Approximants of generalized minimizers and degree of singularity of noncoercive optimal control problems

Manuel Guerra* and Andrey Sarychev**

*CEOC and ISEG, TULisbon (Portugal)

**University of Florence (Italy)

CCQ Bedlewo 2007

Motivation: coercive optimization problem

Theorem

Consider an Hilbert space $\mathcal{H}$ and a functional $J : \mathcal{H} \mapsto \mathbb{R}$ such that:

(1) $\exists C_1 \in \mathbb{R}, C_2 > 0,\ J(u) \geq C_1 + C_2 \|u\|^2, \qquad \forall u \in \mathcal{H};$

(2) *J is weakly lower semicontinuous.*

J **has a minimizer in** $\mathcal{H}$.

Motivation: coercive optimization problem

Theorem

Consider an Hilbert space $\mathcal{H}$ and a functional $J : \mathcal{H} \mapsto \mathbb{R}$ such that:

(1) $\exists C_1 \in \mathbb{R}, C_2 > 0,\ J(u) \geq C_1 + C_2 \|u\|^2, \qquad \forall u \in \mathcal{H};$

(2) *J is weakly lower semicontinuous.*

J has a minimizer in $\mathcal{H}$.

Optimal control problem

$$J(u) = \int_0^T \ell(x(t), u(t))\, dt \to \min$$
$$\dot{x}(t) = f(x(t), u(t)), \qquad u \in \mathcal{U}$$
$$x(0) = x_0, \qquad x(T) = x_T$$

Optimal control problem

$$J^T(u(\cdot)) = \int_0^T x(t)'Px(t)dt \to \min,$$
$$\dot{x} = f(x) + G(x)u, \qquad x(0) = x_0, \qquad x(T) = x_T.$$

$T \in]0, +\infty[$,

$P \in \mathbb{R}^{n\times n}$ *symmetric definite positive,*

f *smooth vector field,*

$G = (g_1, g_2, ..., g_k)$ *array of smooth vector fields.*

Optimal control problem

$$J^T(u(\cdot)) = \int_0^T x(t)'Px(t)dt \to \min,$$
$$\dot{x} = f(x) + G(x)u, \qquad x(0) = x_0, \qquad x(T) = x_T.$$

$T \in]0, +\infty[$,

$P \in \mathbb{R}^{n\times n}$ *symmetric definite positive,*

f *smooth vector field,*

$G = (g_1, g_2, \ldots, g_k)$ *array of smooth vector fields.*

Due to lack of coercivity, "classical" (L_∞) minimizers do not, in general, exist.

Minimizing sequences

Generalized solutions = "limits" of minimizing sequences

Minimizing sequences

Generalized solutions = *"limits" of minimizing sequences*

- Generalized controls typically contain impulses or more complex singularities.

Minimizing sequences

Generalized solutions = "limits" of minimizing sequences

- Generalized controls typically contain impulses or more complex singularities.
- Quasioptimal controls exhibit high-gain and/or highly-oscillatory behavior.
  (Quasioptimal steering of the system requires "large" controls)

Minimizing sequences

Generalized solutions = "limits" of minimizing sequences

- Generalized controls typically contain impulses or more complex singularities.
- Quasioptimal controls exhibit high-gain and/or highly-oscillatory behavior.
  (Quasioptimal steering of the system requires "large" controls)

Questions:

- *What is the generalized solution for a given problem?*
- *How "large" must be a control in order to $\varepsilon-$approximate the optimal solution?*

Minimizing sequences

Remark

The connection between the commutativity/noncommutativity of inputs and generalized minimizers is an established fact.

See e.g.: Bressan (1987), Orlov (1988), Sarychev (1991), Bressan & Rampazzo(1994).

Minimizing sequences

Remark

The connection between the commutativity/noncommutativity of inputs and generalized minimizers is an established fact.

See e.g.: Bressan (1987), Orlov (1988), Sarychev (1991), Bressan & Rampazzo(1994).

Definition

$$\mu = \limsup_{\varepsilon \to 0^+} \frac{\inf \left\{ \ln \|u\|_{L_2} : J^T(u) \leq \inf J^T + \varepsilon, \; |x_u(T) - x_T| \leq \varepsilon \right\}}{\ln \frac{1}{\varepsilon}}.$$

The singular linear-quadratic case

$$\begin{aligned} J^T(u) &= \int_0^T x_u' P x_u + 2u' Q x_u + u' R u \, d\tau \to \min, \\ \dot{x} &= Ax + Bu, \qquad x(0) = x_0, \qquad x(T) = x_T. \end{aligned}$$

$R \in \mathbb{R}^{k \times k}$ symmetric nonnegative, $\ker(R) \neq \{0\}$.

