GEOMETRY OF THE SPACE OF DENSITY STATES

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CONTENTS

- Geometric Quantum Mechanics.
- The space of density states.
- A simple problem.

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Collaboration with Giuseppe Marmo.

Results from: J. F. Cariñena, J. Grabowski, M. Kus, V. I. Manko, G. Sudarshan...

Based on: Kibble, Ashtekar, Brody and Hughston, Cirelly and co, Benvegnu and co

Geometric Quantum Mechanics

Usual Quantum Mechanics	**Geometric QM**
$\psi \in \mathcal{H}$	$\psi \in \mathcal{H}_{\mathbb{R}}$
$\dim_{\mathbb{C}} \mathcal{H} = n$	$\dim \mathcal{H}_{\mathbb{R}} = 2n$
$\langle \cdot, \cdot \rangle$ Hermitian	(g, ω, J) Kähler
$A : \mathcal{H} \to \mathcal{H}$	$\begin{cases} T_A : (\phi, \psi) \to (\phi, A\psi) \\ X_A : \psi \to (\psi, A\psi) \\ Y_A : \psi \to (\psi, JA\psi) \\ f_A(\psi) = \langle \psi, A\psi \rangle \quad e_A(\psi) = \frac{\langle \psi, A\psi \rangle}{\langle \psi, \psi \rangle} \end{cases}$
AB	$f_A \star f_B$
$AB + BA$	$f_{AB+BA} = G(df_A, df_B)$
$[A, B]$	$f_{[A,B]} = \{f_A, f_B\}$
	$f_A f_B$
Schrödinger eq.	Flow of Y_H
$A\lvert\psi\rangle = \lambda\lvert\psi\rangle$	$de_A(\psi) = 0 \quad e_A(\psi) = \lambda$

CONT

In Geometric Quantum Mechanics observables belong to $i\mathfrak{u}(\mathcal{H})$. By using the Killing-Cartan form

$$\langle A, B\rangle = \mathrm{Tr} AB \Rightarrow \langle A, \cdot\rangle \in \mathfrak{u}^*(\mathcal{H}) \Rightarrow \text{ Poisson}$$

This relation is one-to-one.

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A, B	$\hat{A}, \hat{B}$
$AB + BA$	$R(d\hat{A}, d\hat{B})(\xi) = \xi(AB + BA)$
$[A, B]$	$\Lambda(d\hat{A}, s\hat{B})(\xi) = \xi(AB - BA)$
AB	$\widehat{AB}(\xi) = \xi(AB) = \hat{A} \star \hat{B}$
Heisenberg eq.	$\dot{\hat{A}} = \frac{i}{\hbar}\{\hat{H}, \hat{A}\}$

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On $\mathfrak{u}^*(\mathcal{H})$ we can consider thus a Jordan product, a Poisson product and a nonlocal product which translate the properties of the algebra of observables of our quantum system.

UNIFYING FRAMEWORKS

Consider the natural action $U(N) \times \mathcal{H} \to \mathcal{H}$ and the corresponding momentum mapping

$$\mu : \mathcal{H} \to \mathfrak{u}^*(\mathcal{H}) \qquad \mu(\psi) = |\psi\rangle\langle\psi| = \rho_\psi$$

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PROJECTING

If we consider the complex projective space $\pi : \mathcal{H}_0 \to \mathcal{PH}$ everything works

$$\tilde{\mu} : \mathcal{PH} \to \mathfrak{u}^*(\mathcal{H}) \qquad \tilde{\mu}(\psi) = \frac{|\psi\rangle\langle\psi|}{\langle\psi, \psi\rangle} = \rho_{[\psi]}$$

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Lemma:

$$\tilde{\mu}_*(\pi_*(Y_H)) = \{\hat{H}, \cdot\}$$

The space of density matrices

μ embeds the set of pure states on $\mathfrak{u}^*(\mathcal{H})$. We can consider a basis $\{\rho_i\}$ of those, satisfying

$$\rho_k^2 = \rho_k \quad \rho_k^\dagger = \rho_k \quad \mathrm{Tr}\rho_k = 1$$

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We call **density states** to the set of convex combinations

$$\mathcal{D}(\mathcal{H}) = \left\{ \rho = \sum_k p_k \rho_k | \sum_k p_k = 1 \right\}$$

