Controllability of finite-dimensional quantum systems

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Identification, Optimization and Control with Applications in Modern Technologies

Outline

- Motivation: Quantum Control of Spin Systems
- Example: Two-level, spin-$\frac{1}{2}$ system
- Controllability on Homogeneous Spaces
- Classification of Transitive Lie Group Actions
- Main Results A : Accessibility of Quantum Systems
- Main Results B : Accessibility of Double Bracket Flows

Motivation: Quantum Control of Spin Systems

N-Level Quantum Systems

- N-**level quantum systems** are described by their **density operators** ρ satisfying

  (a) $\rho \in \mathbb{C}^{N\times N} \quad , \quad \rho = \rho^\dagger$

  (b) $\mathrm{Tr}(\rho) = 1$

  (c) $\rho \geq 0$

- **Fact:** The set $\mathcal{P}$ of all density operators forms a **convex set**.
- Example : n **spin-$\frac{1}{2}$ systems** are 2^n-level systems.

Motivation: Quantum Control of Spin Systems

Completely positive operators

- A linear map $\Phi : \mathcal{P} \to \mathcal{P}$ is **completely positive** if and only if $I_N \otimes \Phi$ is positivity preserving.
- The set CP of completely positive operators is a **semialgebraic Lie-semigroup** of $GL(N^2)$.
- CP operates **transitively** on $\mathcal{P}$.
- The **Lindblad-Kossakowski** operators are infinitesimal generators of one-parameter semigroups in CP.

Motivation: Quantum Control of Spin Systems

Dynamics of N-Level Open Quantum Systems

- **Lindblad Master Equation**

$$\dot{\rho} = -\mathrm{i}\Big[H_d + \sum_{k=1}^{m} u_k(t) H_k, \rho\Big] + \mathcal{L}(\rho), \quad \rho(0) = \rho_0 \in \mathcal{P}$$

- The Lindblad term $\mathcal{L}$ models relaxation / dissipation,

$$\mathcal{L}(\rho) = \sum_{k=1}^{p} [\lambda_k \rho, \lambda_k^\dagger] + [\lambda_k, \rho\lambda_k^\dagger].$$

- The Lindblad term $\mathcal{L}$ is **unital** iff $\mathcal{L}(I) = 0$, e.g. when

$$\lambda_k \lambda_k^\dagger = \lambda_k^\dagger \lambda_k.$$

Motivation: Quantum Control of Spin Systems

- **Optimal Control Problem :** Maximize the trace function

$$\mathrm{tr}(\mathrm{C}^\dagger \rho(\mathrm{T}))$$

subject to the Lindblad Master Equation

$$\dot{\rho} = -\mathrm{i}\Big[H_d + \sum_{k=1}^{r} u_k(t) H_k, \rho\Big] + \sum_{k=1}^{p} [\lambda_k \rho, \lambda_k^\dagger] + [\lambda_k, \rho \lambda_k^\dagger]$$

Goals:

- **Characterize the reachable sets** $\mathcal{R}(\rho_0)$!
- **Optimize** $\mathrm{tr}(\mathrm{C}^\dagger \rho)$ **on reachable sets** $\mathcal{R}(\rho_0)$!
- **Solve the optimal control problem with terminal constraints**!

Motivation: Quantum Control of Spin Systems

Basic Structural Questions:

- Is the system semigroup of the Lindblad-Kossakowski equation a **semialgebraic** Lie semigroup?
- Are the reachable sets of the Lindblad-Kossakowski equation **semialgebraic**?
- **When is the system semigroup dense in CP?**
  Answer: Never.
- **When is the Lindblad equation controllable?**
  Answer: Never.

Motivation: Quantum Control of Spin Systems

Different Notions of Controllability / Accessibility

- **Unit Sphere / Projective Space:** Pure State Controllability

$$(\Sigma') \qquad \dot{\psi} = -\mathrm{i}\Big(H_d + \sum_{k=1}^{m} u_k(t) H_k\Big)\psi, \quad \psi(0) = \psi_0 \in S^{2N-1}$$

- **Unitary Orbit:** Density Operator Controllability

$$(\widehat{\Sigma}) \qquad \dot{\rho} = -\mathrm{i}\Big[H_d + \sum_{k=1}^{m} u_k(t) H_k, \rho\Big], \quad \rho(0) = \rho_0 \in \mathcal{P}$$

- **Special Unitary Group:** Operator Controllability

$$(\Sigma) \qquad \dot{U} = -\mathrm{i}\Big(H_d + \sum_{k=1}^{m} u_k(t) H_k\Big)U, \quad U(0) = I_N$$

Motivation: Quantum Control of Spin Systems

Known Results: Liouville Equation

$$\dot{\rho} = -\mathrm{i}\Big[H_d + \sum_{k=1}^{m} u_k(t) H_k, \rho\Big].$$

- **Characterization of pure state controllability** (spheres/projective spaces) : Allesandro, Albertini, et al.
- **Characterization of reachable sets**!
  - **Specific n-level systems:** (operator controllability: S.-Herbrüggen, Brockett-Khaneja; general case: Albertini, Allesandro)
- **Riemannian optimization algorithms for the trace function** (Dirr, He, Morar, Schulte-Herbrüggen)

Connections: Brockett's work; gen. C-numerical ranges; tensor SVD

Motivation: Quantum Control of Spin Systems

Controllability of Liouville Equation

- The Liouville Master equation for n spin-$\frac{1}{2}$ systems

$$\dot{\rho} = -\mathrm{i}\Big[H_d + \sum_{j=1}^{m} u_j(t) H_j, \rho\Big]$$

  is **controllable on each isospectral set** $\{U\rho U^* | U \in SU(2^n)\}$, iff the spin-spin coupling graph is connected.