The singular linear-quadratic case

$$\begin{aligned} J^T(u) &= \int_0^T x_u' P x_u + 2u' Q x_u + u' R u \, d\tau \to \min, \\ \dot{x} &= Ax + Bu, \qquad x(0) = x_0, \qquad x(T) = x_T. \end{aligned}$$

$R \in \mathbb{R}^{k \times k}$ symmetric nonnegative, $\ker(R) \neq \{0\}$.

Theorem (Jurdjevic, 1997)

The generalized optimal trajectory consists of:

- *An initial "jump", $x(0^+) - x_0 \in \mathcal{J}$;*
- *An analytical arc, $\quad x(t), \; t \in]0, T[$;*
- *A final "jump", $x_T - x(T^-) \in \mathcal{J}$.*

$\mathcal{J}$: space of jump directions.

The singular linear-quadratic case

Theorem (Guerra, 2000)

If $\inf J^T > -\infty$ *for some boundary conditions and some* $T \in]0, +\infty[$ *then there exists an integer* $r \leq n$ *such that:*

- *the map* $u \mapsto J^T(u)$ *admits one unique continuous extension into a certain subspace* $\mathcal{U} \subset H_{-r}[0, T]$*;*
- *If* $\inf J^T > -\infty$ *then the problem admits a minimizer* $\hat{u} \in \mathcal{U}$.

The singular linear-quadratic case

Theorem (Guerra, 2000)

If $\inf J^T > -\infty$ *for some boundary conditions and some* $T \in]0, +\infty[$ *then there exists an integer* $r \leq n$ *such that:*

- *the map* $u \mapsto J^T(u)$ *admits one unique continuous extension into a certain subspace* $\mathcal{U} \subset H_{-r}[0, T]$*;*
- *If* $\inf J^T > -\infty$ *then the problem admits a minimizer* $\hat{u} \in \mathcal{U}$*.*

Proposition

For generic boundary conditions, the degree of singularity is $\mu = r - \frac{1}{2}$*.*

The singular linear-quadratic case

Proposition

The possible values of μ are:

$$-\infty,\ 0,\ \frac{i+1/2}{2(j-i)-1}, \qquad 0 \le i < j \le r. \tag{1}$$

The numbers (1) correspond to a stratification of the space of boundary conditions $\mathbb{R}^{2n}$. Lower-dimensional strata correspond to smaller values of μ.

The singular linear-quadratic case

Proposition

The possible values of μ are:

$$-\infty,\ 0,\ \frac{i+1/2}{2(j-i)-1}, \qquad 0 \le i < j \le r. \tag{1}$$

The numbers (1) correspond to a stratification of the space of boundary conditions $\mathbb{R}^{2n}$. Lower-dimensional strata correspond to smaller values of μ.

$i =$ *order of Lie brackets between Ax and B;*

$j =$ *index of Sobolev space H_{-j}.*

The singular linear-quadratic case

Proposition

The possible values of μ *are:*

$$-\infty,\ 0,\ \frac{i+1/2}{2(j-i)-1}, \qquad 0 \le i < j \le r. \tag{1}$$

The numbers (1) correspond to a stratification of the space of boundary conditions $\mathbb{R}^{2n}$*. Lower-dimensional strata correspond to smaller values of* μ*.*

$i =$ *order of Lie brackets between* Ax *and* B*;*

$j =$ *index of Sobolev space* H_{-j}*.*

Remark

If $J(u) = \int_0^T x'Px\,dt$*,* $P > 0$*, then* $r = 1$ *and (1) reduces to* $\left\{-\infty, 0, \frac{1}{2}\right\}$*.*

The driftless case

$$J^T(u) = \int_0^T x(t)'Px(t)dt \to \min, \qquad P > 0,$$

$$\dot{x} = \sum_{j=1}^{r} g_j u_j, \qquad x(0) = x_0, \qquad x(T) = x_T, \tag{2}$$

The driftless case

$$J^T(u) = \int_0^T x(t)'Px(t)dt \to \min, \qquad P > 0,$$
$$\dot{x} = \sum_{j=1}^{r} g_j u_j, \qquad x(0) = x_0, \qquad x(T) = x_T, \tag{2}$$

Theorem

Let $\mathcal{A}_{x_0}$ be the orbit of (2) and

$$\alpha = \inf\{x'Px|\ x \in \mathcal{A}_{x_0}\}\ (\alpha \geq 0). \tag{3}$$

Provided $x_T \in \mathcal{A}_{x_0}$:

i) $\inf J^T = \alpha T$;

ii) $\mu \geq \frac{1}{2}$ *unless* $x_0'Px_0 = x_T'Px_T = \alpha$;

iii) *if the infimum (3) is attained then* $\mu \leq \frac{1}{2}$.

