimes U_N^\dagger$$

- The action described via a stationary Markov process

$$p(U_1, \ldots, U_N) = p(U_N | U_{N-1}) \ldots p(U_2 | U_1)$$

What is the natural choice for conditional probability?

$$p(U_i | U_{i-1}) = ?$$

Diffusion on the SU(2) group

- **Isotropic diffusion on SU(2)**

$$\partial_t p(U;t) = \frac{1}{2} D \, \hat{\Delta} p(U;t)$$

Laplace operator on the SU(2) group

- **Solution, with the initial condition:** $p(U;0) = \delta(U - \mathbb{1})$

$$p(U;t) = \sum_{j=0}^{\infty} (2j+1) \exp\left(-\frac{1}{2} j(j+1)\, t\right) \sum_{m=-j}^{j} \mathfrak{D}^j(U)_m^m$$

diffusion strength

rotation matrices

- **Conditional probability**

$$\boxed{p(U_i|U_{i-1}) = p(U_i U_{i-1}^{\dagger};t)}$$

$t \to 0$ perfect noise correlation

$t \to \infty$ no correlation

Action of the channel

- **Probability distribution for unitaries**

$$p(U_1,\ldots,U_N) \;=\; p(U_N|U_{N-1})\cdots p(U_2|U_1) \;=\; p(U_N U_{N-1}^\dagger;t)\ldots p(U_2^\dagger U_1;t)$$

- **The channel action**

$$\mathcal{E}(\rho) = \int \mathrm{d}U_1 \ldots \int \mathrm{d}U_N p(U_1,\ldots,U_N) U_1\otimes\cdots\otimes U_N \;\rho\; U_1^\dagger\otimes\cdots\otimes U_N^\dagger$$

$$\boxed{\mathcal{E}(\rho) = \mathcal{I}_{N-1}\left(\mathcal{I}_{N-2}\left(\ldots\mathcal{I}_1(\mathcal{T}(\rho))\ldots\right)\right)}$$

$$\mathcal{T}(\rho) = \int \mathrm{d}U U^{\otimes N}\rho U^{\dagger\otimes N}$$

$$\mathcal{I}_i(\rho) = \int \mathrm{d}U p(U;t)\,\underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{i}\otimes\underbrace{U\otimes\cdots\otimes U}_{N-i}\;\rho\;\underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{i}\otimes\underbrace{U^\dagger\otimes\cdots\otimes U^\dagger}_{N-i}$$

Action of the channel

- **Output states have a twirled structure ($\mathcal{T}, \mathcal{I}_i$ commute)**

$$\mathcal{E}(\rho) = \mathcal{I}_{N-1}\left(\mathcal{I}_{N-2}\left(\ldots\mathcal{I}_1(\mathcal{T}(\rho))\ldots\right)\right)$$

- **Input states can be restricted to have the twirled structure, so the channel action can be described as**

$$\bigoplus_{j=0}^{N/2} \frac{p_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho_j \xrightarrow{\mathcal{E}} \bigoplus_{j=0}^{N/2} \frac{p'_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho'_j$$

- **Using properties of rotation matrices $\mathfrak{D}^j(U)^m_{m'}$, it is possible to derive analytical expression for the action of the channel**

$$\mathcal{E}(\rho) = \text{lengthy expression involving } e^{-\frac{1}{2}j(j+1)t} \text{ and Wigner } 6j \text{ symbols}$$

Example: Three qubit channel

- **Structure of a three qubit twirled state**

$$\rho = \frac{p}{2}\underbrace{(\mathbb{1}_{\mathcal{H}_{1/2}} \otimes \rho_{1/2})} \oplus \frac{1-p}{4}\mathbb{1}_{\mathcal{H}_{3/2}}$$

effectively two dimensional subspace

$|0\rangle \quad (j_{12} = 0, j_{123} = 1/2)$

$|1\rangle \quad (j_{12} = 1, j_{123} = 1/2)$

one dimensional subspace

$|2\rangle \quad (j_{12} = 1, j_{123} = 3/2)$

- **We have a qutrit channel, with no coherence between $|0\rangle, |1\rangle$ subspace and $|2\rangle$**

- **If correlations of noise were perfect (no diffusion), the channel would allow for $\log_2 3$ bits of classical communication and 1 qubit of quantum communication**