- **Controllability on rank 1 projection operators** holds if and only if the system Lie algebra is isomorphic to $\mathfrak{su}(2^n)$ or to $\mathfrak{sp}(2^n)$ (Albertini, D'Alessandro 2003).

Motivation: Quantum Control of Spin Systems

Known Results: Lindblad Equation

$$\dot{\rho} = -\mathrm{i}\Big[H_d + \sum_{k=1}^{m} u_k(t) H_k, \rho\Big] + \sum_{k=1}^{p} [\lambda_k \rho, \lambda_k^\dagger] + [\lambda_k, \rho \lambda_k^\dagger].$$

- **Characterization of accessibility: unknown!**
- **Wrong Claims: Reachable sets form annulus** (Altafini 04)

  In general meaningless except perhaps

  - for 2-level systems (the simplest case),
  - with additional assumption of "fast-controllability" on the Hamiltonian part (without Lindblad and drift term).
- **Optimization problem: unsolved!**

Example: Spin-$\frac{1}{2}$ System

Example: One-Spin System

Example : two-level systems, Bloch vector equation

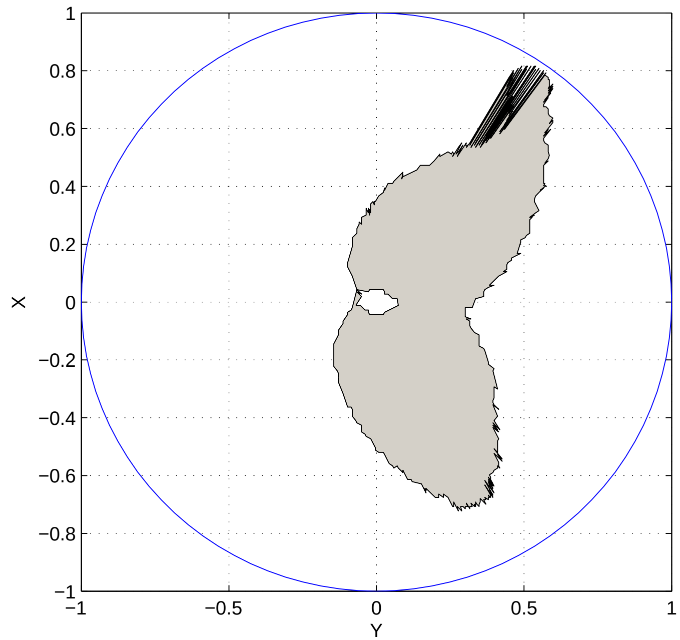
$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\gamma_1 & -\delta & u(t) \\ \delta & -\gamma_2 & 0 \\ -u(t) & 0 & -\gamma_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \gamma_0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

δ : drift $\qquad \gamma_k \geq 0$: Lindblad relaxation

- **Due to relaxation, the system is never controllable!**
- **Accessibility holds if and only if $\delta \neq 0, (\gamma_1, \gamma_2, \gamma_3) \neq 0$.**
- *Generalization: Khaneja/Stefanatos*
- *Special Case: Sugny, Kontz, Jauslin (2007)*

Example: One-Spin System

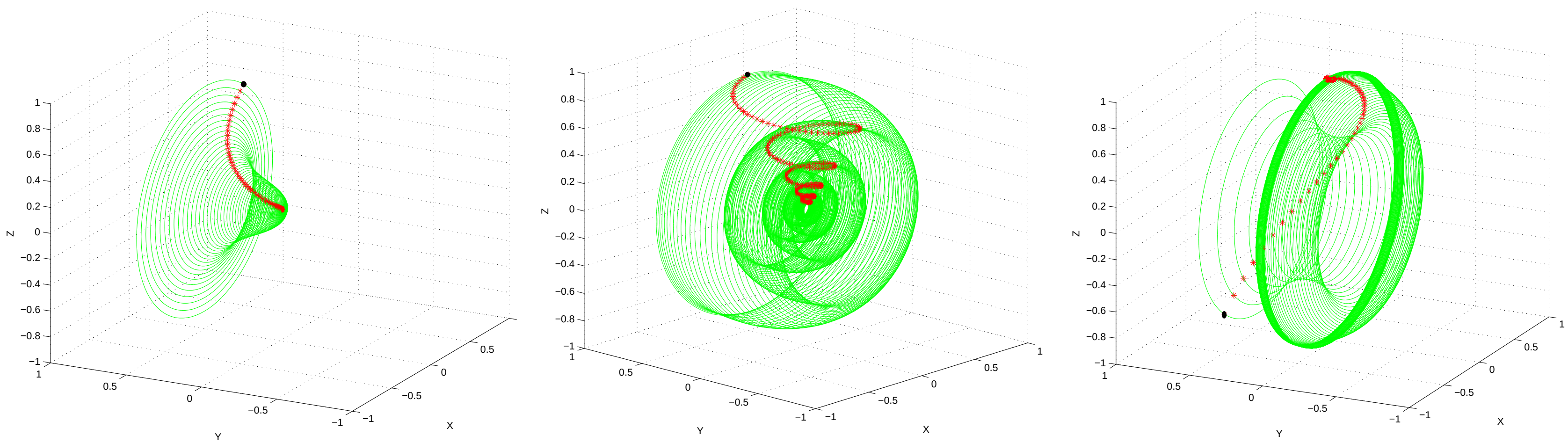
Slice through the reachable set for a single spin system.