Sketch of the proof

(Suppose $\{g^1, \ldots, g^r\}$ has complete Lie rank)
$\inf J^T = 0$, the generalized optimal trajectory consists of three 'pieces':

- an initial 'jump' from $x(0) = x_0$ to $x(0^+) = 0$;
- a constant piece $x(t) \equiv 0, \ t \in]0, T[$;
- a final 'jump' from $x(T^-) = 0$ to the end point $x(T) = x_T$.

Sketch of the proof

(Suppose $\{g^1, \dots, g^r\}$ has complete Lie rank)
$\inf J^T = 0$, the generalized optimal trajectory consists of three 'pieces':

- an initial 'jump' from $x(0) = x_0$ to $x(0^+) = 0$;
- a constant piece $x(t) \equiv 0, \ t \in]0, T[$;
- a final 'jump' from $x(T^-) = 0$ to the end point $x(T) = x_T$.

The jumps can be approximated by describing reference (fixed) trajectories in arbitrarily small intervals (length$=\varepsilon$) of time. Then

$$J^T(u_\varepsilon) = O(\varepsilon), \quad \|u_\varepsilon\|_{L_2[0,T]} = O\left(\frac{1}{\sqrt{\varepsilon}}\right) \qquad as\ \varepsilon \to 0.$$

Sketch of the proof

(Suppose $\{g^1, \ldots, g^r\}$ has complete Lie rank)
$\inf J^T = 0$, the generalized optimal trajectory consists of three 'pieces':

- an initial 'jump' from $x(0) = x_0$ to $x(0^+) = 0$;
- a constant piece $x(t) \equiv 0, \ t \in]0, T[$;
- a final 'jump' from $x(T^-) = 0$ to the end point $x(T) = x_T$.

The jumps can be approximated by describing reference (fixed) trajectories in arbitrarily small intervals (length$= \varepsilon$) of time. Then

$$J^T(u_\varepsilon) = O(\varepsilon), \quad \|u_\varepsilon\|_{L_2[0,T]} = O\left(\frac{1}{\sqrt{\varepsilon}}\right) \qquad as\ \varepsilon \longrightarrow 0.$$

In order to go from x_0 to the set $A = \left\{x \in \mathbb{R}^n : x'Px \leq \frac{1}{2}x_0'Px_0\right\}$ in time $t \leq \varepsilon$ the control must satisfy

$$\|u_\varepsilon\|_{L_2[0,T]} \geq \frac{d(x_0, A)}{\max\limits_{1 \leq i \leq k, |x| \leq |x_0|} |g_i(x)|} \frac{1}{\sqrt{\varepsilon}}.$$

Control-affine systems: commuting inputs

$$J^T(u) = \int_0^T x(t)' P x(t) dt \to \min,$$
$$\dot{x} = f(x) + G(x)u, \qquad x(0) = x_0, \qquad x(T) = x_T.$$

Assumptions

The fields f, g_i, $i = 1, 2, ..., k$ are complete and the controlled fields commute, i.e., $[g_i, g_j] \equiv 0$ holds for all i, j.

Desingularization

Notation

- e^{tF} *flow generated by the smooth field* F

  ($t \mapsto e^{tF}x_0$ *is the unique solution of* $\quad \dot{x} = F(x), \quad x(0) = x_0$);
- $AdPF(x) = (DP(x))^{-1}F(P(x))$;
- $\phi u(t) = \int_0^t u(\tau)\, d\tau$.

Desingularization

Notation

- e^{tF} *flow generated by the smooth field* F

  ($t \mapsto e^{tF}x_0$ *is the unique solution of* $\dot{x} = F(x), \quad x(0) = x_0$*);*
- $AdPF(x) = (DP(x))^{-1}F(P(x))$;
- $\phi u(t) = \int_0^t u(\tau)\,d\tau$.