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LEMMA $\mathcal{D}(\mathcal{H})$ inherits the geometric structure from $\mathfrak{u}^*(\mathcal{H})$:

- The Jordan structure $R_\xi(X_A) = (\xi, [\xi, A]_+)$
- The Poisson structure (which is degenerate) $J_\xi(X_A) = (\xi, [\xi, A])$
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These objects define the corresponding distributions D_Λ and D_R. From them, we can define

$$D_0 = D_\Lambda \cap D_R \qquad D_1 = D_\Lambda + D_R$$

CONT

THEOREM $\mathcal{D}(\mathcal{H})$ is a stratified manifold with respect to the GL–action $(T,\rho) \to \frac{T\rho T^\dagger}{\mathrm{Tr}(T\rho T^\dagger)}$:

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ADVANTAGES

- The use of tools borrowed from Classical Mechanics
- Simplicity from the algebraic point of view
- Formulation of quantum control problems from a classical perspective.

A SIMPLE PROBLEM

Consider the case of a system of two qubits. The Hilbert space in this case is

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We can consider a natural basis for $\mathfrak{u}(4)$ (called sometimes Fano basis)

$$\zeta = \{\mathbb{I}_4, i\sigma_i \otimes \mathbb{I}_2, i\mathbb{I}_2 \otimes \sigma_i, i\sigma_i \otimes \sigma_j | i, j = 1, 2, 3\}$$

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A point $\rho \in \mathcal{D}(\mathbb{C}^4) \subset \mathfrak{u}^*(4)$ will be represented thus by a set $\{\lambda_0, m_i, n_i, r_{ij}\}_{i,j=1,2,3}$ of real numbers:

$$\rho = i\lambda_0 \mathbb{I}_4 + i\sum_i (m_i \sigma_i \otimes \mathbb{I}_2 + n_i \mathbb{I}_2 \otimes \sigma_i) + i\sum_{ij} (m_i n_j + r_{ij})\sigma_i \otimes \sigma_j$$

or also

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On this set we can consider the geometrical structures we defined above: the Jordan and the Poisson tensors and write them in these coordinates (the dual ones, on $\mathfrak{u}^*(4)$)

CONT

Excluding the y_0 coordinate which trivially is a Casimir of the structure we obtain

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	y_{13}	y_{14}	y_{15}
y_1	0	y_3	$-y_2$	0	0	0	0	0	0	y_{13}	y_{14}	y_{15}	y_{10}	y_{11}	y_{12}
y_2	$-y_3$	0	y_1	0	0	0	y_{13}	y_{14}	y_{15}	0	0	0	y_7	y_8	y_9
y_3	y_2	$-y_1$	0	0	0	0	y_{10}	y_{11}	y_{12}	$-y_7$	$-y_8$	$-y_9$	0	0	0
y_4	0	0	0	0	y_6	$-y_5$	0	y_9	y_8	0	y_{12}	$-y_{11}$	0	y_{15}	$-y_{14}$
y_5	0	0	0	$-y_6$	0	y_4	$-y_9$	0	y_7	$-y_{12}$	0	y_{10}	$-y_{15}$	0	y_{13}
y_6	0	0	0	y_5	$-y_4$	0	y_8	$-y_7$	0	y_{11}	$-y_{10}$	0	y_{14}	$-y_{13}$	0
y_7	0	y_{13}	$-y_{10}$	0	y_9	$-y_8$	0	y_6	$-y_5$	y_3	0	0	$-y_2$	0	0
y_8	0	y_{14}	$-y_{11}$	$-y_9$	0	y_7	$-y_6$	0	y_4	0	y_9	0	0	$-y_2$	0
y_9	0	y_{15}	$-y_{12}$	y_8	$-y_7$	0	y_5	$-y_4$	0	0	0	y_9	0	0	$-y_2$
y_{10}	$-y_{13}$	0	y_7	0	y_{12}	$-y_{11}$	$-y_9$	0	0	0	y_6	$-y_5$	y_1	0	0
y_{11}	$-y_{14}$	0	y_8	$-y_{12}$	0	y_{10}	0	$-y_9$	0	$-y_6$	0	y_4	0	y_1	0
y_{12}	$-y_{15}$	0	y_9	y_{11}	$-y_{10}$	0	0	0	y_3	y_5	$-y_4$	0	0	0	y_1
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ENTROPY AND CONCURRENCE