Reachable set is not an annulus!!

Example: One-Spin System

Instantaneous Reachable Sets



- left : $\gamma_1 = \gamma_2 = \gamma_3 = \delta, \;\; \gamma_0 = 0$ (unital)
- center : $\gamma_1 = \gamma_2 = \gamma_3 = 0.1\delta, \;\; \gamma_0 = 0$ (unital)
- right : $\gamma_0 = \gamma_1 = \gamma_2 = 2\gamma_3 = \delta, \;\; \gamma_0 \neq 0$ (nonunital)

Example: One-Spin System

Optimal Control Problem

- Bloch vector equation, $v = [x\ y\ z]^\top$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\gamma_1 & -\delta & u(t) \\ \delta & -\gamma_2 & 0 \\ -u(t) & 0 & -\gamma_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \gamma_0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- **Optimal control task** : given initial condition $v(0)$ and fixed terminal time T, maximize the function

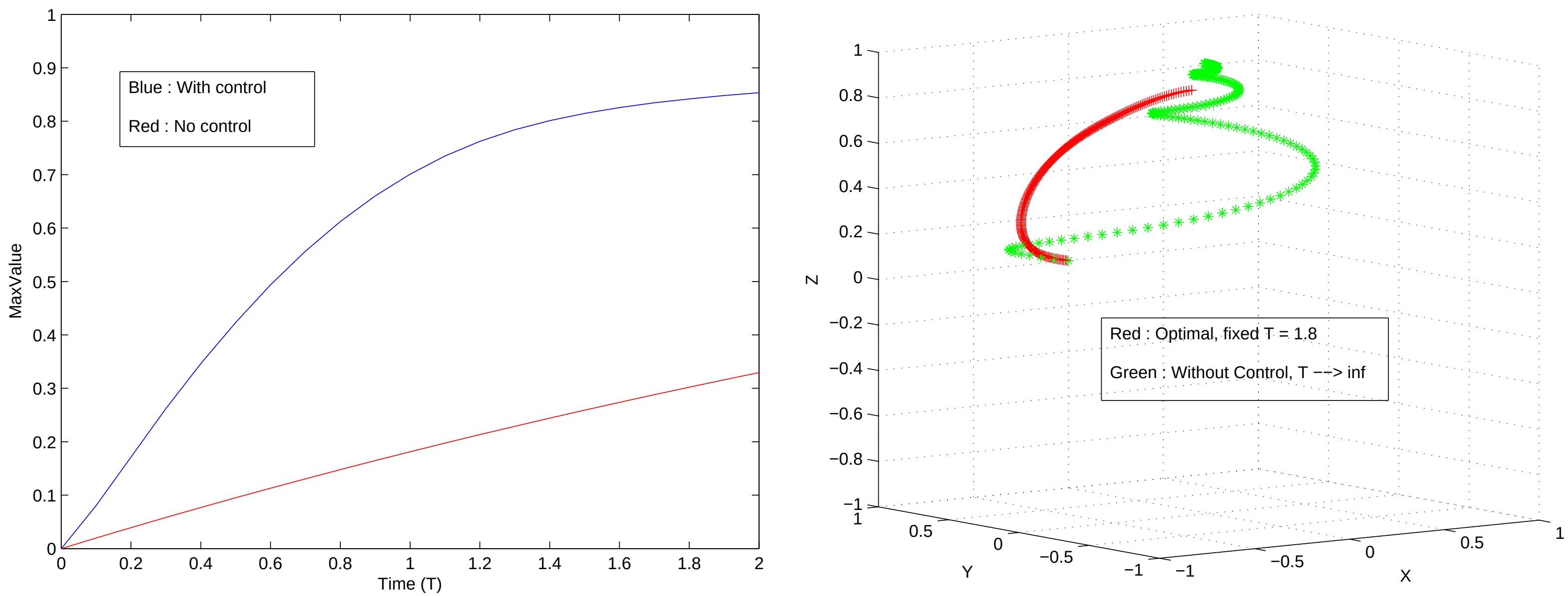
$$f_T = c^\top v(T) \quad , \quad c = [0\ 0\ 1]^\top$$

via an optimal control $u(t)$.

Motivation: Quantum Control of Spin Systems

Numerical optimization for non-unital case

- $\gamma_0 = \gamma_1 = \gamma_2 = \gamma_3 = 0.2,\ \delta = 1,\ v(0) = [-\frac{1}{2} \quad \frac{1}{2} \quad -\frac{\sqrt{2}}{2}]^\top$