Theorem (Agrachev & Sarychev, 1987)

$$x_u(t) = e^{G\phi u(t)} y_{\phi u}(t), \qquad \forall t \in [0, T],\ u \in L_\infty[0, T],$$

$$\dot{x}_u(t) = f(x_u(t)) + G(x_u(t))u(t), \qquad x(0) = x_0,$$

$$\dot{y}_{\phi u}(t) = \left(Ad\left(e^{G\phi u(t)}\right) f\right)(y_{\phi u}(t)), \qquad y(0) = x_0.$$

Reduced problem

The substitution $x = e^{G\phi u(t)}y$ leads to the 'desingularized' problem

$$\begin{aligned} J_r(v) &= \int_0^T \left(e^{Gv(t)}y(t)\right)' P \left(e^{Gv(t)}y(t)\right) dt \rightarrow \min, \\ \dot{y}(t) &= \left(Ad\left(e^{Gv(t)}\right) f\right)(y(t)), \\ y(0) &= x_0, \qquad y(T) = e^{GV}x_T, \quad V \in \mathbb{R}^k. \end{aligned}$$

Reduced problem

The substitution $x = e^{G\phi u(t)}y$ leads to the 'desingularized' problem

$$
\begin{aligned}
J_r(v) &= \int_0^T \left(e^{Gv(t)}y(t)\right)' P \left(e^{Gv(t)}y(t)\right) dt \to \min, \\
\dot{y}(t) &= \left(Ad\left(e^{Gv(t)}\right) f\right)(y(t)), \\
y(0) &= x_0, \qquad y(T) = e^{GV}x_T, \quad V \in \mathbb{R}^k.
\end{aligned}
$$

Remark

When g_i, $i = 1, 2, \ldots, k$ are constant we have:

$$
\begin{aligned}
e^{Gv}y &= y + Gv, \qquad \left(Ad\left(e^{Gv}\right) f\right)(y) = f(y + Gv). \\
J_r(v) &= \int_0^T y(t)'Py(t) + 2v(t)'G'PGv(t) + v(t)'G\prime PGv(t)\, dt.
\end{aligned}
$$

Nonconvexity of the reduced problem

$$\left(\tilde{f}_0(y,v), \tilde{f}(y,v)\right) = \left(\left(e^{Gv}y\right)' P \left(e^{Gv}y\right), \left(Ad\left(e^{Gv}\right)f\right)(y)\right)$$

Remark

For generic $y \in \mathbb{R}^n$ (fixed) the set

$$\Gamma(y) = \left\{\left(y_0, \tilde{f}(y,v)\right) : y_0 \geq \tilde{f}_0(y,v),\ v \in \mathbb{R}^k\right\} \subset \mathbb{R} \times \mathbb{R}^n$$

is nonconvex (even in the case when G is constant).

Nonconvexity of the reduced problem

$$\left(\tilde{f}_0(y,v), \tilde{f}(y,v)\right) = \left(\left(e^{Gv}y\right)' P \left(e^{Gv}y\right), \left(Ad\left(e^{Gv}\right)f\right)(y)\right)$$

Remark

For generic $y \in \mathbb{R}^n$ (fixed) the set

$$\Gamma(y) = \left\{\left(y_0, \tilde{f}(y,v)\right) : y_0 \geq \tilde{f}_0(y,v),\ v \in \mathbb{R}^k\right\} \subset \mathbb{R} \times \mathbb{R}^n$$

is nonconvex (even in the case when G is constant).

Classical minimizers for the reduced problem typically fail to exist.
Instead, existence of *relaxed* minimizers can be expected.

Relaxed minimizers

- A relaxed control is as a family $t \mapsto \eta_t$ of inner regular probability measures with compact support in $\mathbb{R}^k$ such that $t \mapsto \eta_t$ is measurable in the weak sense with respect to $t \in [0, T]$.

Relaxed minimizers

- A relaxed control is as a family $t \mapsto \eta_t$ of inner regular probability measures with compact support in $\mathbb{R}^k$ such that $t \mapsto \eta_t$ is measurable in the weak sense with respect to $t \in [0, T]$.
- Extension of the reduced problem achieves convexification of the epigraphs $\Gamma(y),\ y \in \mathbb{R}^n$.

Relaxed minimizers

- A relaxed control is as a family $t \mapsto \eta_t$ of inner regular probability measures with compact support in $\mathbb{R}^k$ such that $t \mapsto \eta_t$ is measurable in the weak sense with respect to $t \in [0, T]$.
- Extension of the reduced problem achieves convexification of the epigraphs $\Gamma(y), \ y \in \mathbb{R}^n$.

Remark

Extension into the class of relaxed controls **does not preserve coercivity**.

The convex hulls of the epigraphs

$$\Gamma(y) = \left\{ \left(y_0, \tilde{f}(y,v)\right) : y_0 \geq \tilde{f}_0(y,v), \ v \in \mathbb{R}^m \right\}, \quad y \in \mathbb{R}^n$$

may fail to be closed.
In that case a "generalized minimizer" may be achievable only by taking directions lying in $\overline{\mathrm{conv}}\,(\Gamma(y))$ but not in $\mathrm{conv}\,(\Gamma(y))$.