Fidelity of transmitting a qubit

- **Transmitting a qubit**

$$|\psi\rangle = \cos(\theta/2)|e_1\rangle + \sin(\theta/2)e^{i\phi}|e_2\rangle$$

$$|e_1\rangle = (|0\rangle + \sqrt{3}|1\rangle)/2$$
$$|e_2\rangle = (\sqrt{3}|0\rangle - |1\rangle)/2$$

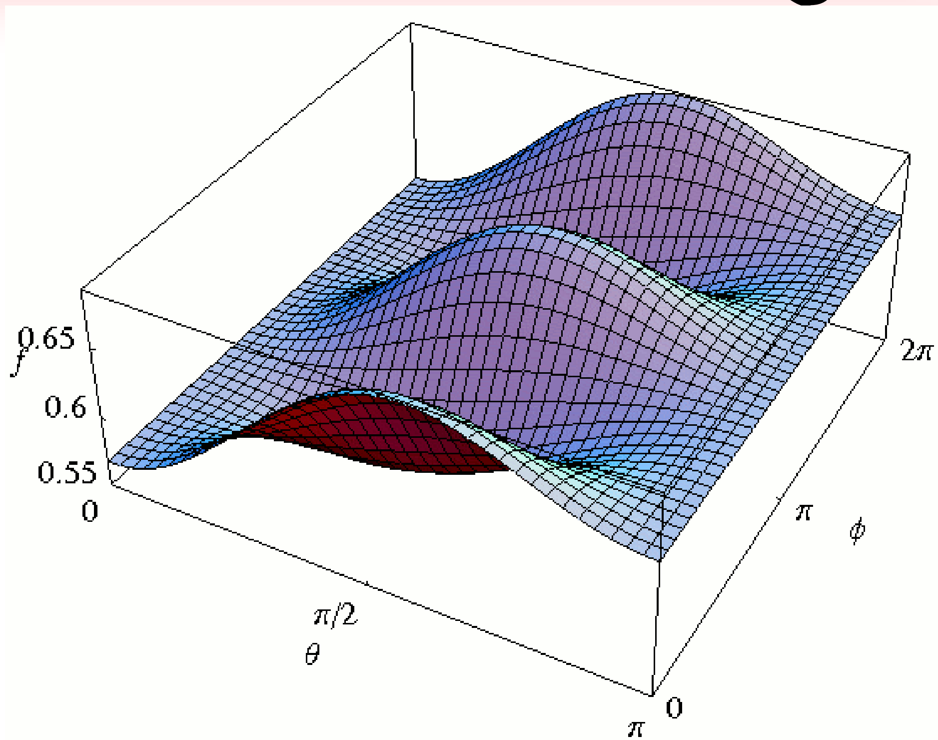
substituting the output state $|2\rangle$, with a maximally mixed state of a qubit we can write the effective qubit channel in terms of evolution of the Bloch vector

$$\mathbf{r}_{\text{out}} = \left(\begin{array}{ccc} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & \frac{2e^{-2t}+e^{-t}}{3} \end{array} \right) \mathbf{r}_{\text{in}} + \left(\begin{array}{c} 0 \\ 0 \\ \frac{e^{-2t}-e^{-t}}{3} \end{array} \right)$$

anisotropic shrinknig with displacement, states with $\phi = 0, \pi$ will tend to have high fidelity (weakest shrinking)

Fidelity of transmitting a qubit

for $t = 1$



$$\mathbf{r}_{\text{out}} = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & \frac{2e^{-2t}+e^{-t}}{3} \end{pmatrix} \mathbf{r}_{\text{in}} + \begin{pmatrix} 0 \\ 0 \\ \frac{e^{-2t}-e^{-t}}{3} \end{pmatrix}$$

anisotropic shrinknig with displacement, states with $\phi = 0, \pi$ will tend to have high fidelity (weakest shrinking)

Summary

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- Introduction of a natural model of an N qubit channel with imperfectly correlated random unitary rotations acting on consecutive qubits

- Derivation of an analytic formula for the action of the channel on an arbitrary input state

- Detailed analysis of the case N=3
  - fidelity of the channel
  - optimal classical capacity, and corresponding states
  - orthogonal states that provide almost optimal classical capacity even for non perfect noise correlations
  - threshold of diffusion strength above which coherent information vanishes

- Future work:
  - develop a perturbative approach for weak dffusion for large number of qubits, find optimal capacities and corresponding states
  - analyse within this framework "estimate and correct" startegy for sending information