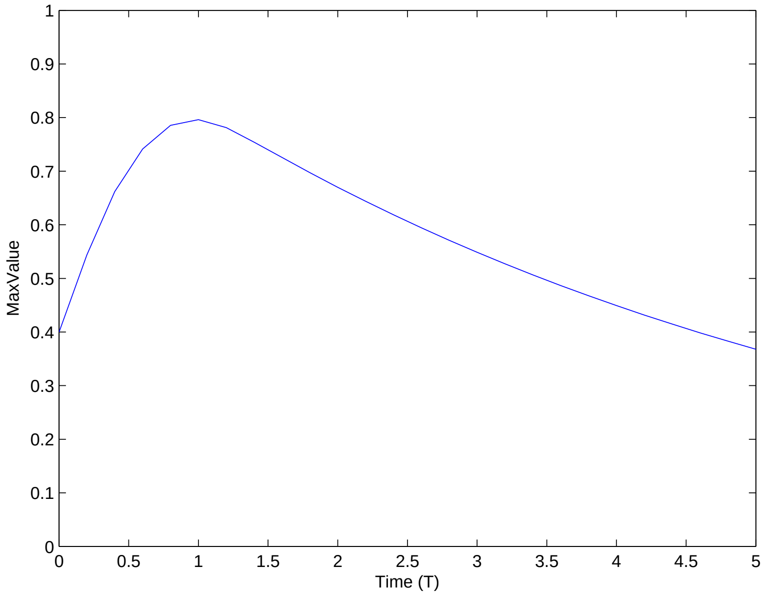


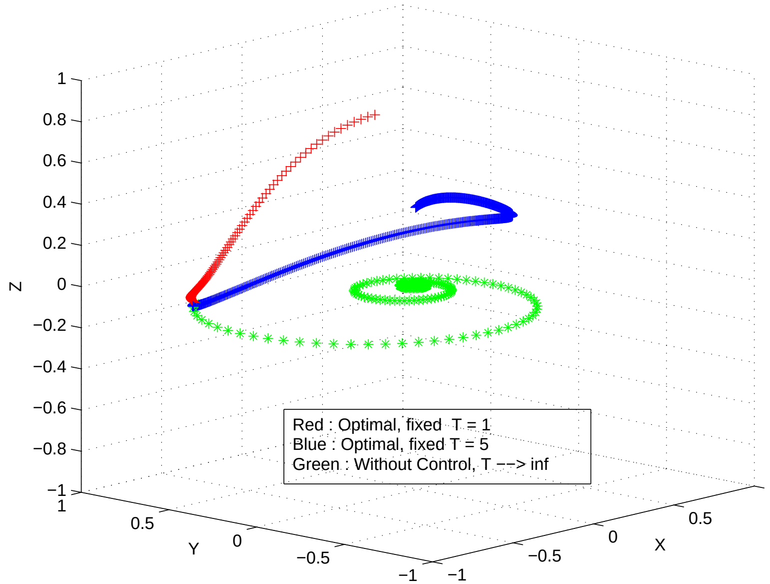
- The value $f_T = 85\%$ can be achieved at $T = 2$
  $f_T = 1$ only as $T \to \infty$ (even with optimal control)

Example: One-Spin System

Numerical optimization for unital case

- $\gamma_0 = 0,\ \gamma_1 = \gamma_2 = \gamma_3 = 0.2,\ \delta = 1,\ v(0) = [-\frac{4}{10} \quad \frac{9}{10} \quad -\frac{\sqrt{3}}{10}]^\top$





- The maximum value $f_T = 80\%$ can only be achieved at $T = 1$
  When $T \neq 1$, the maximum value $f_T < 80\%$

Controllability on Homogeneous Spaces

Controllability on Homogeneous Spaces

History

- **Nonlinear Control Theory**: Controllability, observability, ...
  (Brockett, Krener, Kupka, Sussmann, Jurdjevic, ...)
- **Lie Theory of Semigroups**: Lie wedges; cones in Lie algebras, ...
  (Hilgert, Hoffmann, Lawson, Ol'shanskii, ...)
- **Transformation Groups**: Classification of transitive actions
  (Montgomery/Samelson, Boothby/Wilson, Völklein, Kramer)

Controllability on Homogeneous Spaces

Bilinear Control System

- G connected matrix Lie group with Lie algebra $\mathfrak{g}$.
- **Bilinear control system** on G

$$(\Sigma) \quad \dot{X} = \left(A_d + \sum_{k=1}^{r} u_k(t) A_k \right) X, \quad X(0) = I,$$

where $A_d, A_1, ..., A_r \in \mathfrak{g}$ and $u_k(t) \in \mathbb{R}$ for $k = 1, \ldots, r$.

- **Reachable set**

$$\mathcal{R}_\Sigma(I) = \{X \in G \mid \exists T \geq 0 \text{ and } u_1, ..., u_r : X(T) = X\}$$

Controllability on Homogeneous Spaces

Controllability and Accessibility:

- **Accessibility:** The reachable set $\mathcal{R}_\Sigma(I)$ has an interior point.
- **Controllability:**

$$\mathcal{R}_\Sigma(I) = G.$$

- **System Lie Algebra:**

  $\mathcal{L}_\Sigma$ is the smallest Lie subalgebra of $\mathfrak{g}$ containing $A_1, \ldots, A_r, A_d$, i.e. the smallest subspace containing all the iterated Lie brackets

$$A_d, A_1, \ldots, A_r, [A_d, A_i], [A_i, A_j], [A_d, [A_i, A_j]], \ldots$$

- **System Group:**

  $\mathcal{G}_\Sigma$ is the smallest Lie subgroup of G containing all one-parameter subgroups e^{At}, $A \in \mathcal{L}_\Sigma$, i.e. $\mathcal{G}_\Sigma := \langle \exp(\mathcal{L}_\Sigma) \rangle$.