Existence of relaxed minimizers

Theorem (Cesari, 1983)

(Under suitable growth conditions imposed on $|f|$, $|g_1|$, $|g_2|$, ..., $|g_k|$)
The reduced problem has a **relaxed minimizer**.

Existence of relaxed minimizers

Theorem (Cesari, 1983)

(Under suitable growth conditions imposed on $|f|$, $|g_1|$, $|g_2|$, ..., $|g_k|$)
The reduced problem has a **relaxed minimizer**.

However:

Cesari's optimal trajectories are in general **non-Lipschitzian** *with respect to time!*

Existence of relaxed minimizers

Theorem (Cesari, 1983)

(Under suitable growth conditions imposed on $|f|$, $|g_1|$, $|g_2|$, ..., $|g_k|$)
The reduced problem has a **relaxed minimizer**.

However:

Cesari's optimal trajectories are in general **non-Lipschitzian** *with respect to time!*

$\Rightarrow$ *Cesari's minimizers may fail to satisfy the Pontryagin Maximum Principle.*

Existence of relaxed minimizers

Theorem (Cesari, 1983)

(Under suitable growth conditions imposed on $|f|$, $|g_1|$, $|g_2|$, ..., $|g_k|$)
The reduced problem has a **relaxed minimizer**.

However:

Cesari's optimal trajectories are in general **non-Lipschitzian** *with respect to time!*

- $\Rightarrow$ *Cesari's minimizers may fail to satisfy the Pontryagin Maximum Principle.*
- $\Rightarrow$ *Cesari's minimizers are very difficult to characterize.*

Existence of relaxed minimizers

Assumptions

For any compact set $K \subset \mathbb{R}^n$ *we have*

$$\lim_{|v|\to+\infty} \left|e^{Gv}\right| = +\infty \qquad \textit{and} \qquad \lim_{|v|\to+\infty} \frac{\left|\frac{\partial}{\partial x}\left(\left(e^{Gv}x\right)' P \left(e^{Gv}x\right)\right)\right|}{\left|e^{Gv}x\right|^2} = 0,$$

uniformly with respect to $x \in K$*, and there exists a function* $\gamma : [0, +\infty[\mapsto \mathbb{R}$ *bounded below, such that:*

i) $\displaystyle\lim_{s\to+\infty} \frac{\gamma(s)}{s} = +\infty;$

ii) $\displaystyle\left|e^{Gv}x\right|^2 \geq \gamma\left(\left|\left(Ad\left(e^{Gv}\right)f\right)(x)\right| + \left|\frac{\partial}{\partial x}\left(Ad\left(e^{Gv}\right)f\right)(x)\right|\right),$

$$\forall (x, v) \in K \times \mathbb{R}^k.$$

Existence of relaxed minimizers

Remark

When the fields g_i, $i = 1, 2, ..., k$ *are constant the assumptions reduce to*

$$\begin{aligned} &\mathrm{rank}(G) = k \\ &|v|^2 \geq \gamma \left(|f(x + Gv)| + |Df(x + Gv)| \right), \qquad \forall (x, v) \in K \times \mathbb{R}^k \end{aligned}$$

Existence of relaxed minimizers

Remark

When the fields g_i, $i = 1, 2, ..., k$ are constant the assumptions reduce to

$$\text{rank}(G) = k$$
$$|v|^2 \geq \gamma \left(|f(x + Gv)| + |Df(x + Gv)| \right), \qquad \forall (x, v) \in K \times \mathbb{R}^k$$

Remark

It is sufficient to consider Gamkrelidze generalized controls

$$\eta_t = \sum_{j=1}^{n+2} p_j(t)\delta_{v^j(t)}, \qquad \sum_{j=1}^{n+2} p_j \equiv 1, \qquad p_j(t) \geq 0,$$

$$\dot{y}(t) = \sum_{j=1}^{n+2} p_j(t) \left(Ad \left(e^{Gv^j(t)} \right) f \right) (y(t)).$$

Existence of relaxed minimizers

Theorem

(under the assumptions above)

- *The reduced problem has a minimizer in the class of Gamkrelidze controls.*

Existence of relaxed minimizers

Theorem

(under the assumptions above)

- *The reduced problem has a minimizer in the class of Gamkrelidze controls.*
- *All Gamkrelidze minimizers satisfy PMP.*