Consider two functions defined on our set of density matrices $\mathcal{D}(\mathcal{H})$:

- Von Neumann entropy $S(\rho) = \mathrm{Tr}\rho \log \rho$
- concurrence $C(\rho) = \max(0, 2\lambda_{max}(\hat{\rho}) - \mathrm{Tr}(\hat{\rho})$, where $\hat{\rho}$ is defined as $\hat{\rho} = \sqrt{\rho}\sqrt{\rho(\sigma_2 \otimes \sigma_2)\rho^*(\sigma_2 \otimes \sigma_2)}\sqrt{\rho}$.

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From the geometrical point of view they are just functions

$$S : \mathcal{D}(\mathcal{H}) \to \mathbb{R} \qquad C : \mathcal{D}(\mathcal{H}) \to \mathbb{R},$$

thus we can consider the problem of their independence.

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DEFINITION: Two functions $f_1, f_2 \in \mathcal{F}(\mathcal{D}(\mathcal{H}))$ are said to be independent at a point $p \in M$ iff

$$(df_1 \wedge df_2)(p) \neq 0 \Leftrightarrow \left(\frac{\partial f_1}{\partial \lambda_i}\frac{\partial f_2}{\partial \lambda_j} - \frac{\partial f_1}{\partial \lambda_j}\frac{\partial f_2}{\partial \lambda_i}\right)(p) \neq 0 \quad \forall i, j$$

This is the problem we would like to study now.

CONT

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$$\mathcal{S} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & \frac{1}{2}ce^{i\phi} & 0 \\ 0 & \frac{1}{2}ce^{-i\phi} & b & 0 \\ 0 & 0 & 0 & 1-a-b \end{pmatrix} ; 0 \leq a+b \leq 1 \quad 0 \leq c \leq 1 \quad 4ab \geq c^2 \right\}$$

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$\mathcal{S}$ is clearly a 4–dimensional submanifold of $\mathcal{D}(\mathcal{H})$. We can take an adapted basis for S, considering the matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$
\begin{aligned}
2S(\rho) = &-2(-1+a+b)\log[1-a-b]+ \\
&\left(a+b-\sqrt{(a-b)^2+c^2}\right)\log\left[\frac{1}{2}\left(a+b-\sqrt{(a-b)^2+c^2}\right)\right]+ \\
&\left(a+b+\sqrt{(a-b)^2+c^2}\right)\log\left[\frac{1}{2}\left(a+b+\sqrt{(a-b)^2+c^2}\right)\right]. \\
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$$\left(a+b+\sqrt{(a-b)^2+c^2}\right)\log\left[\frac{1}{2}\left(a+b+\sqrt{(a-b)^2+c^2}\right)\right].$$
$$C(\rho_t) = c$$

LEMMA

The concurrence and the von Neumann entropy on $\mathcal{S}$ are functionally independent everywhere but on a submanifold $\mathcal{I}$ of dimension 2.

PROOF If we compute the expression $(dS \wedge dC)(\rho)$ for $\rho \in \mathcal{S}$ we obtain

$$dS(\rho) = \frac{\partial S}{\partial a}da + \frac{\partial S}{\partial b}db + \frac{\partial S}{\partial c}dc$$
$$dC(\rho) = \frac{\partial C}{\partial c}dc$$

Thus

$$(dS \wedge dC)(\rho) = 0 \Leftrightarrow \frac{\partial S(\rho)}{\partial a} = 0 = \frac{\partial S(\rho)}{\partial b}$$