Controllability on Homogeneous Spaces

Controllability and Accessibility:

Theorem.(Jurdjevic/Sussmann)

- (Σ) is **accessible** if and only if the system Lie algebra is $\mathcal{L}_\Sigma = \mathfrak{g}$.
- If G is compact, than **controllability** of (Σ) is equivalnet to **accessibility**.

Controllability on Homogeneous Spaces

Definition A **homogeneous space** is a manifold M on which G acts transitively via $\alpha : (X, p) \mapsto X \cdot p$. The homogeneous spaces are thus the quotient spaces

$$G/H$$

by a closed Lie subgroup $H \subset G$.

Example: Grassmannian

The **complex Grassmannian** is the unitary orbit

$$\mathrm{Grass}(\mathrm{k}, \mathrm{n}) := \{\rho \in \mathbb{C}^{\mathrm{n} \times \mathrm{n}} | \rho = \rho^*, \rho^2 = \rho, \mathrm{tr}(\rho) = \mathrm{k}\}.$$

Can be identified with manifold of k-dimensional complex subspaces of $\mathbb{C}^n$.

Controllability on Homogeneous Spaces

Lie Group Actions and Induced / Lifted Systems

- **Induced Control System on G/H:**

$$\dot{p} = \mathrm{D}_1\alpha(I,p)\left(A_d + \sum_{k=1}^{r} u_k(t)A_k\right), \quad p(0) = p_0.$$

- **Fact:** Each solution $X(t)$ of the bilinear system yields a solution $p(t)$ of the induced system via

$$p(t) := X(t) \cdot p_0$$

and vice versa.

Controllability on Homogeneous Spaces

Transitive Group Actions and Accessibility

Theorem.

(a) If the system is **controllable** on G, then the induced system is **controllable** on G/H.

(b) The system group $\mathcal{G}_\Sigma$ acts **transitively** on G/H if and only if the induced system is is **accessible** on G/H.

Two Cases of Interest:

- $G/H = \mathbf{P}^{n-1} := \mathrm{Grass}(1, \mathrm{n})$ **projective space**
- $G/H = \mathrm{Grass}(\mathrm{k}, \mathrm{n}), \mathrm{k} \geq 2,$ **Grassmannian**

Controllability on Homogeneous Spaces

Sufficient Condition for Controllability

Theorem. The induced bilinear system is **controllable** on G/H, provided

(a) The induced system is is **accessible** on G/H.

(b) There exist **constant controls** such that the induced system is **weakly Poisson stable** on G/H.

Corollary. Let G be a **compact** connected Lie group and H a closed Lie subgroup. Then the induced bilinear system is **controllable on** G/H **if and only if the system group** $\mathcal{G} \subset G$ **acts transitively on** G/H.

Classification of Transitive Lie Group Actions

Classification of Transitive Lie Groups Actions

Transitive Lie Group Actions on $\mathbb{R}^m \setminus \{0\}$

Theorem. (Boothby/Wilson; Tits; Kramer) Let $A_d, A_1, \ldots, A_r \in \mathfrak{gl}(m, \mathbb{R})$ with $m \geq 2$. The bilinear system

$$(\Sigma) \qquad \dot{x} = \Big(A_d + \sum_{k=1}^{r} u_k(t) A_k\Big) x, \quad x(0) = x_0$$

is accessible on $\mathbb{R}^m \setminus \{0\}$ if and only if the system Lie algebra $\mathcal{L}_\Sigma \subset \mathfrak{gl}(m, \mathbb{R})$ is conjugate to one of the following types:

(1) $\mathfrak{so}(m) \oplus \mathbb{R}$, if $m \geq 3$.

(2) $\mathfrak{su}(m/2) \oplus \mathrm{e}^{\mathrm{i}\alpha}\mathbb{R}$ and $\mathfrak{su}(m/2) \oplus \mathbb{C}$, if m is even and $m \geq 3$.

(3) $\mathfrak{sp}(m/4) \oplus \mathrm{e}^{\mathrm{i}\alpha}\mathbb{R}$, $\mathfrak{sp}(m/4) \oplus \mathbb{C}$ and $\mathfrak{sp}(m/4) \oplus \mathbb{H}$, if $m = 4k$.

(4) $\mathfrak{g}_2 \oplus \mathbb{R}$, if $m = 7$.

(5) $\mathfrak{spin}(7) \oplus \mathbb{R}$, if $m = 8$.

(6) $\mathfrak{spin}(9) \oplus \mathbb{R}$, if $m = 16$.

Classification of Transitive Lie Groups Actions

Transitive Lie Group Actions on $\mathbb{R}^m \setminus \{0\}$ (cont'd)

(7) $\mathfrak{sl}(m,\mathbb{R})$ and $\mathfrak{gl}(m,\mathbb{R})$, if $m \geq 2$.

(8) $\mathfrak{sl}(m/2,\mathbb{C})$, $\mathfrak{sl}(m/2,\mathbb{C}) \oplus \mathrm{e}^{\mathrm{i}\beta}\mathbb{R}$ and $\mathfrak{gl}(m/2,\mathbb{C})$, if m is even and $m \geq 2$.

(9) $\mathfrak{sl}(m/4,\mathbb{H})$, $\mathfrak{sl}(m/4,\mathbb{H}) \oplus \mathrm{e}^{\mathrm{i}\beta}\mathbb{R}$ and $\mathfrak{sl}(m/4,\mathbb{H}) \oplus \mathbb{C}$, if $m = 4k$.