Existence of relaxed minimizers

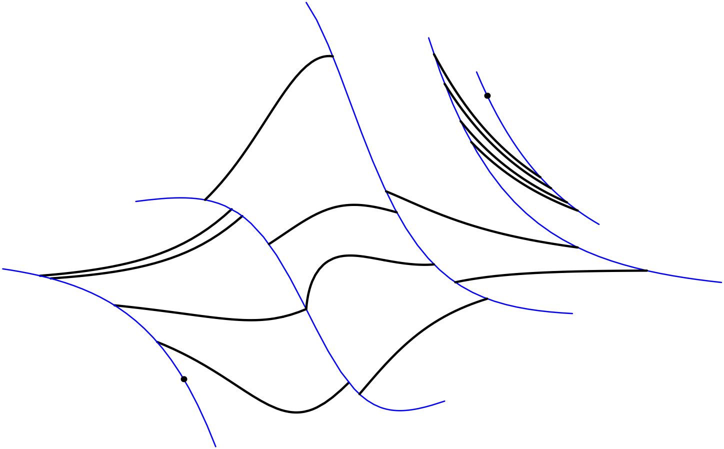
Theorem

(under the assumptions above)

- *The reduced problem has a minimizer in the class of Gamkrelidze controls.*
- *All Gamkrelidze minimizers satisfy PMP.*
- *All Gamkrelidze minimizers which are not Lipschitzian must correspond to strictly abnormal extremals.*

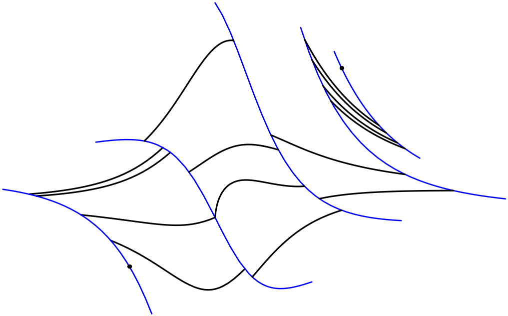
$$\sum_{j=1}^{n+2} p_j(t) \left| \left(Ad \left(e^{Gv^j(t)} \right) f \right) (y_\eta(t)) \right| < M \qquad a.e. t \in [0, T]$$

Generalized optimal trajectories



(jumps along curves tangent to $\{g_1, g_2, \ldots, g_k\}$)

Generalized optimal trajectories



$$\tilde{x}(t) = (J^t,\, x(t))$$

$$\tilde{f}(x) = (x'Px,\, f(x))$$

$$\tilde{g}_i(x) = (0,\, g_i(x))$$

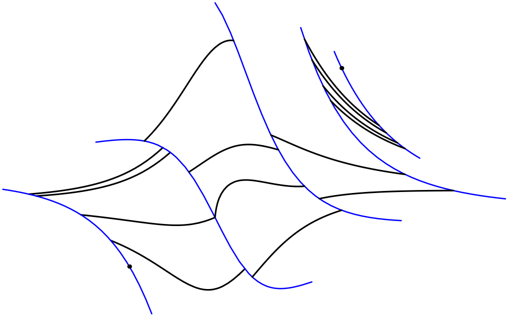
$$\tilde{q} = \left(q^0,\, q\right), \quad q^0 \in \{0, -1\}$$

$$H(\tilde{x}, \tilde{q}, u) = \left\langle \tilde{q}, \tilde{f}(x) + \tilde{G}(x)u \right\rangle$$

$$H_{Gv}(\tilde{x}, \tilde{q}) = \langle q, G(x)v \rangle$$

$$\langle q, g_i \rangle = 0, \quad 0 \leq i \leq k$$

Generalized optimal trajectories



$$\tilde{x}(t) = (J^t, x(t))$$
$$\tilde{f}(x) = (x' P x,\ f(x))$$
$$\tilde{g}_i(x) = (0,\ g_i(x))$$
$$\tilde{q} = \left(q^0,\ q\right), \quad q^0 \in \{0, -1\}$$
$$H(\tilde{x}, \tilde{q}, u) = \langle \tilde{q}, \tilde{f}(x) + \tilde{G}(x) u \rangle$$
$$H_{Gv}(\tilde{x}, \tilde{q}) = \langle q, G(x) v \rangle$$
$$\langle q, g_i \rangle = 0, \quad 0 \le i \le k$$