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And these conditions become

$$2\text{Log}[1-a-b]+\text{Log}\left[ab - \frac{c^2}{4}\right] = 0 \quad \text{Log}\left[4ab - c^2\right]-2\text{Log}\left[a + b + \sqrt{(a-b)^2 + c^2}\right] = 0$$

Thus

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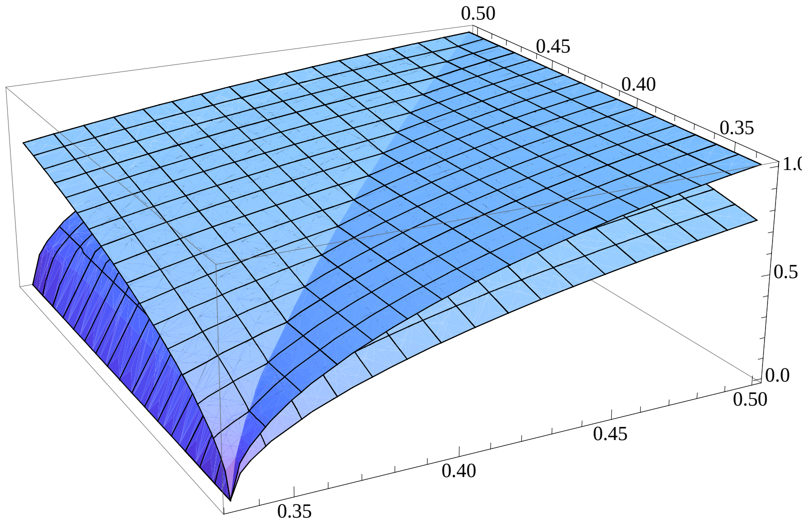
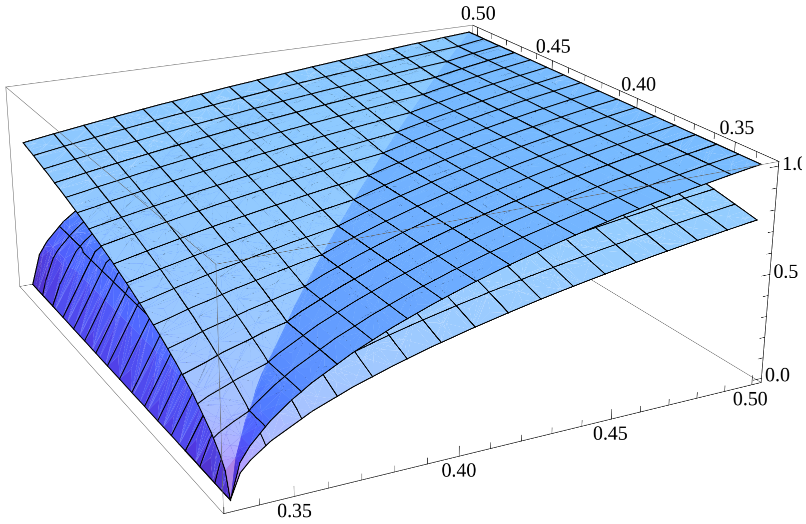
These equations have a solution on

$$\frac{1}{3} < a < \frac{1}{2}; \qquad b = a; \qquad c = \sqrt{-4 + 8a - 4a^2 + 8b - 4ab - 4b^2}$$

If we represent the condition for c as a function of a and b we verify that it is well defined for all values of a and b (we represent the function $c = \sqrt{-4 + 8a - 4a^2 + 8b - 4ab - 4b^2}$ and the function $c = \sqrt{-4 + 16a - 12a^2}$, which corresponds to the evaluation on the submanifold $a = b$).



Thus there is a submanifold $\mathcal{I}$ (cylinder-like) on $\mathcal{S}$ where the two functions are dependent. Everywhere else, they are independent functions.



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LEMMA The two functions $C(\rho)$ and $S(\rho)$ Poisson-commute on $\mathcal{S}$

PROOF: Direct computation.

FUTURE WORK

- Study the relation of this notion of independence with the action of local unitary transformations
- The behavior with respect to the Poisson/Jordan algebra structures.
- Simple test for pure states.