(10) $\mathfrak{sl}(m/4,\mathbb{H}) \oplus \mathfrak{sp}(1)$ and $\mathfrak{sl}(m/4,\mathbb{H}) \oplus \mathbb{H}$, if $m = 4k$.

(11) $\mathfrak{sp}(m/2,\mathbb{R})$ and $\mathfrak{sp}(m/2,\mathbb{R}) \oplus \mathbb{R}$, if m is even and $m \geq 3$.

(12) $\mathfrak{sp}(m/4,\mathbb{C})$, $\mathfrak{sp}(m/4,\mathbb{C}) \oplus \mathrm{e}^{\mathrm{i}\beta}\mathbb{R}$ and $\mathfrak{sp}(m/4,\mathbb{C}) \oplus \mathbb{C}$, if $m = 4k$.

(13) $\mathfrak{spin}(9,1,\mathbb{R})$ and $\mathfrak{spin}(9,1,\mathbb{R}) \oplus \mathbb{R}$, if $m = 16$.

Comments

- Boothby/Wilson: $\mathfrak{spin}(9,1,\mathbb{R})$ and $\mathfrak{spin}(9,1,\mathbb{R}) \oplus \mathbb{R}$ missing.
- Complete proof: L. Kramer (2003) .
- Open Problem: Affine Systems.

Transitive Lie Groups Actions

Transitive Lie Group Actions on Projective Space

Theorem (compact case). Let $A_d, A_1, \ldots, A_m \in \mathfrak{so}(N, \mathbb{R})$ with $N \geq 2$. The bilinear system

$$\dot{\rho} = \left[A_d + \sum_{k=1}^{r} u_k(t) A_k, \rho \right]$$

is controllable on projective space $\mathbf{P}^{N-1}(\mathbb{R})$ if and only if the system Lie algebra $\mathcal{L} \subset \mathfrak{so}(N, \mathbb{R})$ is conjugate to one of the following types:

(1) $\mathfrak{so}(N, \mathbb{R})$

(2) $\mathfrak{su}(N/2)$ or $\mathfrak{u}(N/2)$, if N is even.

(3) $\mathfrak{sp}(N/4)$, $\mathfrak{sp}(N/4) \oplus \mathfrak{u}(1)$ or $\mathfrak{sp}(N/4) \oplus \mathfrak{sp}(1)$, if $N = 4k$.

(4) $\mathfrak{g}_2$, if $N = 7$.

(5) $\mathfrak{spin}(7)$, if $N = 8$.

(6) $\mathfrak{spin}(9)$, if $N = 16$.

Transitive Lie Groups Actions

Transitive Lie Group Actions on odd-dimensional Spheres

Corollary (compact case). Let $A_d, A_1, ..., A_m \in \mathfrak{so}(2n, \mathbb{R})$, $n > 1$. The bilinear system

$$\dot{x} = \left(A_d + \sum_{k=1}^{m} u_k(t) A_k\right) x, \quad x(0) = x_0 \in S^{2n-1}$$

is controllable on the sphere $\mathbf{S}^{2n-1}$ of odd dimension $2n-1$ if and only if the system Lie algebra $\mathcal{L} \subset \mathfrak{so}(2n)$ is **conjugate** to one of the following types:

(1) $\mathfrak{so}(2n)$

(2) $\mathfrak{su}(n)$ or $\mathfrak{u}(n)$.

(3) $\mathfrak{sp}(n/2)$, $\mathfrak{sp}(n/2) \oplus \mathfrak{u}(1)$ or $\mathfrak{sp}(n/2) \oplus \mathfrak{sp}(1)$, if $n = 2k$.

(4) $\mathfrak{spin}(7)$, if $n = 4$.

(5) $\mathfrak{spin}(9)$, if $n = 8$.

Transitive Lie Groups Actions

Comments

- Conjugate can not be replaced by isomorphic! Counterexample: $\mathfrak{so}(3), \mathfrak{su}(2) \subset \mathfrak{so}(4)$ are isomorphic, but not conjugate to each other. $\mathfrak{so}(3)$ does not act transitively on S^3!
- Montgomery/Samelson: Classification of transitive compact group actions on spheres. The above result does not appear there.
- Brockett: incomplete characterization; $\mathfrak{sp}(N/4) \oplus \mathfrak{u}(1)$ and $\mathfrak{sp}(N/4) \oplus \mathfrak{sp}(1)$ missing.

Transitive Lie Groups Actions

Transitive Lie Group Actions on projective space

Theorem (arbitrary complex case). Let $A_d, A_1, ..., A_m \in \mathfrak{sl}(N, \mathbb{C})$ with $N \geq 2$. The linear induced bilinear system is controllable on complex projective space $\mathbf{P}^{N-1}(\mathbb{C})$ if and only if the system Lie algebra $\mathcal{L} \subset \mathfrak{sl}(N, \mathbb{C})$ is conjugate to one of the following types:

(1) $\mathfrak{sl}(N, \mathbb{C})$

(2) $\mathfrak{su}(N)$.

(3) $\mathfrak{sp}(N/2, \mathbb{C})$, $\mathfrak{sp}(N/2)$ for N even.

(4) $\mathfrak{sl}(N/2, \mathbb{H})$ for N even.