$$(\tilde{x}_i, \tilde{q}_i) = e^{\overrightarrow{H}_{G(v_i - v_j)}} (\tilde{x}_j, \tilde{q}_j), \qquad H(\tilde{x}_i, \tilde{q}_i, 0) = \max_{v \in \mathbb{R}^k} H\left(e^{\overrightarrow{H}_{Gv}} (\tilde{x}_j, \tilde{q}_j), 0\right)$$
$$\left\langle \tilde{q}_i, [\tilde{f}, \tilde{g}_s](x_i) \right\rangle = 0, \qquad \left(\left\langle \tilde{q}_i, [\tilde{g}_s, [\tilde{f}, \tilde{g}_m]](x_i) \right\rangle\right)_{1 \le s \le k, 1 \le m \le k} \ge 0$$
$$(\dot{\tilde{x}}_i, \dot{\tilde{q}}_i) = \sum_{j=1}^{n+2} p_j Ad\left(e^{\overrightarrow{H}_{G(v_j - v_i)}}\right) \overrightarrow{H}(\tilde{x}_i, \tilde{q}_i, u^j)$$
$$\left\langle \tilde{q}_i, [f + G u^i, [f, g_s]](x_i) \right\rangle = 0, \quad 1 \le s \le k$$

Degree of singularity: the commutative case

Theorem

If the generalized optimal trajectory is bounded, then

$$\mu \leq 3/2.$$

Approximation of generalized minimizers

To prove that $\mu \leq \frac{3}{2}$ we use two approximation steps:

- approximate the relaxed minimizer of the reduced problem by piecewise continuous controls w_ε such that the trajectory and the functional driven by w_ε, are ε-close to the trajectory and the functional driven by the relaxed minimizer.

  Number of discontinuities of w_ε: $\sim \frac{1}{\varepsilon}$.

Approximation of generalized minimizers

To prove that $\mu \leq \frac{3}{2}$ we use two approximation steps:

- approximate the relaxed minimizer of the reduced problem by piecewise continuous controls w_ε such that the trajectory and the functional driven by w_ε, are ε-close to the trajectory and the functional driven by the relaxed minimizer.

  Number of discontinuities of w_ε: $\sim \frac{1}{\varepsilon}$.

- approximate w_ε by an *absolutely continuous* control $v_\varepsilon(\cdot)$ such that the trajectory and the functional driven by v_ε, is ε-close to the trajectory and the functional driven by w_ε.

  v_ε differs from w_ε at $\sim \frac{1}{\varepsilon}$ intervals of length ε^2.

Approximation of generalized minimizers

To prove that $\mu \leq \frac{3}{2}$ we use two approximation steps:

- approximate the relaxed minimizer of the reduced problem by piecewise continuous controls w_ε such that the trajectory and the functional driven by w_ε, are ε-close to the trajectory and the functional driven by the relaxed minimizer.

  Number of discontinuities of w_ε: $\sim \frac{1}{\varepsilon}$.

- approximate w_ε by an *absolutely continuous* control $v_\varepsilon(\cdot)$ such that the trajectory and the functional driven by v_ε, is ε-close to the trajectory and the functional driven by w_ε.

  v_ε differs from w_ε at $\sim \frac{1}{\varepsilon}$ intervals of length ε^2.

$$\mu \leq \lim_{\varepsilon \to 0^+} \frac{\ln \|\dot{v}_\varepsilon\|_{L_2[0,T]}}{\ln \frac{1}{\varepsilon}}.$$

Conjecture

Suppose that:

- *the fields* f*,* g_i*,* $i = 1, 2, \ldots, k$ *are complete and* $[g_i, g_j] \equiv 0$ *holds for all* i, j*.*
- *the generalized trajectory is bounded.*

Then $\mu \leq 1$*.*

Input-commutative control-affine systems: a conjecture

- Our two-step approximation procedure can be improved?
  - there exists a piecewise continuous control w_ε with $\leq O(\varepsilon^{-1})$ intervals of continuity, such that the **end-point** of the trajectory and the value of the functional driven by w_ε are ε^2-close to the **end-point** of the optimal trajectory and the corresponding value of the functional?

If yes, then by modifying w_ε in intervals of length ε^3 instead of ε^2 we obtain a family of square-integrable controls $u_\varepsilon = \frac{dv_\varepsilon}{dt}$ satisfying the estimate $\|u_\varepsilon\|_{L_2} = O(\varepsilon^{-2})$.

- Our two-step approximation procedure can be improved?
  - there exists a piecewise continuous control w_ε with $\leq O(\varepsilon^{-1})$ intervals of continuity, such that the **end-point** of the trajectory and the value of the functional driven by w_ε are ε^2-close to the **end-point** of the optimal trajectory and the corresponding value of the functional?