Transitive Lie Groups Actions

Transitive Lie Group Actions on Grassmannians

Theorem (arbitrary complex case). Let $A_d, A_1, ..., A_m \in \mathfrak{sl}(N, \mathbb{C})$ with $N \geq 2$. The linear induced bilinear system is controllable on **complex Grassmann manifold** $\mathbf{Grass}(k, N), k \geq 2$ if and only if the system Lie algebra $\mathcal{L} \subset \mathfrak{sl}(N, \mathbb{C})$ is conjugate to one of the following types:

(1) $\mathfrak{sl}(N, \mathbb{C})$.

(2) $\mathfrak{su}(N)$

The same classification holds true for an arbitrary **complex flag manifold**!

Transitive Lie Groups Actions

Applications: Generalization of Albertini-Alessandro result

Corollary. The **Liouville Master equation**

$$\dot{\rho} = -\mathrm{i}\big[H_d + \sum_{k=1}^{r} u_k(t) H_k, \rho\big]$$

is controllable on

$$\mathrm{Grass}(\mathrm{k}, \mathrm{N}) := \{\rho \in \mathbb{C}^{\mathrm{N}\times\mathrm{N}} | \rho = \rho^*, \rho^2 = \rho, \mathrm{tr}(\rho) = \mathrm{k}\}.$$

if and only if the system Lie algebra $\mathcal{L} \subset \mathfrak{su}(N, \mathbb{C})$ is conjugate to one of the following types:

- $k = 1$ or $N - 1$. $\mathcal{L} = \mathfrak{sp}(N/2)$, for N even, or $\mathcal{L} = \mathfrak{su}(N)$.
- $\mathcal{L} = \mathfrak{su}(N)$ for $1 < k < N - 1$.

Transitive Lie Groups Actions

Comments. k=1

- Proof: **trivial consequence of Theorem (arbitrary complex case)**.
- Alessandro-Albertini proof:
  - **gap** in the proof, as $\mathfrak{sp}(N/4) \oplus \mathfrak{sp}(1)$ case is not discussed.
  - **complicated** proof, using representation theoretic arguments, and weak version of Corollary (compact case)–only necessary; isomorphic instead conjugate.
  - **prove equivalence of pure state and projective state controllability of quantum systems**. Follows also immediately by combining our proof with theirs.

Transitive Lie Groups Actions

Alternative Proof.

(1) $\mathfrak{so}(2n)$: too high dimension

(2) $\mathfrak{u}(n)$: too high dimension

(3) $\mathfrak{sp}(n/2) \oplus \mathfrak{u}(1)$, $\mathfrak{sp}(n/2) \oplus \mathfrak{sp}(1)$: no Lie algebra strictly between $\mathfrak{sp}(n/2)$ and $\mathfrak{su}(n)$ ($\mathfrak{sp}(n/2) =$ Cartan-summand of $\mathfrak{su}(n)$)

(4) $\mathfrak{spin}(7)$: too high dimension

(5) $\mathfrak{spin}(9)$: extra argument

Main Results A :
Accessibility of Quantum Systems

Main Results: Accessibility of Quantum Systems

N-level Systems:

Theorem A (N-level systems).

The (unital) N-level **Lindblad Master Equation**

$$\dot{\rho} = -\mathrm{i}\Big[H_d + \sum_{k=1}^{r} u_k(t) H_k, \rho\Big] + \mathcal{L}(\rho), \quad \rho(0) = \rho_0 \in \mathcal{P}$$

is **accessible** if and only if the system Lie algebra $\mathcal{L}_\Sigma$ is conjugate to

- For $N > 2$ even: $\mathfrak{gl}(m, \mathbb{R})$
- For N odd: $\mathfrak{gl}(m, \mathbb{R})$, $\mathfrak{gl}(m/2, \mathbb{C})$, $\mathfrak{sl}(m/2, \mathbb{C}) \oplus \mathrm{e}^{\mathrm{i}\beta}\mathbb{R}$, $\mathfrak{sl}(m/4, \mathbb{H}) \oplus \mathrm{e}^{\mathrm{i}\beta}\mathbb{R}$, $\mathfrak{sl}(m/4, \mathbb{H}) \oplus \mathbb{C}$, $\mathfrak{sl}(m/4, \mathbb{H}) \oplus \mathbb{H}$, $\mathfrak{sp}(m/2, \mathbb{R}) \oplus \mathbb{R}$, $\mathfrak{sp}(m/4, \mathbb{C}) \oplus \mathrm{e}^{\mathrm{i}\beta}\mathbb{R}$, $\mathfrak{sp}(m/4, \mathbb{C}) \oplus \mathbb{C}$

Main Results: Accessibility of Quantum Systems

Spin-$\frac{1}{2}$ Systems: $m = 2^{2n} - 1$

Theorem B (special case : spin-$\frac{1}{2}$ systems).

(a) For arbitrary $\boldsymbol{n}$ **spin-$\frac{1}{2}$ systems**, $n \geq 2$, the (unital) Lindblad Master Equation is **accessible** if and only if $\mathcal{L}_\Sigma = \mathfrak{gl}(2^{2n} - 1, \mathbb{R})$

(b) Exceptional case $(n = 1\ ,\ N = 2)$: a 2-level system is accessible if and only if

- $\mathcal{L}_\Sigma = \mathfrak{gl}(3, \mathbb{R})$
- $\mathcal{L}_\Sigma = \mathfrak{so}(3) \oplus \mathbb{R}\, I_3$

Main Results: Accessibility of Quantum Systems

Spin-$\frac{1}{2}$ Systems: $m = 2^{2n} - 1$

Sketch of the proof.