  If yes, then by modifying w_ε in intervals of length ε^3 instead of ε^2 we obtain a family of square-integrable controls $u_\varepsilon = \frac{dv_\varepsilon}{dt}$ satisfying the estimate $\|u_\varepsilon\|_{L_2} = O(\varepsilon^{-2})$.

- Can the second approximation step be improved?

Input-commutative control-affine systems: a conjecture

Example

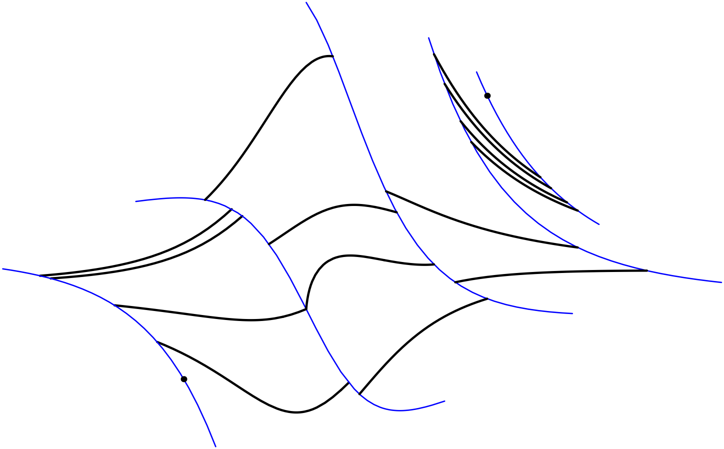
$$
\begin{aligned}
J^1 &= \int_0^1 (x_1^2 + x_2^2 + x_3^2)\,dt \to \min, \\
f(x_1, x_2, x_3) &= x_1 \frac{\partial}{\partial x_2} + \gamma(x_1)(x_1^2 - 1)\frac{\partial}{\partial x_3}, \qquad g_1 = \frac{\partial}{\partial x_1} \\
x(0) &= 0, \qquad x(1) = 0,
\end{aligned}
$$

$\gamma : \mathbb{R} \mapsto [0, 1]$ is smooth, $\mathrm{supp}(\gamma) \subset [-2, 2]$ and $\gamma(x) \equiv 1$ on $[-3/2, 3/2]$.

There exists a piecewise continuous control w_ε with $\leq O(\varepsilon^{-1})$ intervals of continuity that ε^2-approximate the optimal trajectory and the infimum of the functional.

Hence $\mu \leq 1$.

The non-commutative case



(jumps along curves tangent to $\mathcal{L}\{g_1, g_2, ..., g_k\}$)

Example

$$J^1 = \int_0^1 \sum_{i=1}^{5} x_i^2 \, dt \to \min,$$

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2, \qquad x = (x_1, x_2, x_3, x_4, x_5),$$

$$f(x) = x_5 \frac{\partial}{\partial x_2} + \gamma(x_5)(x_5^2 - 1)\frac{\partial}{\partial x_3}, \quad g_1(x) = \frac{\partial}{\partial x_1},$$

$$g_2(x) = \frac{\partial}{\partial x_4} + x_1 \frac{\partial}{\partial x_5},$$

$$x(0) = 0, \qquad x(1) = 0.$$

$$span\mathcal{L}(G) = span\{g_1, g_2, [g_1, g_2]\} = span\{e_1, e_4, e_5\}$$

$$\dot{y} = f(y + e_1 w_1 + e_4 w_4 + e_5 w_5)$$

$$\dot{y}_2 = y_5 + w_5 \qquad \dot{y}_3 = \gamma(y_5 + w_5)\left(y_5^2 + 2y_5 w_5 + w_5^2 - 1\right).$$

Non-commutative control-affine case: Example 2

Example

$$J(u) = \int_0^1 \sum_{i=1}^{4} x_i(t)^2 \, dt;$$

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = u_2, \qquad \dot{x}_3 = x_2 u_1, \qquad \dot{x}_4 = a_1 x_1 + a_2 x_2 + a_3 x_3,$$

$$x(0) = x_0, \qquad x(1) = x_T.$$

$$a_1, a_2, a_3 \in \mathbb{R}.$$

Non-commutative control-affine case: Example 2

Example

$$J(u) = \int_0^1 \sum_{i=1}^{4} x_i(t)^2 \, dt;$$

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = u_2, \qquad \dot{x}_3 = x_2 u_1, \qquad \dot{x}_4 = a_1 x_1 + a_2 x_2 + a_3 x_3,$$

$$x(0) = x_0, \qquad x(1) = x_T.$$

$$a_1, a_2, a_3 \in \mathbb{R}.$$

For both examples we obtain the same estimate: $\mu \leq 1$!