- For n spin-$\frac{1}{2}$ systems, the dimension $m = (2^n)^2 - 1$ is odd. Thus, the only transitive algebras left on previous list are $\mathfrak{so}(m) \oplus \mathbb{R}\, I_3$, $\mathfrak{sl}(m, \mathbb{R})$ and $\mathfrak{gl}(m, \mathbb{R})$.
- $\mathfrak{sl}(m, \mathbb{R})$ can be excluded due to the condition Tr$(\mathcal{L}) < 0$.
- $\mathfrak{so}(3) \oplus \mathbb{R}$ is only possible for 2-level systems due to the Lie algebra isomorphism between $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$.

Comment:

(Altafini'03) states incorrectly that $\mathcal{L}_\Sigma = \mathfrak{gl}(m, \mathbb{R})$ is necessary and sufficient for accessibility **for arbitrary N-level systems**.

Generalized
Double Bracket Flow

Controllability of Double Bracket Equation

Generalized Double Bracket Flow:

A Hermitian, B skew-Hermitian.

- Controlled Isospectral Flow

$$\dot{X} = \Big[\Big[A(u), X\Big], X\Big] + \Big[B(u), X\Big]$$

  on Hermitian matrices.

- Generalizes **Brockett's double bracket flow.**
- Generalizes **Matrix Riccati Differential Equation.**
- Lie -algebraic generalization: Mittenhuber.

Controllability of Riccati Equation

Generalized double bracket flow on projection operators is equivalent to Riccati differential equation:

$$\dot{K} = -KM_{11}(u) + M_{22}(u)K - KM_{12}(u)K + M_{21}(u)$$

- **Associated "Hamiltonian" matrix:**

$$M(u) := A(u) + B(u) = \begin{bmatrix} M_{11}(u) & M_{12}(u) \\ M_{21}(u) & M_{22}(u) \end{bmatrix}$$

- **System Lie algebra $\mathcal{L}_\Sigma$, generated by $M(u)$, $u \in \mathbb{R}^m$.**

Controllability of Riccati Equation

Riccati Equation:

Theorem The Riccati equation is **accessible** if and only if one of the following conditions is satisfied for the system Lie algebra:

(a) n is odd or $1 < k < n-1$. $\mathcal{L}_\Sigma$ is conjugate to $\mathfrak{sl}_n(\mathbb{C})$ or $\mathfrak{su}_n(\mathbb{C})$.

(b) n is even and $k = 1$ or $k = n-1$. $\mathcal{L}_\Sigma$ is conjugate to $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{su}_n(\mathbb{C})$, $\mathfrak{sl}_{n/2}(\mathbb{H})$, $\mathfrak{sp}_{n/2}(\mathbb{C})$ or $\mathfrak{sp}_{n/2}$.

Theorem Let $M(u) \in \mathfrak{sp}_n(\mathbb{C})$ be skew-Hermitian and Hamiltonian. The Riccati equation is **accessible** if and only if the system Lie algebra $\mathcal{L}_\Sigma$ is conjugate to $\mathfrak{sp}_n(\mathbb{C})$ or to $\mathfrak{sp}_n$.

Controllability of Double Bracket Flows

Generalized Double Bracket Flow:

- **When is the Generalized Double Bracket Flow**

$$\dot{P} = [[A(u), P], P] + [B(u), P]$$

accessible?

- **Isospectral Projection operator:**

$$P := \begin{bmatrix} I_k \\ K \end{bmatrix} \left(I_k + K^* K \right)^{-1} \begin{bmatrix} I_k & K^* \end{bmatrix},$$

Controllability of Double Bracket Flow

Theorem The generalized double bracket equation is **accessible on constant rank projection operators** if and only if one of the following conditions is satisfied:

(a) n is odd or $1 < k < n-1$. The Lie algebra $\mathcal{L}_\Sigma$ is conjugate to $\mathfrak{sl}_n(\mathbb{C})$ or $\mathfrak{su}_n(\mathbb{C})$.

(b) n is even and $k=1$ or $k=n-1$. The Lie algebra $\mathcal{L}_\Sigma$ is conjugate to $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{su}_n(\mathbb{C})$, $\mathfrak{sl}_{n/2}(\mathbb{H})$, $\mathfrak{sp}_{n/2}(\mathbb{C})$ or $\mathfrak{sp}_{n/2}$.

Controllability of Double Bracket Flow

- **Accessibility of GDBE does not imply controllability:**
  Counter example $B = 0, A_0 + uA_1$ exists.

- **Bedlewo Conjecture:**
  accessibility implies controllability, provided the Lie algebra of $B(u)$ equals $\mathfrak{su}(n)$.

Controllability of Double Bracket Flow

Conclusions:

- Showed how to derive controllability results from the theory of transitive Lie group actions.
- **Application 1:** Characterized controllability of the Liouville Master Equation for N-coupled spin $1/2$ systems. (New proof of results by Altafini, Schirmer, Brockett and others.)
- **Application 2:** A necessary and sufficient condition for accessibility of the generalized double flow on Grassmann manifolds.
- **Application 3:** A necessary and sufficient condition for accessibility of the controlled Riccati differential equation

$$\dot{K} = A_{11}(u)K + KA_{22}(u) - KA_{12}(u)K + A_{21}(u)$$

  on symmetric matrices K.
- **Application 4:** Generalizations for nested Riccati differential equations are possible.